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Présentée par

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Les équations elliptiques du quatrième ordre sur une variété riemannienne

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Dedication

I am dedicating this modest thesis to both my parents, who taught me and made everything well known in my mind and even the largest task can be accomplished if it is done one step at a time. Since my mother has been a source of motivation and I could not forget her support when she always told me these simple words "**Do mathematics**, **Do mathematics**..."

Next, it is dedicated to five beloved people who have meant and continue to mean so much to me. Although they are no longer of this word, their memories continue to regulate my life. First and foremost, to my unique brother "**Yehya**" who enabled and gave me this opportunity and support with money, advice and so on. Thank you so much and I will never forget you.

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Finally, I dedicate this work and give special thanks to my best friends: **Ikram S**, **Soumia Z**, **Mohammed B.S**, **Seif Amir B, Youssef T** and **Noureddine M** who I will wish never forget them.

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0.1 Introduction

L'étude des équations aux dérivées partielles elliptiques est l'un des sujets de recherche de grande importance dans l'analyse sur les variétés développée ces dernières années dans de nombreux travaux [5]-[6]-[7]-[8] -[9]-[10].

La résolution des équations aux dérivées partielles elliptiques et les problèmes liés à la géométrie conforme a conduit à développer des outils d'analyse non-linéaire, comme par exemple "la méthode variationnelle" pour résoudre *le problème de Yamabé*, *le problème de la courbure scalaire* et *la Q-courbure prescrites.*

Le problème de Yamabé s'énonce comme suit: *étant donnée (M, g) une variété riemannienne compacte de classe C^∞ de dimension $n \geq 2$, on note par S_g la courbure scalaire de g. Existe-t-il une métrique $\tilde{g}$ conforme à g qui est de courbure scalaire constante ?*

On a deux cas:

• Si $\dim M = 2$ et $\tilde{g} = e^{2u}g$ où $u \in C^\infty(M)$: les deux métriques $\tilde{g}$ et g sont reliées par l'expression suivante:

$$\Delta_g u + \frac{1}{2} S_g u = \frac{1}{2} S_{\tilde{g}} e^{2u} \tag{1}$$

• Si $\dim M \geq 3$ et $\tilde{g} = u^{\frac{4}{n-2}} g$ avec $u \in C^\infty(M)$, $u > 0$, on obtient

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = \frac{n-2}{4(n-1)} S_{\tilde{g}} u^{2^*-1} \tag{2}$$

L'existence d'une métrique conforme à courbure scalaire constante est équivalent à l'existence d'une solution positive de l'équation aux dérivées partielles du second ordre (1) et (2). La difficulté de cette question provient essentiellement de l'exposant critique de Sobolev $2^* = \frac{2n}{n-2}$ pour l'injection $H_1^2(M) \subset L^{2^*}(M)$ qui rend les méthodes variationnelles classiques inefficaces. En 1960, Yamabé dans [33] avait donné les outils essentiels pour attaquer la question, malheureusement ou heureusement sa démonstration est incomplète: en effet en 1968 Trudinger dans [32] remarqua un sérieux gap dans la preuve de Yamabé et résolva la question dans le cas où l'invariant de Yamabé est inférieur ou égal à zéro, huit ans plus tard T. Aubin dans [3] avec une idée géniale qui consistait à isoler les points de concentration avait donné une démonstration satisfaisante dans le cas où la variété est non conformément plate et de dimension $n \geq 6$. Pour les variétés conformément plates et les petites dimensions le problème a été complètement résolu par Schoen dans [27] en 1984 via le théorème de la masse positive trouvé en collaboration avec Yau.

Si on définit l'opérateur conforme de Yamabé par

$$\begin{aligned} L_g^n : \quad & C^2(M) \to C^0(M) \\ & u \to L_g^n(u) = \Delta_g u + \tfrac{n-2}{4(n-1)} S_g u \end{aligned}$$

alors il est invariant conforme dans le sens suivant:

- Si $\dim M = 2$ et $\tilde{g} = e^{2u} g$, $u \in C^\infty(M)$:

$$\forall \varphi \in C^2(M) : \Delta_g \varphi = e^{2u} \Delta_{\tilde{g}} \varphi$$

- Si $\dim M \geq 3$ et $\tilde{g} = u^{\frac{4}{n-2}} g$, $u \in C^\infty(M)$, $u > 0$:

$$\forall \varphi \in C^2(M) : L_g^n(u\varphi) = u^{\frac{n+2}{n-2}} L_{\tilde{g}}^n(\varphi)$$

L'équation (2) se généralise aux équations dites de courbure scalaire prescrite et qui s'exprime par:

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = f(x) u^{2^*-1} \text{ où } f \in C^\infty(M) \tag{3}$$

La positivité de la solution de l'équation (3) s'obtient grâce au principe de maximum et le fait que si $u \in H_1^2(M)$, alors $|u| \in H_1^2(M)$.

En 1983 un important opérateur conforme du quatrième a été découvert par Paneitz en dimension 4 et ensuite étendu aux dimensions supérieures par Branson. Par analogie avec le problème de Yamabé, on considère les équations contennant l'opérateur de Paneitz-Branson qui sont des équations elliptiques du $4^{ème}$ et qui dans des situations particulières elles correspondent au problème de la *Q-courbure prescrite* et qui s'énonce comme suit:

Etant donnée (M,g) *une variété riemannienne compacte de classe* C^∞ *de dimension* $n \geq 5$*, on note* Q_g *est la Q-courbure associée à* g *et on se donne* $f \in C^\infty(M)$.
Par analogie avec le problème de Yamabé, on peut se poser la question de savoir s'il existe une métrique $\tilde{g}$ *conforme à* g *telle que la* $Q_{\tilde{g}} = f$ *?*
L'opérateur de Paneitz-Branson est définit par:

$$P_g^n(u) = \Delta_g^2 u - div_g\left((\frac{(n-2)^2+4}{2(n-2)(n-1)} S_g.g - \frac{4}{n-2} Ric_g).du \right) + \frac{n-4}{2} Q_g u$$

où

$$Q_g^n u = \frac{1}{2(n-1)} \Delta_g S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |Ric_g|^2$$

L'invariant conforme de P_g^n se traduit par: soit $\tilde{g} = u^{\frac{n+4}{n-4}} g$, $u > 0$, une métrique conforme à g, alors l'opérateur P_g^n est conformément invariant dans le sens suivant: pour tout $\varphi \in C^\infty(M)$,

$$P_g^n(\varphi.u) = u^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(\varphi).$$

Des équations elliptiques de type de *Q-courbure prescrite* sont de la forme:

$$P_g^n(u) = f(x) |u|^{N-2} u \tag{4}$$

Chercher des solutions positives de l'équation (4) est problème majeur pour deux raisons: la première est le fait que si $u \in H_2^2(M)$, $|u|$ n'est pas dans $H_2^2(M)$, la deuxième il n'y a pas de principe de maximum. Dans cette thèse on se propose de résoudre les équations de type de la $Q-$courbure prescrite.

Notre manuscript est structuré comme suit

Dans le premier chapitre on présente les éléménts de la géométrie rièmanienne nécéssaires à la comprehension des chapitres suivants. Dans le deuxième chapitre on étudie le problème suivant:

Existe-t-il une fonction $u \in H_2^2(M)$ solution de l'équation ?

$$\Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u = \lambda \left|u\right|^{q-2} u + f(x)\left|u\right|^{N-2} u \tag{5}$$

où a, b et f trois fonctions de classe C^∞ sur M avec f strictement positive, $\lambda > 0$ suffisamment petit et $1 < q < 2$ ($N = \frac{2n}{n-4}$) l'exposent critique de Sobolev.
Les résultats obtenus s'énoncent comme suit:

Théorème 0.1 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 6$ et a, b et f trois fonctions de classes C^∞ sur M telles que:*

1. *$f(x) > 0$ sur toute la variété M.*
2. *Au point $x_\circ$ où f atteint son maximum, on suppose que:*

$$S_g(x_\circ) + 3a(x_\circ) > 0 \text{ pour } n = 6$$

et

$$\left(\frac{(n^2+4n-20)}{2(n+2)} S_g(x_\circ) + \frac{(n-1)}{(n+2)} a(x_\circ) - \frac{(n-6)}{8} \frac{\Delta f(x_\circ)}{f(x_\circ)}\right) > 0 \text{ pour } n > 6.$$

Alors, l'équation (5) admet une solution u non triviale de classe $C^{4,\alpha}, \alpha \in (0,1)$.

En cas particulier, quand $a(x) = \alpha$ et $b(x) = \beta$ deux fonctions constantes sur M, on trouve:

Théorème 0.2 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$, on suppose que:*

1. *$P_g(u) := \Delta_g^2 u - \alpha \Delta_g u + \beta u$ est coercif.*
2. *$J_\lambda(u) \leq c < \frac{2}{n K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}$.*

3. $\alpha^2 - 4\beta > 0$ *et* $\alpha > 0$.

   Alors, il existe $\lambda^* > 0$*, telle que pour tout* $\lambda \in (0, \lambda^*)$*, l'équation (5) possède trois solutions distinctes dans* $H_2^2(M)$.

Le troisième chapitre concerne les équations du type de la Q-courbure singulière: plus précisément on cherche une fonction $u \in H_2^2(M)$ solution de l'équation

$$\Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u = \lambda\left|u\right|^{q-2}u + f(x)\left|u\right|^{N-2}u \tag{6}$$

où $a \in L^r(M)$ et $b \in L^s(M)$ où $r > \frac{n}{2}$ et $s > \frac{n}{4}$ et f une fonction de classe C^∞ sur M strictement positive, $\lambda > 0$ suffisamment petit et $1 < q < 2$ ($N = \frac{2n}{n-4}$) l'exposent critique de Sobolev. On obtient les résultats suivants

Théorème 0.3 *Soit* (M, g) *une variété riemannienne compacte de dimension* $n \geq 6$ *et* a, b , f *trois fonctions de classes* C^∞ *sur* M *telle que* $f(x) > 0$ *sur toute la variété* M*. Supposons que:*

1. $u \to P_g(u) = \Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u$ *est coercif.*

2. *Au point* $x_\circ$ *où* f *atteint son maximum, on suppose que:*

$$S_g(x_\circ) > 0 \ \textit{pour}\ n = 6$$

et

$$\left(\frac{n\left(n^2+4n-20\right)}{(n-2)(n-4)(n-6)\left(1+\|b\|_s+\|a\|_r\right)^{\frac{4}{n}}} - \frac{n-2}{(n-1)}\right)S_g(x_\circ) - \frac{3\Delta f(x_\circ)}{f(x_\circ)} > 0 \ \textit{pour}\ n > 6$$

alors, l'équation (6) admet une solution u *non triviale.*

Le chapitre 4, a pour objectif l'étude de l'équation suivante:

$$\Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^\sigma}\nabla_g u\right) + \frac{b(x)}{\rho^\mu}u = f(x)\left|u\right|^{N-2}u + \lambda\frac{h(x)}{\rho^\beta}\left|u\right|^{q-2}u \tag{7}$$

Où a, b et h trois fonctions de classe $C^\infty(M)$ et $2 < q < N$ et $\lambda > 0$ un paramètre réel. On obtient plus précisément les principaux résultats suivants:

Théorème 0.4 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 6$ et a, b , f trois fonctions de classes C^∞ sur M telle que $f(x) > 0$ sur toute la variété M. Supposons que:*

1. $u \to P_g(u) = \Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^\sigma}\nabla_g u\right) + \frac{b(x)}{\rho^\beta}u$ *est coercif.*

2. *Au point $x_\circ$ où f atteint son maximum, on suppose que:*

$$S_g(x_\circ) > 0 \textit{ pour } n = 6$$

et

$$\left(\frac{n(n-2\sqrt{6}+2)(n+2\sqrt{6}+2)-(n-6)(n-4)^3(n+2)}{12(n+2)(n-4)^2(n-6)\left(1+\|a\|_r+\|b\|_s\right)^{\frac{4}{n}}}S_g(x_\circ)\right.$$
$$\left.-\frac{(n-4)\Delta f(x_\circ)}{8f(x_\circ)}\right) > 0 \quad \textit{pour} \quad n > 6.$$

Alors, il existe $\lambda^* > 0$, telle que pour tout $\lambda \in (0,\ \lambda^*)$, l'équation (7) admet une solution u non triviale dans $H_2^2(M)$.

Le chapitre 5, concerne la multiplicité des solutions du problème singulier de type Q-courbure dans le cas critique: plus précisément on considère l'équation suivante

$$\Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^2}\nabla_g u\right) + \frac{b(x)}{\rho^4}u = f(x)\left|u\right|^{N-2}u + \lambda\left|u\right|^{q-2}u \tag{8}$$

Où a, b et f trois fonctions de classe C^∞ sur M avec f strictement positive, $0 < \sigma < 2$ et $0 < \beta < 4$.

Les résultats obtenus s'énnoncent comme suit:

Théorème 0.5 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 6$ et a, b , f trois fonctions de classes C^∞ sur M telle que $f(x) > 0$ sur toute la variété M. Supposons que:*

1. $u \to P_g(u) = \Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^\sigma}\nabla_g u\right) + \frac{b(x)}{\rho^\beta}u$ *est coercif.*

2. Au point $x_\circ$ *où* f *atteint son maximum, on suppose que:*

$$S_g(x_\circ) > 0 \text{ pour } n = 6$$

et

$$\left(\frac{S_g(x_\circ)}{6(n-1)} + \frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)}\right) < 0 \text{ et } S_g(x_\circ) > 0 \text{ pour } n > 6.$$

Alors, il existe $\lambda^* > 0$*, telle que pour tout* $\lambda \in (0, \lambda^*)$*, l'équation*

$$\Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^\sigma}\nabla_g u\right) + \frac{b(x)}{\rho^\beta}u = f(x)\,|u|^{N-2}\,u + \lambda\,|u|^{q-2}\,u$$

possède trois solutions distinctes dans $H_2^2(M)$*.*

Théorème 0.6 *Soient* $\lambda \in (0, \lambda^*)$ *et* $\left(u^+_{m,\sigma,\beta}\right)_{m\in\mathbb{N}} \subset N^+_{\lambda,\sigma,\beta}$ *telle que*

$$\begin{cases} J_{\lambda,\sigma,\beta}(u^+_{m,\sigma,\beta}) = c^+_{\lambda,\sigma,\beta} + o(1) \\ \nabla J_{\lambda,\sigma,\beta}(u^+_{m,\sigma,\beta}) = o(1), \text{ dans } \left(H_2^2(M)\right)^* \end{cases}$$

Supposons que

$$\begin{cases} \left|c^+_{\lambda,\sigma,\beta}\right| < \frac{2}{nK_\circ^{\frac{n}{4}}\left(\max_{x\in M} f(x)\right)^{\frac{n-4}{4}}} \\ \frac{1}{2} + a^- K^2(n,1,2) + b^- K^2(n,2,4) > 0 \end{cases}$$

Alors l'équation (8) possède une solution non triviale $u^+ \in N^+_{\lambda,\sigma,\beta}$ *dans* $H_2^2(M)$*.*

Les travaux de cette thèse ont fait l'objet des publications suivantes:

1. M. Benalili, K. Tahri, Nonlinear elliptic fourth order equations existence and multiplicity results, Nonlinear Differ. Equ. Appl. 18, 539-556, (2011).

2. M. Benalili, K. Tahri, Existence of solutions to singular fourth-order elliptic equations. Electron. J. Differ. Equ; 1-23, (2013).

3. M. Benalili, K. Tahri, Multiple solutions to singular fourth order elliptic equations on compact manifolds, Complex Variables and Elliptic Equations; 1-28, (2014).

4. K. Tahri, On singular elliptic equations involving critical Sobolev exponent, Journal of Physics: Conference Series 482 ,1-8 (2014).

Chapitre 1

Notions Préliminaires

Dans ce chapitre, nous présentons quelques outils nécessaires d'analyse non linéaire sur les variétés qui seront utilisés dans la suite de notre travail.
Pour plus d'information sur le sujet on renvoie le lecteur au référence [4].

1.1 Courbures riemanniennes

Définition 1.1 *Soit* (M, g) *une variété riemannienne de dimension* $n \geq 1$ *et* ∇ *la connexion de Levi-Civita donnée par son expression locale*

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{li}}{\partial xj} - \frac{\partial g_{ij}}{\partial xl} \right)$$

où g_{ij} *et* g^{ij} *désignent respectivement le tenseur riemannien et son inverse.*

1. *Le tenseur de courbure* R *relatif à la connexion* ∇ *s'écrit:*

$$R_{ijk}^l = \frac{\partial \Gamma_{ki}^l}{\partial x_j} - \frac{\partial \Gamma_{ji}^l}{\partial x_k} + \Gamma_{j\alpha}^l \Gamma_{ki}^\alpha - \Gamma_{k\alpha}^l \Gamma_{ji}^\alpha$$

2. *Le tenseur de Riemann* Rm_g *est alors donné par:*

$$R_{ijkl} = g_{i\alpha} R_{jkl}^\alpha$$

3. *Le tenseur de courbure de Ricci est obtenu par contraction du tenseur de courbure riemannienne:*

$$R_{ij} = R^{\alpha}_{i\alpha j} = g^{\alpha\beta} R_{i\alpha j\beta}$$

4. *La courbure scalaire est la fonction numérique de classe* C^{∞} *sur* M *notée par* S_g *est définie par:*

$$S_g = R_{ij} g^{ij}$$

En particulier si $g \in H_2^p(M, T^*M \otimes T^*M)$, alors:

$$S_g \in L^p(M)$$

On aura l'occasion d'utiliser ces propriétés dans les chapitres qui suivent

1.2 Développement de la mesure riemannienne

Considérons un système de coordonnées normales géodésiques $(y^1, ..., y^n)$ centré en $x_\circ$.

Soit $S(r)$ l'ensemble des points situés à la distance r de $x_\circ$ ($r < d$ le rayon d'injectivité) et $d\Omega$ l'élément volume de la sphère $S^{n-1}(1)$ unité de dimension $(n-1)$.

Posons:

$$G(r) = \frac{1}{\omega_{n-1}} \int_{S(r)} \sqrt{|g(x)|} d\Omega$$

ω_{n-1} désigne l'aire de $S^{n-1}(1)$ et $|g(x)|$ le déterminant de la métrique g.

Un développement limité de $G(r)$ au voisinage de $r = 0$ est donné par:

Proposition 1 *[3]*

$$G(r) = 1 - \frac{S_g(x_\circ)}{6n} r^2 + o(r^2)$$

où $S_g(x_\circ)$ *est la courbure scalaire de* M *au point* $x_\circ$.

Un développement limité de la mesure riemannienne $dv(g)$ au voisinage de $r = 0$ est donné par:

Proposition 2 *[3]*

$$dv(g) = 1 - \frac{1}{6}R_{ij}(x_\circ)y^i y^j + o(r^2)$$

Pour les calculs qui suivent nous utilisons les facteurs suivants: p et q étant deux réels positifs, posons $p - q > 1$,

$$I_p^q = \int_0^{+\infty} \frac{t^q}{(1+t)^p} dt$$

Proposition 3 *Avec les notations ci-dessous, nous avons les relations:*

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \quad , \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q$$

1.3 Opérateur de Laplace-Beltrami

Définition 1.2 *Soit (M, g) une variété riemannienne de classe C^∞ et soit $u \in C^\infty(M)$,*

$$\Delta_g u := -\frac{1}{\sqrt{|g(x)|}} \frac{\partial}{\partial y_j} \left(\sqrt{|g(x)|} g^{ij} \frac{\partial u}{\partial y_i} \right)$$

En particulier, dans les coordonnées géodésiques polaires et si u est radiale:

$$\Delta_g u := -u^{''}(r) - \frac{n-1}{r} u^{'}(r) - u^{'}(r)\partial_r \log \sqrt{|g(x)|}$$

1.4 Classe conforme de g

Définition 1.3 *Si $f > 0$, est une fonction de classe C^∞sur M strictement positive, $\tilde{g} = \varphi.g$ est dite métrique conforme à g. La classe conforme de g est notée*

$$[g] = \{\varphi.g, \ \varphi \in C^\infty(M) \ et \ \varphi > 0\}$$

Un résultat important dans l'étude des EDP elliptiques par la méthode variationelle du à Rellich-Kondrakov s'énonce comme suit:

1.5 Théorème de Rellich-Kondrakov

Théorème 1.1 *Soit* (M, g) *une variété riemannienne compacte de dimension* $n \geq 5$, $q \geq 1$ *réel, et* $m < k$ *deux entiers.*

1. *Si* $\frac{1}{q} > \frac{k-m}{n}$, *alors* $H_k^q(M) \subset H_m^P(M)$ *pour tout* $p \geq 1$ *tel que* $\frac{1}{p} \geq \frac{1}{q} - \frac{k-m}{n}$.
2. *Si* $\frac{1}{p} > \frac{1}{q} - \frac{k-m}{n}$, *alors l'inclusion est compacte.*
3. *Pour* $N = \frac{2n}{n-4}$ *l'inclusion* $H_2^2(M)$ *dans* $L^N(M)$ *cesse d'être compacte.*

Un invariant conforme a été découvert par Paneitz en dimension 4 et généralisé par Branson pour les dimensions supérieures [13]. L'étude des EDP du $4^{ème}$ ordre est motivé par la découverte d'un invariant conforme due à Paneitz-Branson [13].

1.6 Opérateur de Paneitz-Branson

Définition 1.4 *Pour* $n \geq 5$, *l'opérateur de Paneitz -Branson est donné par*

$$P_g^n(u) = \Delta_g^2 u - div_g\left(\left(\frac{(n-2)^2+4}{2(n-2)(n-1)} S_g.g - \frac{4}{n-2} Ric_g \right).du \right) + \frac{n-4}{2} Q_g u$$

où

$$Q_g^n u = \frac{1}{2(n-1)} \Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |Ric_g|^2$$

L'invariant conforme de P_g^n se traduit par: soit $\tilde{g} = u^{\frac{n+4}{n-4}} g$, $u > 0$, une métrique conforme à g, alors l'opérateur P_g^n est conformément invariant dans le sens suivant: pour tout $\varphi \in C^\infty(M)$,

$$P_g^n(\varphi.u) = u^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(\varphi)$$

1.7 Solutions faibles

Dans les méthodes variationnelles, les solutions obtenues, points critiques de fonctionnelles, sont dans les espaces fonctionnelles de Sobolev, elles sont dites distributionnelles ou faibles.

Définition 1.5 *Soient (M, g) une variété riemannienne, a et b deux fonctions réelles de classe C^∞ sur M et $f \in L^1_{loc}(M)$, alors u est dite solution faible de l'équation*

$$\Delta_g^2 u - div_g\left(a(x)\nabla_g u\right) + b(x)u = \lambda\left|u\right|^{q-2}u + f(x)\left|u\right|^{N-2}u$$

si pour tout $\phi \in C^\infty(M)$,

$$\int_M (\langle \Delta u, \Delta\phi\rangle_g + a\left\langle \nabla u, \nabla\phi\right\rangle_g + bu\phi)dv_g = \int_M \left(\lambda\left|u\right|^{q-2}u + f(x)\left|u\right|^{N-2}u\right)\phi dv_g$$

Le théorème suivant est clé dans la recherche des solutions faibles.

1.8 Théorème des multiplicateurs de Lagrange

Théorème 1.2 *[4] Soient $(E, \|.\|)$ un espace de Banach, Ω un ouvert de E et $f : \Omega \longrightarrow R$ une fonction différentiable sur Ω et $\Phi : \Omega \longrightarrow \mathbb{R}^n$ une application de classe C^1 sur Ω de composantes $\Phi_1, .., \Phi_n$. Etant donné a un point de $\mathbb{R}^n$, on pose $K = \Phi^{-1}(a)$ que l'on suppose non vide, si en un point $x_\circ \in K$*

$$f(x_\circ) = \inf_{x\in K} f(x) \tag{1.1}$$

et si de plus la différentielle $d\Phi(x_\circ) \in L(E\ ,\mathbb{R}^n)$ est surjective alors ils existent des réels $\lambda_1, .., \lambda_n$ pour lesquels

$$df(x_\circ) = \lambda_1 d\Phi_1(x_\circ) + ... + \lambda_n d\Phi_n(x_\circ).$$

Cette équation est l'équation d'Euler-Lagrange associée au problème de minimisation considéré (1.1), les λ_i sont les coefficients de Lagrange de cette équation.

Une fois les solutions faibles obtenues on les régularise afin qu'elles deviennent des solutions classiques des équations.

1.9 Théorème de régularité

Théorème 1.3 *Soit L un opérateur linéaire elliptique du second ordre à coefficients de classe C^∞ et soit u une solution faible de l'équation :*

$$L(u) = f$$

avec $f \in L^1(M)$ alors:

1. *Si $f \in C^{k,\alpha}(M)$, $k \in \mathbb{N}$ et $\alpha \in (0,1)$, alors $u \in C^{k+2,\alpha}_{loc}(\ M)$.*

2. *Si $f \in H^p_k(M)$, $k \in \mathbb{N}$ et $p > 1$, alors $u \in H^p_{k+2,loc}(M)$.*

Un récent résultat de régularité notamment obtenu pour les équations du $4^{ème}$ordre est celui trouvé par Djadli,Hebey et Ledoux dans [17]

Lemme 1.1 *Etant donnée une variété riemannienne compacte (M, g) de dimension $n \geq 5$ et soit $\alpha \in \mathbb{R}$ positive et b une fonction définie sur M à valeur dans $\mathbb{R}$ et soit de plus $u \in H^2_2(M)$ une solution faible de l'équation*

$$\Delta_g^2 u + \alpha \Delta_g u + \frac{\alpha^2}{4} u = b(x)u$$

Si $b \in L^{\frac{n}{4}}(M)$ alors $u \in L^s(M)$, $\forall s \geq 1$.

Dans les applications et particulièrement à la géométrie différentielle, on cherche des solutions positives qui sont obtenues grâce au principe suivant

1.10 Principe du maximum (forme locale)

Théorème 1.4 *Soit Ω un ouvert borné de $\mathbb{R}^n$ et*

$$L(u) = a^{ij}(x)D_{ij}u(u) + b^i(x)D_iu(x) + c(x)u$$

un opérateur elliptique sur Ω à coefficients continus sur $\bar{\Omega}$ et tel que $c(x) \leq 0$, $\forall x \in \Omega$, si une fonction $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ et telle que $L(u) \geq 0$ sur Ω et si le maximum de u est M sur $\bar{\Omega}$ est à la fois positif ou nul et atteint en un point de Ω alors u est nécessairement constante égale à M sur Ω.

Un résultat important en analyse du type du lemme de Fatou généralisé est donné par:

1.11 Lemme de Brézis-Lieb

Lemme 1.2 *[14] Soient Ω un ouvert borné de $\mathbb{R}^n$ et $1 \leq p < +\infty$, $(f_n)_n$ une suite bornée de fonctions de $L^p(\Omega)$ convergeant p.p vers f , alors:*

$$f \in L^p(\Omega) \text{ et } \|f\|_p^p = \|f_n\|_p^p - \|f_n - f\|_p^p + o(1)$$

On cite une inégalité du type de Sobolev obtenue dans [17] et [7]

1.12 Inégalité de Sobolev

1.12.1 Cas régulier

Théorème 1.5 *[16] Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$. Pour tout $\varepsilon > 0$ il existe une constante $A_\varepsilon \in \mathbb{R}$ telle que :*

$$\forall u \in H_2^2(M) : \|u\|_N^2 \leq (1+\varepsilon)K_\circ \int_M |\Delta_g u|^2 + |\nabla_g u|^2 \, dv(g) + A_\varepsilon \int_M |u|^2 \, dv(g)$$

avec $N = \frac{2n}{n-4}$ *et* $\frac{1}{K_\circ} = \pi^2 n(n-4)(n^2-4)\{\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\}^{\frac{4}{n}}$ *où* Γ *est la fonction de Gamma l'Euler.*

1.12.2 Cas singulier

Dans cette partie, on cite une inégalité de type de Hardy-Sobolev obtenue dans [7].

Définition 1.6 *Soit* P *un point d'une variété riemannienne* (M,g). ρ_P *est la fonction définie par:*

$$\rho_P(Q) = \begin{cases} d(P,Q) \ si \ d(P,Q) < \delta(M) \\ \delta(M) \ si \ d(P,Q) \geq \delta(M) \end{cases}$$

avec $\delta(M)$ *le rayon d'injectivité de la variété* (M,g)*, et* d *la distance sur* M.
La fonction ρ *dépend évidemment du point* $P \in M$ *que l'on omettra parfois dans les notations.*

Définition 1.7 *Sur une variété riemannienne* (M,g) *on définit* $L^p(M,\rho^\gamma)$ *comme étant l'espace des fonctions* u *telles que* $\rho^\gamma |u|^p$ *soit intégrable. On le munit de la norme*

$$\|u\|^p_{p,\rho^\gamma} := \int_M \rho^\gamma |u|^p dv(g)$$

où $p \geq 1$ *et* ρ *est la fonction introduite dans la définition précédente.*

Théorème 1.6 *[7] Soit* (M,g) *une variété riemannienne compacte de dimension* $n \geq 5$ *et* p, q *et* γ *des nombres réels qui satisfont* $\frac{\gamma}{p} = \frac{n}{q} - \frac{n}{p} - 2$ *et* $2 \leq p \leq \frac{2n}{n-4}$.
Pour tout $\epsilon > 0$ *,il existe* $A(\epsilon,q,\gamma)$ *tel que*

$$\forall u \in H_2^q(M) : \|u\|^q_{p,\rho^\gamma} \leq (1+\epsilon)K^q(n,q,\gamma)^2 \left\|\nabla_g^2 u\right\|^q_q + A(\epsilon,q,\gamma)\|u\|^q_q$$

en particulier $K(n,2,0)^2 = K_\circ$ *la meilleure constante dans l'inégalité de Sobolev.*

Théorème 1.7 *[7] Soit* (M,g) *une variété riemannienne compacte de dimension* $n \geq 5$ *et* p, q *et* γ *des nombres réels qui satisfont:* $1 \leq q \leq p \leq \frac{nq}{n-2q}$ *et* $\gamma < 0$.

1. *Si* $\frac{\gamma}{p} = n(\frac{1}{q} - \frac{1}{p}) - 2$, *alors l'inclusion* $H_2^q(M) \subset L^p(M, \rho^\gamma)$ *est continue.*

2. *Si* $\frac{\gamma}{p} > n(\frac{1}{q} - \frac{1}{p}) - 2$, *alors l'inclusion* $H_2^q(M) \subset L^p(M, \rho^\gamma)$ *est compacte*

On établit l'existence de solution faible par la méthode variationnelle, via le théorème suivant

1.13 Théorème du col (Ambrosetti-Rabinowitz)

Théorème 1.8 *Soit J une fonctionnelle de classe C^1 sur un espace de Banach E, telle que:*

1. *Il existe $\bar{u} \in E$, $r > 0$, et $\alpha > 0$ tels que $\|u - \bar{u}\| = r$, alors*

$$J(u) > J(\bar{u}) + \alpha$$

2. *Il existe un point $u_1 \in E$ tel que $\|\bar{u} - u_1\| > 0$ et*

$$J(u_1) < J(\bar{u}) + \alpha$$

Soit alors Γ l'ensemble des chemins reliant $\bar{u}$ à u_1, c'est-à-dire

$$\Gamma = \{\gamma \in C([0,1], E) \text{ tels que } \gamma(0) = \bar{u}, \ \gamma(1) = u_1\}$$

Soit

$$\beta = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t))$$

Alors,

$$\beta \geq J(\bar{u}) + \alpha$$

et β est la valeur critique généralisée de J. Si J vérifie la condition de Palais-Smale au niveau β, alors β est donc une valeur critique de J.

1.14 Les conditions de Palais-Smale avec contrainte

Soit E un espace de Banach. Dans toute la suite lorsque l'on considère une contrainte du type

$$S := \{u \in E;\ \Phi(u) = 0\} \tag{1.2}$$

On suppose toujours que

$$\Phi \in C^1(E, \mathbb{R}),\ \text{et } \forall u \in S,\ \nabla\Phi \neq 0 \tag{1.3}$$

Définition 1.8 *[2] Soient E un espace de Banach, Φ vérifiant (1.3), S une variété définie par (1.2) et $J \in C^1(E, \mathbb{R})$. Si $c \in \mathbb{R}$, on dit que J vérifie la condition de Palais-Smale sur la contrainte S au niveau c, si toute suite $(u_n,\, \mu_n)_n \in S \times \mathbb{R}$ telle que*

$$J(u_n) \to c \text{ dans } \mathbb{R} \text{ et } \nabla J(u_n) - \mu_n \nabla\Phi(u_n) \to 0 \text{ dans } E^{'}$$

contient une sous suite notée $(u_n,\, \mu_n)_n$ convergente vers $(u,\, \mu)$ dans $S \times \mathbb{R}$.

Nous avons dit auparavant que si J est une fonctionnelle de classe C^1 bornée inférieurement, en général il n'est pas vrai que pour toute suite minimisante $(u_n)_n$, la dérivée $\nabla J(u_n)$ tend vers zéro. Cepandant on a le lemme suivant:

1.15 Lemme d'Ekeland

Lemme 1.3 *Soit $(X,\, d)$ un espace métrique complet et J une fonctionnelle s.c.i de X dans $\mathbb{R}$. On suppose que J est bornée inférieurement et soit*

$$c = \inf_{x \in X} J\left(x\right).$$

Alors, pour tout $\epsilon > 0$, *il exsite* u_ϵ *tel que:*

$$\begin{cases} c \leq J(u_\epsilon) \leq c + \epsilon \\ \forall x \in X,\ x \neq u_\epsilon : J(x) - J(u_\epsilon) + \epsilon d(x, u_\epsilon) > 0 \end{cases}$$

Chapitre 2

Problème elliptique non-linéaire avec des termes réguliers

2.1 Introduction

Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$. Soient a, b et f trois fonctions de classe C^∞ sur M avec f strictement positive. On considère l'équation suivante:

$$\Delta_g^2 u + div_g \left(a(x) \nabla_g u \right) + b(x) u = f(x) \left| u \right|^{N-2} u + \lambda \left| u \right|^{q-2} u \tag{2.1}$$

où $1 < q < 2$ et $N = \frac{2n}{n-4}$ l'exposent critique de Sobolev et λ un réel strictement positif. Dans cette section, on démontre l'existence d'une solution non triviale, en procédant par la technique variationnelle.

On considère sur $H_2^2(M)$ la fonctionnelle

$$J_\lambda(u) = \frac{1}{2} \int_M (\Delta_g u)^2 - a(x) \left| \nabla_g u \right|^2 + b(x) u^2 dv(g) - \frac{1}{N} \int_M f(x) \left| u \right|^N dv(g) - \frac{\lambda}{q} \int_M \left| u \right|^q dv(g)$$

On pose

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u),\ u \rangle$$

$$\Phi_\lambda(u)=\int_M |\Delta_g u|^2 - a(x)\,|\nabla_g u|^2 + b(x)u^2 dv(g) - \int_M f(x)\,|u|^N\, dv(g) - \lambda \int_M |u|^q\, dv(g)$$

On considère l'ensemble M_λ donné par

$$M_\lambda = \left\{ u \in H_2^2(M) : \ \Phi_\lambda(u) = 0 \text{ et } \ \|u\| \geq \rho > 0 \right\}$$

On prendra les fonctions $a(x)$ et $b(x)$ de telle manière que:

$$\|u\|^2 = \int_M (\Delta_g u)^2 - a(x)\,|\nabla_g u|^2 + b(x)u^2 dv(g)$$

soit une norme équivalente à celle de $H_2^2(M)$.

Exemple 2.1 *Pour avoir une norme équivalente à celle de $H_2^2(M)$, on peut prendre par exemple $a(x)$ et $b(x)$ comme suit*

$$\sup_{x\in M} a(x) < 0 \ \textit{et} \ \inf_{x\in M} b(x) > 0.$$

Définition 2.1 *On dit que l'opérateur $u \to \Delta_g^2 u + div_g\,(a(x)\nabla_g u) + b(x)u$ est coercif s'il existe $\Lambda > 0$ telle que pour tout $u \in H_2^2(M)$*

$$\int_M (\Delta_g^2 u + div_g\,(a(x)\nabla_g u) + b(x)u)u dv(g) \geq \Lambda\, \|u\|^2_{H_2^2(M)}$$

Proposition 4

$$\|u\| = (\int_M (\Delta_g u)^2 - a(x)\,|\nabla_g u|^2 + b(x)u^2 dv(g))^{\frac{1}{2}}$$

est une norme équivalente à celle de $H_2^2(M)$ si et seulement si l'opérateur

$$u \to \Delta_g^2 u + div_g\,(a(x)\nabla_g u) + b(x)u$$

est coercif.

Preuve:

1. $(\Rightarrow)$

   Si $\|.\|$ est une norme équivalente à celle de $H_2^2(M)$ i.e. ils existent deux constantes α et $\beta > 0$ telles que pour tout $u \in H_2^2(M)$

   $$\alpha \|u\|_{H_2^2(M)} \leq \|u\| \leq \beta \|u\|_{H_2^2(M)}$$

   alors l'opérateur

   $$u \rightarrow \Delta_g^2 u + div_g \left(a(x)\nabla_g u\right) + b(x)u$$

   est coercif.

2. $(\Leftarrow)$

   Si on suppose que l'opérateur

   $$u \rightarrow \Delta_g^2 u + div_g \left(a(x)\nabla_g u\right) + b(x)u$$

   est coercif, il existe $\Lambda > 0$ tel que pour tout $u \in H_2^2(M)$:

   $$\int_M (\Delta_g^2 u + div_g \left(a(x)\nabla_g u\right) + b(x)u) u dv(g) \geq \Lambda \|u\|^2_{H_2^2(M)}$$

   Comme M est compacte et $a(x)$ et $b(x)$ deux fonctions de classe $C^\infty(M)$,

   $$\int_M (\Delta_g^2 u + div_g \left(a(x)\nabla_g u\right) + b(x)u) u dv(g) \leq$$

   $$\underbrace{\max\left(1, -\min_{x\in M} a(x), \max_{x\in M} b(x)\right)}_{>0} \|u\|^2_{H_2^2(M)}$$

   D'où le résultat.

■

Lemme 2.1 *L'ensemble M_λ est non vide pour $\lambda \in (0,\, \lambda_\circ)$ où*

$$\lambda_\circ = \frac{\left(2^{q-2} - 2^{q-N}\right) \Lambda^{\frac{N-q}{N-2}}}{V(M)^{(1-\frac{q}{N})} \left(\max_{x\in M} f(x)\right)^{\frac{2-q}{N-2}} \left(\max((1+\varepsilon)K_\circ, A_\varepsilon)\right)^{\frac{N-q}{N-2}}}$$

avec $1 < q < 2$.

Preuve: Soient $t > 0$ et $u \in H_2^2(M)$ tels que $\|u\| \geq \rho > 0$,
alors

$$\Phi_\lambda(tu) = t^2 \|u\|^2 - t^N \int_M f(x) |u|^N dv(g) - \lambda t^q \|u\|_q^q .$$

Posons

$$\alpha(t) = \|u\|^2 - t^{N-2} \int_M f(x) |u|^N dv(g)$$

et

$$\beta(t) = \lambda t^{q-2} \|u\|_q^q$$

par l'inégalité de Sobolev, on obtient

$$\alpha(t) \geq \|u\|^2 - \max_{x\in M} f(x) (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \|u\|_{H_2^2(M)}^N t^{N-2}$$

Par la coercitivité de l'opérateur $u \to \Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u$, il existe une constante $\Lambda > 0$ telle que:

$$\alpha(t) \geq \|u\|^2 - \Lambda^{-\frac{N}{2}} \max_{x\in M} f(x) (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \|u\|^N t^{N-2}$$

En posant:

$$\alpha_1(t) = \|u\|^2 - \Lambda^{-\frac{N}{2}} \max_{x\in M} f(x) (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \|u\|^N t^{N-2}$$

Par l'inégalités de Hölder et de Sobolev, on obtient

$$\beta(t) \leq \lambda V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q_{H_2^2(M)} t^{q-2}$$

et par la coercitivité de l'opérateur $u \to \Delta_g^2 u + div_g(a(x)\nabla_g u) + b(x)u$, il existe une constante $\Lambda > 0$ telle que

$$\beta(t) \leq \lambda \Lambda^{-\frac{q}{2}} V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q t^{q-2}.$$

Posons

$$\beta_1(t) = \lambda \Lambda^{-\frac{q}{2}} V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q t^{q-2}.$$

$\alpha_1(t)$ est une fonction décroissante en t et concave et $\beta_1(t)$ est une fonction décroissante en t et convexe.
$\alpha_1(t)$ s'annule au point

$$t_\circ = \frac{\Lambda^{\frac{N}{2(N-2)}}}{\|u\| (\max_{x\in M} f(x))^{\frac{1}{N-2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2(N-2)}}}$$

et en prenant $u \in H_2^2(M)$, telle que

$$\|u\| = \frac{\Lambda^{\frac{N}{2(N-2)}}}{(\max_{x\in M} f(x))^{\frac{1}{N-2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2(N-2)}}}$$

ce qui est possible pour un $\rho > 0$ suffisament petit, ce qui donne $t_\circ = 1$. Alors

$$\min_{t\in(0,\ \frac{1}{2}]} \alpha_1(t) = \alpha_1(\frac{1}{2}) = \frac{\Lambda^{\frac{N}{(N-2)}}}{(\max_{x\in M} f(x))^{\frac{2}{N-2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{(N-2)}}}(1-2^{2-N})$$

$$\alpha_1(\frac{1}{2}) \geq \rho^2(1-2^{2-N}) > 0$$

et

$$\min_{t\in(0,\,\frac{1}{2}]}\beta_1(t)=\beta_1(\frac{1}{2})>0$$

où

$$\beta_1(\frac{1}{2})=\frac{\lambda 2^{2-q}V(M)^{(1-\frac{q}{N})}\Lambda^{\frac{q}{(N-2)}}}{(\max((1+\varepsilon)K_\circ,A_\varepsilon))^{\frac{q}{(N-2)}}(\max_{x\in M}f(x))^{\frac{q}{N-2}}}$$

L'équation $\Phi_\lambda(tu)=0$ possède une solution si

$$\min_{t\in(0,\frac{1}{2}]}\alpha_1(t)\geq\min_{t\in(0,\frac{1}{2}]}\beta_1(t)$$

i.e:

$$0<\lambda<\frac{\left(2^{q-2}-2^{q-N}\right)\Lambda^{\frac{N-q}{N-2}}}{V(M)^{(1-\frac{q}{N})}(\max_{x\in M}f(x))^{\frac{2-q}{N-2}}(\max((1+\varepsilon)K_\circ,A_\varepsilon))^{\frac{N-q}{N-2}}}=\lambda_\circ.$$

L'ensemble M_λ est alors non vide pour tout $\lambda\in(0,\lambda_\circ)$. ■

2.2 Etude de la fonctionnelle J_λ sur M_λ

Lemme 2.2 *Soit (M,g) une variété riemannienne compacte de dimension $n\geq 5$. Il existe $A>0$ telque $J_\lambda(u)\geq A>0$ pour tout $u\in M_\lambda$ où $\lambda\in(0,\min(\lambda_\circ,\lambda_1))$ et*

$$\lambda_1=\frac{\frac{(N-2)q}{2(N-q)}\Lambda^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ,A_\varepsilon))^{\frac{q}{2}}\rho^{q-2}}.$$

Preuve: Soit $u\in M_\lambda$

$$\|u\|^2=\int_M f(x)\,|u|^N\,dv(g)+\lambda\int_M|u|^q\,dv(g)$$

Pour $u\in M_\lambda$

$$J_\lambda(u)=\frac{N-2}{2N}\|u\|^2-\lambda\frac{N-q}{Nq}\int_M|u|^q\,dv(g)$$

Les inégalités de Hölder et Sobolev donnent

$$J_\lambda(u) \geq \frac{N-2}{2N}\|u\|^2 - \lambda\frac{N-q}{Nq}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|u\|^q_{H_2^2(M)}$$

et par la coercitivité de l'opérateur $u \to \Delta_g^2 u + div_g(a(x)\nabla_g u) + b(x)u$, il existe une constante $\Lambda > 0$ telle que

$$J_\lambda(u) \geq \frac{N-2}{2N}\|u\|^2 - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|u\|^q$$

$$\geq \|u\|^2\left(\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|u\|^{q-2}\right) > 0$$

Comme $\|u\| \geq \rho > 0$, nous obtenons

$$J_\lambda(u) \geq \|u\|^2\left(\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\rho^{q-2}\right).$$

maintennat si:

$$0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\rho^{q-2}} := \lambda_1$$

Alors, $J_\lambda(u) > 0$ pour tout $u \in M_\lambda$. ■

Lemme 2.3 *Soit (M,g) une variété riemannienne compacte de dimension $n \geq 5$, alors les deux assertions suivantes sont vraies:*

1. *$\langle \nabla\Phi_\lambda(u),\ u\rangle < 0$ pour tout $u \in M_\lambda$ et pour tout $\lambda \in (0, \min(\lambda_\circ, \lambda_1))$.*

2. *Les points critiques de J_λ sont des points de M_λ.*

Preuve:

1. Soit $u \in M_\lambda$, alors

$$\|u\|^2 = \lambda\int_M |u|^q\, dv(g) + \int_M f(x)\,|u|^N\, dv(g)$$

et

$$\langle \nabla \Phi_\lambda(u), u\rangle = 2\|u\|^2 - \lambda q \int_M |u|^q \, dv(g) - N \int_M f(x)\,|u|^N \, dv(g)$$

$$= 2\|u\|^2 - \lambda q \int_M |u|^q \, dv(g) - N(\|u\|^2 - \lambda \int_M |u|^q \, dv(g))$$

$$\langle \nabla \Phi_\lambda(u),\ u\rangle = (2-N)\|u\|^2 + \lambda (N-q)\|u\|_q^q$$

Par les inégalités de Hölder et Sobolev, on obtient

$$\langle \nabla \Phi_\lambda(u), u\rangle \leq$$

$$(2-N)\|u\|^2 + \lambda (N-q) V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|_{H_2^2(M)}^q$$

et en tenant compte de la coercitivité de l'opérateur $u \rightarrow \Delta_g^2 u + div_g\,(a(x)\nabla_g u) + b(x)u$, il existe une constante $\Lambda > 0$ telle que

$$\langle \nabla \Phi_\lambda(u), u\rangle \leq$$

$$\leq (2-N)\|u\|^2 + \lambda (N-q)\,\Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q$$

$$\leq \|u\|^2 \left[(2-N) + \lambda (N-q)\,\Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^{q-2}\right]$$

$$\langle \nabla \Phi_\lambda(u), u\rangle \leq \|u\|^2 \left[(2-N) + \lambda (N-q)\,\Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \rho^{q-2}\right]$$

et puisque

$$0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \rho^{q-2}}$$

on obtient

$$\langle \nabla \Phi_\lambda(u),\ u\rangle < 0.$$

2. $(\Rightarrow)$

   Soit $u \in M_\lambda \Leftarrow d\acute{e}f \Rightarrow \langle \nabla J_\lambda(u),\ u\rangle = 0$ et u un point critique de J_λ sur M_λ.

   $(\Leftarrow)$

En appliquant le théorème des multiplicateurs de Lagrange, il existe $\mu \in R$ tel que pour tout $u \in M_\lambda$

$$\nabla J_\lambda(u) = \mu \nabla \Phi_\lambda(u) \tag{2.2}$$

En testant au point $u \in M_\lambda$ l'équation (2.2), on obtient

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u), u \rangle = \mu \langle \nabla \Phi_\lambda(u), u \rangle = 0$$

et alors, pour tout $u \in M_\lambda$

$$\mu \langle \nabla \Phi_\lambda(u), u \rangle = 0.$$

Et comme

$$\langle \nabla \Phi_\lambda(u), u \rangle < 0$$

alors

$$\mu = 0.$$

D'où pour tout $u \in M_\lambda$

$$\nabla J_\lambda(u) = 0$$

■

2.3 Existence d'une solution non triviale de l'équation (2.1) sur $\mathbf{M}_\lambda$

On va voir dans ce qui suit que la fonctionnelle d'énergie J_λ vérifie les conditions de Palais-Smale avec la contrainte M_λ.

Lemme 2.4 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$. Soit*

$(u_n)_n$ une suite dans M_λ telle que:

$$\begin{cases} J_\lambda(u_n) \leq c \\ \nabla J_\lambda(u_n) - \mu_n \nabla \Phi_\lambda(u_n) \to 0 \end{cases}$$

Supposons que

$$c < \frac{2}{n K_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}}.$$

Alors, il existe une sous-suite de $(u_n)_n$ convergente fortement dans $H_2^2(M)$.

Preuve: Soit $(u_n)_n \subset M_\lambda$

$$J_\lambda(u_n) = \frac{N-2}{2N} \|u_n\|^2 - \lambda \frac{N-q}{Nq} \int_M |u_n|^q \, dv(g)$$

Dans un premier temps on montre que la suite$(u_n)_n$ est bornée dans $H_2^2(M)$.

Par le Lemme **2.2**, on obtient que:

$$J_\lambda(u_n) \geq \frac{N-2}{2N} \|u_n\|^2 - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u_n\|^q$$

$$J_\lambda(u_n) \geq \|u_n\|^2 (\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \rho^{q-2}) > 0$$

et comme $0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)} \Lambda^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \rho^{q-2}}$ et $J_\lambda(u_n) \leq c$, alors

$$c \geq J_\lambda(u_n)$$

$$\geq [\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \rho^{q-2}] \|u_n\|^2 > 0,$$

d'où

$$0 \leq \|u_n\|^2 \leq \frac{c}{\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \rho^{q-2}} < +\infty$$

Donc $(u_n)_n$ est bornée dans $H_2^2(M)$. La réflexivité de l'espace $H_2^2(M)$ et la compacité de l'inclusion $H_2^2(M) \subset H_p^k(M)$ ($k = 0, 1$; $p < N$), implique existence d'une sous-suite notée $(u_n)_n$ telle que:

1. $u_n \to u$ faiblement dans $H_2^2(M)$.
2. $u_n \to u$ et $\nabla u_n \to \nabla u$ fortement dans $L^p(M)$ où $p < N$
3. $u_n \to u$ presque partout dans M.

D'après le lemme de Brézis-Lieb, on peut écrire

$$\int_M (\Delta_g u_n)^2 \, dv(g) = \int_M (\Delta_g u)^2 \, dv(g) + \int_M (\Delta_g (u_n - u))^2 \, dv(g) + o(1)$$

et aussi

$$\int_M f(x) \left| u_n \right|^N dv(g) = \int_M f(x) \left| u \right|^N dv(g) + \int_M f(x) \left| u_n - u \right|^N dv(g) + o(1)$$

On va montrer que $u \in M_\lambda$.

Comme $u_n \to u$ faiblement dans $H_2^2(M)$ i.e. si pour tout $\phi \in H_2^2(M)$,

$$\int_M \left(\Delta_g u_n \Delta_g \phi - a(x) \left\langle \nabla u_n, \nabla \phi \right\rangle_g + a u_n \phi \right) dv(g) =$$

$$\int_M \left(\Delta_g u \Delta_g \phi - a(x) \left\langle \nabla u, \nabla \phi \right\rangle_g + a u \phi \right) dv(g) + o(1)$$

En particulier pour $\phi = u$, on obtient,

$$\int_M \left(\Delta_g u_n \Delta_g u - a(x) \left\langle \nabla u_n, \nabla u \right\rangle_g + a u_n u \right) dv(g) = \left\| u \right\|^2 + o(1)$$

et aussi pour $\phi = u_n$,

$$\int_M \left(\Delta_g u_n \Delta_g u - a(x) \left\langle \nabla u_n, \nabla u \right\rangle_g + a u_n u \right) dv(g) = \left\| u_n \right\|^2 + o(1)$$

et puisque $(u_n)_n \subset M_\lambda$ i.e.

$$\int_M \left(\lambda |u_n|^{q-2} u_n u + f(x) |u_n|^{N-2} u_n u\right) dv(g) \ = \|u\|^2 + o(1)$$

mais quand $n \to +\infty$

$$\int_M \left(\lambda |u_n|^{q-2} u_n u + f(x) |u_n|^{N-2} u_n u\right) dv(g) \to \int_M \left(\lambda |u|^q + f(x) |u|^N\right) dv(g)$$

ce qui donne

$$\Phi_\lambda(u_n) = \Phi_\lambda(u) = \|u\|^2 - \lambda \int_M |u|^q \, dv(g) - \int_M f(x) |u|^N \, dv(g) = 0$$

ou encore

$$\|u\| + o(1) = \|u_n\| \geq \rho$$

D'où $u \in M_\lambda$.

On va montrer que $\mu_n \to 0$ quand $n \to +\infty$

En testant avec u_n, on obtient

$$\langle \nabla J_\lambda(u_n) - \mu_n \nabla \Phi_\lambda(u_n), u_n \rangle = o(1)$$

$$= \underbrace{\langle \nabla J_\lambda(u_n), u_n \rangle}_{=0} - \mu_n \langle \nabla \Phi_\lambda(u_n), u_n \rangle = o(1)$$

Donc,

$$\mu_n \langle \nabla \Phi_\lambda(u_n), u_n \rangle = o(1)$$

D'après le Lemme **2.3**, on a

$$\limsup \langle \nabla \Phi_\lambda(u_n), u_n \rangle < 0$$

et par consequent

$$\mu_n \to 0 \text{ quand } n \to +\infty$$

On va montrer que $u_n \to u$ converge fortement dans $H_2^2(M)$, nous avons

$$J_\lambda(u_n) - J_\lambda(u)$$

$$= \frac{1}{2}\int_M (\Delta_g(u_n - u))^2 \, dv(g) - \frac{1}{N}\int_M f(x) \left|u_n - u\right|^N dv(g) + o(1). \tag{2.3}$$

Puisque $u_n - u \to 0$ faiblement dans $H_2^2(M)$, on teste avec $\nabla J_\lambda(u_n) - \nabla J_\lambda(u)$

$$\langle \nabla J_\lambda(u_n) - \nabla J_\lambda(u), u_n - u \rangle = o(1)$$

$$= \int_M (\Delta_g(u_n - u))^2 \, dv(g) - \int_M f(x) \left|u_n - u\right|^N dv(g) = o(1) \tag{2.4}$$

De sorte que

$$\int_M (\Delta_g(u_n - u))^2 \, dv(g) = \int_M f(x) \left|u_n - u\right|^N dv(g) + o(1)$$

et en tenant compte de (2.3), on obtient

$$J_\lambda(u_n) - J_\lambda(u) = \frac{1}{2}\int_M (\Delta_g(u_n - u))^2 \, dv(g) - \frac{1}{N}\int_M (\Delta_g(u_n - u))^2 \, dv(g) + o(1)$$

i.e.

$$J_\lambda(u_n) - J_\lambda(u) = \frac{2}{n}\int_M (\Delta_g(u_n - u))^2 \, dv(g).$$

Independament, d'après l'inégalité de Sobolev, on obtient pout tout $u \in H_2^2(M)$

$$\|u\|_N^2 \leq (1+\varepsilon)K_\circ \int_M (\Delta_g u)^2 + |\nabla_g u|^2 \, dv(g) + A_\varepsilon \int_M u^2 dv(g).$$

En testant l'inégalité de Sobolev par $u_n - u$, on obtient

$$\|u_n - u\|_N^2 \leq (1+\varepsilon)K_\circ \int_M (\Delta_g(u_n - u))^2 \, dv(g) + o(1). \tag{2.5}$$

Comme

$$\int_M f(x) \left|u_n - u\right|^N dv(g) \leq \max_{x \in M} f(x) \int_M \left|u_n - u\right|^N dv(g)$$

et en remplaçant dans (2.5), on obtient:

$$\int_M f(x) \left|u_n - u\right|^N dv(g) \leq (1+\varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta_g(u_n - u)\|_2^N + o(1)$$

En faisant appel à l'égalité (2.4), on trouve:

$$o(1) \geq \|\Delta_g(u_n - u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta_g(u_n - u)\|_2^N + o(1)$$

$$\geq \|\Delta_g(u_n - u)\|_2^2 (1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta_g(u_n - u)\|_2^{N-2}) + o(1)$$

et par conséquent si

$$\limsup \|\Delta_g(u_n - u)\|_2^{N-2} < \frac{1}{(1+\varepsilon)^{\frac{n}{n-4}} K_\circ^{\frac{n}{n-4}} \max_{x \in M} f(x)}$$

alors

$$\frac{2}{n} \int_M \left|\Delta_g(u_n - u)\right|^2 dv(g) < c.$$

Comme

$$c < \frac{2}{n K_\circ^{\frac{n}{4}} (\max_{x \in M} f(x))^{\frac{n-4}{4}}}$$

Nous obtenons

$$\int_M \left|\Delta_g(u_n - u)\right|^2 dv(g) < \frac{1}{K_\circ^{\frac{n}{4}} (\max_{x \in M} f(x))^{\frac{n-4}{4}}}.$$

Par conséquent

$$o(1) \geq \|\Delta_g(u_n - u)\|_2^2 \underbrace{(1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta_g(u_n - u)\|_2^{N-2})}_{>0} + o(1)$$

ou encore

$$\|\Delta_g(u_n - u)\|_2^2 = o(1)$$

i.e. $u_n \to u$ converge fortement dans $H_2^2(M)$. ■

Théorème 2.1 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$. Soit f une positive fonction de classe C^∞ sur M et on suppose que $u \to \Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u$ est coercif et que*

$$c < \frac{2}{nK_\circ^{\frac{n}{4}} \left(f(x_\circ)\right)^{\frac{n-4}{4}}}$$

Alors, il existe $\lambda^* > 0$ telle que pour tou $\lambda \in (0, \lambda^*)$, l'équation possède une solution non triviale.

Preuve: D'après les Lemmes **2.2; 2.3** et **2.4**, on a démontré qu'il existe $v \in M_\lambda$ telle que:

$$J_\lambda(v) = \max_{u\in M_\lambda} J_\lambda(u)$$

D'après le théorème des multiplicateurs de Lagrange il existe $\mu \in \mathbb{R}$:

$$\nabla J_\lambda(v) = \mu \nabla\Phi_\lambda(v). \tag{2.6}$$

En testant au point $u \in M_\lambda$ l'équation (2.6), on obtient

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u), u\rangle = \mu \langle \nabla\Phi_\lambda(u), u\rangle = 0$$

et alors, pour tout $u \in M_\lambda$

$$\mu \langle \nabla \Phi_\lambda(u), u \rangle = 0.$$

Comme

$$\langle \nabla \Phi_\lambda(u), u \rangle < 0$$

alors,

$$\mu = 0.$$

D'où pour tout $u \in M_\lambda$

$$\nabla J_\lambda(u) = 0$$

D'après les Lemmes **2.3**, on obtient que v est une solution faible non-triviale. ■

2.4 Multiplicité de la solution pour l'équation (2.7) à coefficients constants

On considère l'équation suivante:

$$\Delta_g^2 u - \alpha \Delta u + \beta u = f(x) |u|^{N-2} u + \lambda |u|^{q-2} u \tag{2.7}$$

où α, $\beta \in \mathbb{R}$.

On considère sur $H_2^2(M)$ les fonctionnelles suivantes:

$$J_\lambda^+(u) = \frac{1}{2} \|u\|^2 - \frac{1}{N} \int_M f(x)(u^+)^N dv(g) - \frac{\lambda}{q} \int_M \left(u^+\right)^q dv(g)$$
$$J_\lambda^-(u) = \frac{1}{2} \|u\|^2 - \frac{1}{N} \int_M f(x) \left(u^-\right)^N dv(g) - \frac{\lambda}{q} \int_M \left(u^-\right)^q dv(g)$$

où

$$\|u\|^2 := \int_M (\Delta_g u)^2 + \alpha |\nabla_g u|^2 + \beta u^2 dv(g)$$

avec

$$u^{+}=\max_{x\in M}(u(x);0) \text{ et } u^{-}=\min_{x\in M}(u(x);0).$$

On pose

$$\Phi_{\lambda}^{\pm}(u)=\left\langle \nabla J_{\lambda}^{\pm}(u),\ u\right\rangle$$

$$\Phi_{\lambda}^{\pm}(u)=\|u\|^{2}-\int_{M}f(x)\left(u^{\pm}\right)^{N}dv(g)-\lambda\int_{M}\left(u^{\pm}\right)^{q}dv(g)$$

On considère l'ensemble $M_{\lambda}^{\pm}$ donné par

$$M_{\lambda}^{\pm}=\left\{u\in H_{2}^{2}(M):\ \Phi_{\lambda}^{\pm}(u)=0 \text{ et } \|u\|\geq\rho>0\right\}$$

Lemme 2.5 *Pour tou $\lambda\in(0,\ \lambda^{*})$, l'équation (2.7) possède deux minimas locaux.*

Preuve: D'après les Lemmes **2.2; 2.3** et **2.4**, on a démontré qu'ils existent $v_1\in M_{\lambda}^{+}$ et $v_2\in M_{\lambda}^{-}$ telle que:

$$J_{\lambda}^{+}(v_1)=\min_{u\in M_{\lambda}^{+}}J_{\lambda}^{+}(u) \text{ et } J_{\lambda}^{-}(v_2)=\min_{u\in M_{\lambda}^{-}}J_{\lambda}^{-}(u)$$

D'après le théorème des multiplicateurs de Lagrange ils existent $\mu,\eta\in\mathbb{R}$:

$$J_{\lambda}^{+}(v_1)=\mu\nabla\Phi_{\lambda}^{+}(v_1) \tag{2.8}$$

et

$$J_{\lambda}^{-}(v_2)=\eta\nabla\Phi_{\lambda}^{-}(v_2) \tag{2.9}$$

En testant au point $v_1\in M_{\lambda}^{+}$ l'équation (2.8), on obtient

$$\Phi_{\lambda}^{+}(v_1)=\left\langle\nabla J_{\lambda}^{+}(v_1),v_1\right\rangle=\mu\left\langle\nabla\Phi_{\lambda}^{+}(v_1),v_1\right\rangle=0$$

et alors, pour tout $v_1\in M_{\lambda}^{+}$

$$\mu\left\langle\nabla\Phi_{\lambda}^{+}(v_1),v_1\right\rangle=0.$$

Comme

$$\left\langle \nabla \Phi_\lambda^+(v_1), v_1 \right\rangle = 0 < 0$$

alors

$$\mu = 0$$

d'où pour tout $v_1 \in M_\lambda^+$

$$\nabla J_\lambda^+(v_1) = 0.$$

De la même façon pour la fonctionnelle J_λ^-, on obtient que v_1 et v_2 sont deux minimas locaux. ■

Théorème 2.2 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$, Soit f une positive fonction de classe C^∞ sur M et on suppose que:*

1. *$u \to \Delta_g^2 u - \alpha \Delta u + \beta u$ est coercif.*
2. *$J_\lambda(u) \leq c < \frac{2}{n\ K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}$.*
3. *$\alpha^2 - 4\beta > 0$ et $\alpha > 0$.*

   Alors, il existe $\lambda^* > 0$ telle que pour tou $\lambda \in (0,\ \lambda^*)$, l'équation (2.7) possède trois solutions distinctes.

Preuve: D'après le lemme **2.7**, on a montré que u^+ et u^- deux solutions de l'équation (2.7). Les conditions géométriques du lemme du Col sont satisfaites.
Posons,

$$\Gamma = \left\{ \eta \in C^1 \left([0; 1]\,;\ M_\lambda \right) :\ \eta\left(0\right) = u^+,\ \eta\left(1\right) = u^- \right\}$$

on remarque que $\Gamma \neq \emptyset$ car

$$\eta\left(t\right) = (1-t)u^+ + tu^- \in \Gamma.$$

Posons

$$c_\lambda = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} \left(J_\lambda \left(\eta(t) \right) \right).$$

En appliquant le théorème du col, il existe une suite de Palais-Smale $(v_m)_m$ au niveau c_λ dans M_λ.

Donc,d'après les lemmes **2.2 et 2.3** et le théorème **2.1**, c_λ est une valeur critique généralisée de la fonctionnelle J_λ de valeur critique $v \in M_\lambda$.

Alors, v, u^+ et u^- trois solutions distinctes. ■

2.5 Fonctions tests

Pour vérifier l'hypothèse du théorème générique **2.1**, on considère les fonctions tests suivantes:

$$u_\epsilon(x) = (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{8}} \frac{\eta(r)}{(r^2+\epsilon^2)^{\frac{n-4}{2}}}$$

où

$$f(x_\circ) = \max_{x \in M} f(x)$$

ici η est une fonction de classe C^∞ égale à 1 sur $B(x_\circ, \delta)$ et 0 sur $M - B(x_\circ, 2\delta)$ où $r = d(x_\circ, .)$ désigne la distance géodésique au point $x_\circ$ et d est le rayan d'injectivité au point $x_\circ \in M$.

2.5.1 Application aux variétés riemanniennes compactes de dimensions $n > 6$

Théorème 2.3 *Soit (M, g) une variété riemannienne compacte de dimension $n > 6$, si en un point $x_\circ$ où f atteint son maximum, la condition*

$$\left(\frac{(n^2+4n-20)}{2(n+2)} S_g(x_\circ) + \frac{(n-1)}{(n+2)} a(x_\circ) - \frac{(n-6)}{8} \frac{\Delta f(x_\circ)}{f(x_\circ)} \right) > 0$$

est vérifiée alors l'équation (2.1) admet une solution u non triviale de classe $C^{4,\alpha}$ avec $\alpha \in (0,1)$ vérifiant

$$J_\lambda(u) < \frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}$$

Preuve: Pour calculer le terme $\int_M f(x)\,|u_\epsilon(x)|^N\,dv(g)$, on considère le développement de f au point $x_\circ$:

$$f(x) = f(x_\circ) + \frac{\partial^2 f(x_\circ)}{2\partial y^i \partial y^j} y^i y^j + o(r^2)$$

Alors

$$f(x).dv(g) = f(x_\circ) + (\frac{\partial^2 f(x_\circ)}{2\partial y^i \partial y^j} - \frac{f(x_\circ)}{6} R_{ij}(x_\circ)) y^i y^j + o(r^2)$$

Maintenant on calcule

$$\int_M f(x)\,|u_\epsilon(x)|^N\,dv(g) = \int_{B(x_\circ,\delta)} f(x)\,|u_\epsilon(x)|^N\,dv(g) + \int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} f(x)\,|u_\epsilon(x)|^N\,dv(g)$$

Le premier terme de droite s'écrit

$$\int_{B(x_\circ,\delta)} f(x)\,|u_\epsilon(x)|^N\,dv(g) = \int_0^\delta r^{n-1}\,|u_\epsilon(x)|^N\,(\int_{S(r)} f(x)\sqrt{|g(x)|}d\Omega)dr$$

$$= (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n}{4}}\omega_{n-1}\int_0^\delta \frac{r^{n-1}}{(r^2+\epsilon^2)^n}\left[f(x_\circ) - (\frac{\Delta f(x_\circ)}{2n} + \frac{f(x_\circ)}{6n}S_g(x_\circ))r^2 + o(r^2)\right]$$

$$= (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n}{4}}\omega_{n-1}\times$$

$$\left(f(x_\circ)\int_0^\delta \frac{r^{n-1}}{(r^2+\epsilon^2)^n}dr - (\frac{\Delta f(x_\circ)}{2n} + \frac{f(x_\circ)}{6n}S_g(x_\circ))\int_0^\delta \frac{r^{n+1}}{(r^2+\epsilon^2)^n}dr + o(r^2)\right)$$

En faisant le changement de variable suivant

$$\left\langle x = (\frac{r}{\epsilon})^2,\ dr = \frac{\epsilon dx}{2\sqrt{x}} \text{ et } r = \epsilon\sqrt{x}\right\rangle$$

on obtient pour $\epsilon \to 0$

$$A = \int_{B(x_\circ,\delta)} f(x)\left|u_\epsilon(x)\right|^N dv(g) = (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n}{4}}\frac{\omega_{n-1}}{2}\times$$

$$\left(f(x_\circ)\int_0^{(\frac{\delta}{\epsilon})^2}\frac{x^{\frac{n}{2}-1}}{(x+1)^n}dx - (\frac{\Delta f(x_\circ)}{2n}+\frac{f(x_\circ)}{6}S_g(x_\circ))\epsilon^2\int_0^{(\frac{\delta}{\epsilon})^2}\frac{x^{\frac{n}{2}}}{(x+1)^n}dx + o(\epsilon^2)\right)$$

$$= (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n}{4}}\frac{\omega_{n-1}}{2}(f(x_\circ)\int_0^{+\infty}\frac{x^{\frac{n}{2}-1}}{(x+1)^n}dx -$$
$$(\frac{\Delta f(x_\circ)}{2n}+\frac{f(x_\circ)}{6n}S_g(x_\circ))\epsilon^2\int_0^{+\infty}\frac{x^{\frac{n}{2}}}{(x+1)^n}dx + o(\epsilon^2))$$

$$= (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n}{4}}\frac{\omega_{n-1}}{2}\left(f(x_\circ)I_n^{\frac{n}{2}-1} - (\frac{\Delta f(x_\circ)}{2n}+\frac{f(x_\circ)}{6n}S_g(x_\circ))\epsilon^2 I_n^{\frac{n}{2}} + o(\epsilon^2)\right)$$

et puisque

$$I_n^{\frac{n}{2}} = \frac{n}{n-2}I_n^{\frac{n}{2}-1}$$

on obtient

$$A = (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n}{4}}\frac{w_{n-1}}{2}\left(f(x_\circ)I_n^{\frac{n}{2}-1} - (\frac{\Delta f(x_\circ)}{2(n-2)}+\frac{f(x_\circ)}{6(n-2)}S_g(x_\circ))\epsilon^2 I_n^{\frac{n}{2}-1} + o(\epsilon^2)\right)$$

$$= (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n}{4}}\frac{w_{n-1}}{2}I_n^{\frac{n}{2}-1}\left(f(x_\circ) - (\frac{\Delta f(x_\circ)}{2(n-2)}+\frac{f(x_\circ)}{6(n-2)}S_g(x_\circ))\epsilon^2 + o(\epsilon^2)\right).$$

Sachant que,

$$K_\circ = \frac{16}{(n-4)n(n^2-4)2^{n-1}\left(I_n^{\frac{n}{2}-1}\omega_{n-1}\right)^{\frac{4}{n}}}$$

on obtient

$$\int_{B(x_\circ,\delta)} f(x)\left|u_\epsilon(x)\right|^N dv(g) = \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1 - (\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)}+\frac{S_g(x_\circ)}{6(n-2)})\epsilon^2 + o(\epsilon^2)\right).$$

Il nous reste à calculer l'intégrale

$$B=\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} f(x)\left|u_\epsilon(x)\right|^N dv(g)=\eta\left(x_\circ\right)^N\left(\frac{(n-4)n(n^2-4)}{f(x_\circ)}\right)^{\frac{n}{4}}\frac{w_{n-1}}{2}\times$$

$$\left(f(x_\circ)\int_{(\frac{\delta}{\epsilon})^2}^{(\frac{2\delta}{\epsilon})^2}\frac{x^{\frac{n}{2}-1}}{(x+1)^n}dx-\left(\frac{\Delta f(x_\circ)}{2n}+\frac{f(x_\circ)}{6}S_g(x_\circ)\right)\epsilon^2\int_{(\frac{\delta}{\epsilon})^2}^{(\frac{2\delta}{\epsilon})^2}\frac{x^{\frac{n}{2}}}{(x+1)^n}dx+o(\epsilon^2)\right)$$

alors,

$$\left|\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} f(x)\left|u_\epsilon(x)\right|^N dv(g)\right|\leq c\left(\int_{(\frac{\delta}{\epsilon})^2}^{(\frac{2\delta}{\epsilon})^2}\frac{x^{\frac{n}{2}-1}}{x^n}dx+\epsilon^2\int_{(\frac{\delta}{\epsilon})^2}^{(\frac{2\delta}{\epsilon})^2}\frac{x^{\frac{n}{2}}}{x^n}dx+o(\epsilon^2)\right)$$

$$\leq c\left(\left(\frac{\delta}{\epsilon}\right)^{-n}+\epsilon^2\left(\frac{\delta}{\epsilon}\right)^{-n+2}\right).$$

$$\leq c(\epsilon^n+\epsilon^{n-4})$$

où c une constante positive universelle.

Comme $n\geq 6$ alors,

$$\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} f(x)\left|u_\epsilon(x)\right|^N dv(g)=o(\epsilon^2).$$

donc,

$$\int_M f(x)\left|u_\epsilon(x)\right|^N dv(g)=\frac{1}{K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left(1-\left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)}+\frac{S_g(x_\circ)}{6(n-2)}\right)\epsilon^2+o(\epsilon^2)\right).$$

Maintenant

$$\left|\frac{\partial u_\epsilon}{\partial r}\right|=\left|\nabla u_\epsilon\right|=(n-4)\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{8}}\frac{r}{(r^2+\epsilon^2)^{\frac{n-2}{2}}}$$

et donc

$$\int_M a(x)\left|\nabla u_\epsilon\right|^2 dv(g)=\int_{B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g)+\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g).$$

On calcule à présent le terme

$$\int_M a(x)\left|\nabla u_\epsilon\right|^2 dv(g) = (n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}}\times$$

$$\int_0^\delta \frac{r^2}{(r^2+\epsilon^2)^{n-2}}\left(\int_{S(r)} a(x)\sqrt{|g(x)|}d\Omega\right)dr + o\left(r^2\right)$$

pour cela on utilise le développement limité de la fonction $a(x)$

$$a(x) = a(x_\circ) + \frac{\partial^2 a(x_\circ)}{2\partial y^i \partial y^j} y^i y^j + o(r^2)$$

$$\int_{S(r)} a(x)\sqrt{|g(x)|}d\Omega = \int_{S(r)}\left(a(x_\circ) + (\frac{\partial^2 a(x_\circ)}{2\partial y^i\partial y^j} - \frac{a(x_\circ)R_{ij}(x_\circ)}{6})y^i y^j + o(r^2)\right)d\Omega$$

$$= \left(a(x_\circ) - (\frac{\Delta a(x_\circ)}{2n} + \frac{a(x_\circ)}{6n}S_g(x_\circ))r^2 + o(r^2)\right)\omega_{n-1}$$

$$\int_M a(x)\left|\nabla u_\epsilon\right|^2 dv(g) = (n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}}\omega_{n-1}$$

$$\int_0^\delta \frac{r^2}{(r^2+\epsilon^2)^{n-2}}\left(a(x_\circ) - (\frac{\Delta a(x_\circ)}{2n} + \frac{a(x_\circ)}{6n}S_g(x_\circ))r^2 + o(r^2)\right)dr.$$

En faisant le changement de variable suivant

$$\left\langle x = (\frac{r}{\epsilon})^2,\ dr = \frac{\epsilon dx}{2\sqrt{x}} \text{ et } r = \epsilon\sqrt{x}\right\rangle$$

on obtient

$$\int_{B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g) = (n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}}\frac{w_{n-1}}{2\epsilon^{n-6}}\times$$

$$\left(a(x_\circ)\int_0^{(\frac{\delta}{\epsilon})^2}\frac{x^{\frac{n}{2}}}{(x+1)^{n-2}}dx - (\frac{\Delta a(x_\circ)}{2n} + \frac{a(x_\circ)}{6n}S_g(x_\circ))\epsilon^2\int_0^{(\frac{\delta}{\epsilon})^2}\frac{x^{\frac{n}{2}+1}}{(x+1)^{n-2}}dx + o(\epsilon^2)\right)$$

Pour $\epsilon \to 0$ on a

$$\int_{B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g) = \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(\frac{4(n-1)a(x_\circ)}{(n^2-4)(n-6)}\epsilon^2 + o(\epsilon^2)\right).$$

Il nous reste à calculer l'intégrale $\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g)$.

Toutes les intégrales sont du type

$$\left|\int_{(\frac{\delta}{\epsilon})^2}^{(\frac{2\delta}{\epsilon})^2} h(x)\frac{x^q}{(x+1)^p}dx\right| \leq C\left(\frac{1}{\epsilon}\right)^{2(q-p+1)} = C\epsilon^{2(p-q-1)}$$

et comme $p-q = n-4 \geq 3$, on obtient

$$\int_{(\frac{\delta}{\epsilon})^2}^{(\frac{2\delta}{\epsilon})^2} h(x)\frac{x^q}{(x+1)^p}dx = o(\epsilon^2)$$

et par conséquent

$$\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g) = o(\epsilon^2).$$

Finalement, on a

$$\int_M a(x)\left|\nabla u_\epsilon\right|^2 dv(g) = \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(\frac{4(n-1)a(x_\circ)}{(n^2-4)(n-6)}\epsilon^2 + o(\epsilon^2)\right)$$

Maintenant on calcule

$$\int_M b(x)u_\epsilon^2 dv(g) = \int_{B(x_\circ,\delta)} b(x)u_\epsilon^2 dv(g) + \int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} b(x)u_\epsilon^2 dv(g)$$

Le premier terme devient

$$\int_{B(x_\circ,\delta)} b(x)u_\epsilon^2 dv(g) = (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}}\int_0^\delta \frac{r^{n-1}}{(r^2+\epsilon^2)^{n-4}}\left(\int_{S(r)} b(x)\sqrt{|g(x)|}d\Omega\right)dr$$

$$= (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}} \omega_{n-1} \int_0^\delta \frac{r^{n-1}}{(r^2+\epsilon^2)^{n-4}} \times$$
$$\left(b(x_\circ) - (\frac{\Delta b(x_\circ)}{2n} + \frac{b(x_\circ)}{6n} S_g(x_\circ)) r^2 + o(r^2) \right) dr$$

et avec les mêmes calculs que ci-dessus, on obtient

$$\int_{B(x_\circ,\delta)} b(x) u_\epsilon^2 dv(g) = o(\epsilon^2).$$

Pour le calcul de

$$\int_M (\Delta u_\epsilon)^2 \, dv(g) = \int_{B(x_\circ,\delta)} (\Delta u_\epsilon)^2 \, dv(g) + \int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} (\Delta u_\epsilon)^2 \, dv(g)$$

on rappelle d'abord l'expression radiale du laplacien

$$-\Delta u_\epsilon = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} (r^{n-1} \frac{\partial u_\epsilon}{\partial r}) + \frac{\partial}{\partial r} \log \sqrt{|g(x)|} \frac{\partial u_\epsilon}{\partial r}$$

$$= (4-n) (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{8}} \left(\frac{n\epsilon^2 + 2r^2}{(r^2+\epsilon^2)^{\frac{n}{2}}} + \frac{\partial}{\partial r} \log \sqrt{|g(x)|} \frac{r}{(r^2+\epsilon^2)^{\frac{n-2}{2}}} \right).$$

Et alors

$$\int_{B(x_\circ,\delta)} (\Delta u_\epsilon)^2 \, dv(g) = (n-4)^2 (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}} \omega_{n-1}$$

$$\int_0^\delta (\frac{n\epsilon^2 + 2r^2}{(r^2+\epsilon^2)^{\frac{n}{2}}} + \frac{\partial}{\partial r} \log \sqrt{|g(x)|} \frac{r}{(r^2+\epsilon^2)^{\frac{n-2}{2}}})^2 (1 - \frac{S_g(x_\circ)}{6n} r^2 + o(r^2)) r^{n-1} dr$$

$$= (n-4)^2 (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}} \omega_{n-1}$$

$$\left(\int_0^\delta \frac{(n\epsilon^2+2r^2)^2 r^{n-1}}{(r^2+\epsilon^2)^n} + \left(\frac{\partial}{\partial r} \log \sqrt{|g(x)|} \right)^2 \frac{r^{n+1}}{(r^2+\epsilon^2)^{n-2}} + 2 \frac{(n\epsilon^2+2r^2)^2 r^n}{(r^2+\epsilon^2)^{n-1}} \frac{\partial}{\partial r} \log \sqrt{|g(x)|} \right) \times$$

$$(1 - \frac{S_g(x_\circ)}{6n} r^2 + o(r^2)) dr$$

ou bien

$$= \frac{1}{2\epsilon^{n-4}} \left[n^2 I_n^{\frac{n}{2}-1} + 4n I_n^{\frac{n}{2}} + 4 I_n^{\frac{n}{2}+1} - \frac{S_g(x_\circ)}{6n} \epsilon^2 (n^2 I_n^{\frac{n}{2}} + 4n \ I_n^{\frac{n}{2}+1} + 4 I_n^{\frac{n}{2}+2}) + o(\epsilon^2) \right]$$

$$= \frac{1}{2\epsilon^{n-4}} I_n^{\frac{n}{2}-1} \left(\frac{n\,(n^2-4)}{(n-4)} - \frac{(n^2+4)\,S_g(x_\circ)}{6\,(n-6)} \epsilon^2 + o(\epsilon^2) \right).$$

Pour la troisième intégrale, rappelons que

$$\frac{\partial}{\partial r} \log \sqrt{|g(x)|} = -\frac{S_g(x_\circ)}{3n} r + o(r^2)$$

ce qui donne

$$\int_0^\delta \frac{2(n\epsilon^2 + 2r^2)^2 r^n}{(r^2+\epsilon^2)^{n-1}} \frac{\partial}{\partial r} \log \sqrt{|g(x)|} (1 - \frac{S_g(x_\circ)}{6n} r^2 + o(r^2)) dr$$

$$= \frac{S_g(x_\circ)}{3n\epsilon^{n-6}} (\int_0^{\left(\frac{\delta}{\epsilon}\right)^2} \frac{(n+2x)\,.x^{\frac{n}{2}}}{(1+x)^{n-1}} dx) + o(\epsilon^2))$$

$$= \frac{S_g(x_\circ)}{3n\epsilon^{n-4}} I_n^{\frac{n}{2}-1} \left(\frac{2n(n-1)(n-2)}{(n-4)(n-6)} \epsilon^2 + o(\epsilon^2) \right).$$

La dernière intégrale s'écrit

$$\int_0^\delta \left(\frac{\partial}{\partial r} \log \sqrt{|g(x)|} \right)^2 \frac{r^{n+1}}{(r^2+\epsilon^2)^{n-2}} (1+o(r^2)) dr = \frac{1}{2\epsilon^{n-8}} \frac{S_g^2(x_\circ)}{9n^2} \int_0^{\left(\frac{\delta}{\epsilon}\right)^2} \frac{x^{\frac{n}{2}+1}}{(1+x)^{n-2}} (1+o(\epsilon^2)) dx$$

$$= \frac{1}{\epsilon^{n-4}} o(\epsilon^4)$$

ce qui permet d'écrire

$$\int_{B(x_\circ,\delta)} (\Delta u_\epsilon)^2 \, dv(g) = \frac{1}{K_\circ^{\frac{n}{4}} \left(f(x_\circ) \right)^{\frac{n-4}{4}}} \left(1 - \frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_\circ) \epsilon^2 + o(\epsilon^2) \right).$$

Les intégrales sur $B(x_\circ,2\delta)-B(x_\circ,\delta)$ sont toutes du type

$$\left|\int_{(\frac{\delta}{\epsilon})^2}^{(\frac{2\delta}{\epsilon})^2} h(x)\frac{x^q}{(x+1)^p}dx\right| \leq cste\left(\frac{1}{\epsilon}\right)^{2(q-p+1)} = cst.\epsilon^{2(p-q-1)}$$

et puisque $p-q=n-4\geq 3$, alors

$$\int_{(\frac{\delta}{\epsilon})^2}^{(\frac{2\delta}{\epsilon})^2} h(x)\frac{x^q}{(x+1)^p}dx = o(\epsilon^2).$$

En fin

$$\int_M (\Delta u_\epsilon)^2\,dv(g) = \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1-\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2+o(\epsilon^2)\right)$$

Récapitulant, on obtient

$$\int_M (\Delta u_\epsilon)^2 - a(x)\left|\nabla u_\epsilon\right|^2 + b(x)u_\epsilon^2 dv(g) = \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\times$$

$$\left(1-\left(\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_\circ)+\frac{4(n-1)}{(n^2-4)(n-6)}a(x_\circ)\right)\epsilon^2+o(\epsilon^2)\right).$$

Tenant compte de l'expression de J_λ

$$J_\lambda(u_\epsilon) = \frac{1}{2}\left\|u_\epsilon\right\|^2 - \frac{\lambda}{q}\left\|u_\epsilon\right\|_q^q - \frac{1}{N}\int_M f(x)\left|u_\epsilon(x)\right|^N dv(g)$$

où

$$\left\|u_\epsilon\right\|^2 = \int_M \left|\Delta u_\epsilon\right|^2 - a(x)\left|\nabla u_\epsilon\right|^2 + b(x)u_\epsilon^2 dv(g)$$

et $\lambda>0$, on obtient

$$J_\lambda(u_\epsilon) \leq J_0(u_\epsilon) = \frac{1}{2}\left\|u_\epsilon\right\|^2 - \frac{1}{N}\int_M f(x)\left|u_\epsilon(x)\right|^N dv(g)$$

$$J_0(u_\epsilon) \leq \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\times$$

$$\left[\frac{2}{n} - \left(\frac{n^2+4n-20}{(n^2-4)(n-6)}S_g(x_\circ) + \frac{2(n-1)}{(n^2-4)(n-6)}a(x_\circ) - \frac{1}{4(n-2)}\frac{\Delta f(x_\circ)}{f(x_\circ)}\right)\epsilon^2 + o(\epsilon^2)\right]$$

$$\leq \frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \times$$

$$\left[1 - \left(\frac{(n^2+4n-20)\,n}{2(n^2-4)(n-6)}S_g(x_\circ) + \frac{(n-1)n}{(n^2-4)(n-6)}a(x_\circ) - \frac{n}{8(n-2)}\frac{\Delta f(x_\circ)}{f(x_\circ)}\right)\epsilon^2 + o(\epsilon^2)\right].$$

Pour assurer

$$J_\lambda(u_\epsilon) < \frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}$$

on prend

$$\left(\frac{(n^2+4n-20)}{2(n+2)}S_g(x_\circ) + \frac{(n-1)}{(n+2)}a(x_\circ) - \frac{(n-6)}{8}\frac{\Delta f(x_\circ)}{f(x_\circ)}\right) > 0$$

Ce qui achève la preuve. ■

2.5.2 Application aux variétés riemanniennes compactes de dimensions $n = 6$

Théorème 2.4 *Lorsque $n = 6$, s'il existe un point $x_\circ \in M$ où $S_g(x_\circ) > -3a(x_\circ)$ alors (2.1) admet une solution u non triviale de classe $C^{4,\alpha}, \alpha \in (0,1)$.*

Preuve: Le developpement de l'intégrale reste le même

$$\int_M f(x)\left|u_\epsilon(x)\right|^N dv(g) = \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1 - (\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-2)})\epsilon^2 + o(\epsilon^2)\right)$$

et

$$\int_{B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g) = (n-4)^2(\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}}\frac{w_{n-1}}{2}\left(a(x_\circ)\epsilon^2\int_0^{(\frac{\delta}{\epsilon})^2}\frac{x^{\frac{n}{2}}}{(x+1)^{n-2}}dx + o(\epsilon^2)\right)$$

on fait le changement de variable

$$y = x + 1$$

ce qui permet d'écrire

$$\int_{B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g) = (n-4)^2 (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \times$$

$$\left(a(x_\circ)\epsilon^2 \int_1^{(\frac{\delta}{\epsilon})^2+1} \frac{(y-1)^{\frac{n}{2}}}{y^{n-2}} dx + o(\epsilon^2) \right)$$

$$= (n-4)^2 (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \left(a(x_\circ)\epsilon^2 + \int_1^{(\frac{\delta}{\epsilon})^2+1} \frac{(y-1)^{\frac{n}{2}}}{y^{n-2}} dx \ + o(\epsilon^2) \right)$$

$$= (n-4)^2 (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \left[a(x_\circ)\epsilon^2 \left(O(1) + \int_k^{(\frac{\delta}{\epsilon})^2+1} \frac{y^{\frac{n}{2}}}{y^{n-2}} dx \right) + o(\epsilon^2) \right]$$

$$= (n-4)^2 (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \left[a(x_\circ)\epsilon^2 \left(O(1) + \log\left((\frac{\delta}{\epsilon})^2 + 1 \right) \right) + o(\epsilon^2) \right]$$

$$= (n-4)^2 (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \left(a(x_\circ)\epsilon^2 \log(\frac{1}{\epsilon^2}) + O(\epsilon^2) \right)$$

Donc,

$$\int_M a(x)\left|\nabla u_\epsilon\right|^2 dv(g) = (n-4)^2 (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \left(a(x_\circ)\epsilon^2 \log(\frac{1}{\epsilon^2}) + O(\epsilon^2) \right)$$

$$\int_M \left|\Delta u_\epsilon\right|^2 dv(g) = (n-4)^2 (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \times$$

$$\left(\frac{n(n+2)(n-2)}{(n-4)} I_n^{\frac{n}{2}-1} - \frac{2}{n} S_g(x_\circ)\epsilon^2 \log(\frac{1}{\epsilon^2}) + O(\epsilon^2) \right).$$

Le developpement du premier terme de la fonctionnelle J_λ reste inchangé et par suite

$$\int_M \left|\Delta u_\epsilon\right|^2 - a(x)\left|\nabla u_\epsilon\right|^2 + b(x)u_\epsilon^2 dv(g) = (n-4)^2 (\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \times$$

$$\left[\frac{n(n+2)(n-2)}{(n-4)} I_n^{\frac{n}{2}-1} - \left(\frac{2}{n} S_g(x_\circ) + a(x_\circ) \right) \epsilon^2 \log(\frac{1}{\epsilon^2}) + O(\epsilon^2) \right]$$

$$\int_M \left|\Delta u_\epsilon\right|^2 - a(x)\left|\nabla u_\epsilon\right|^2 + b(x)u_\epsilon^2 dv(g) = \frac{1}{K_\circ^{\frac{n}{4}} \left(f(x_\circ)\right)^{\frac{n-4}{4}}} \times$$

$$\left(1 - \frac{(n-4)}{n\,(n^2-4)\,I_n^{\frac{n}{2}-1}}\left(\frac{2}{n}S_g(x_\circ) + a(x_\circ)\right)\epsilon^2\log(\frac{1}{\epsilon^2}) + O(\epsilon^2)\right)$$

$$\int_M f(x)\left|u_\epsilon(x)\right|^N dv(g) = \frac{1}{K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left(1 - (\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-2)})\epsilon^2 + O(\epsilon^2)\right)$$

$$J_\lambda\left(u_\epsilon\right) \le \frac{1}{2}\left\|u_\epsilon\right\|^2 - \frac{1}{N}\int_M f(x)\left|u_\epsilon(x)\right|^N dv(g)$$

$$J_\lambda\left(u_\epsilon\right) \le \frac{1}{K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\times$$

$$\left(\frac{1}{2} - \frac{1}{N} - \frac{(n-4)}{2n\,(n^2-4)\,I_n^{\frac{n}{2}-1}}\left(\frac{2}{n}S_g(x_\circ) + a(x_\circ)\right)\epsilon^2\log(\frac{1}{\epsilon^2}) + O(\epsilon^2)\right)$$

$$\le \frac{2}{nK_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left(1 - \frac{(n-4)}{4\,(n^2-4)\,I_n^{\frac{n}{2}-1}}\left(\frac{2}{n}S_g(x_\circ) + a(x_\circ)\right)\epsilon^2\log(\frac{1}{\epsilon^2}) + O(\epsilon^2)\right)$$

$$J_\lambda\left(u_\epsilon\right) \le \frac{2}{nK_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left(1 - \frac{(n-4)}{4\,(n^2-4)\,I_n^{\frac{n}{2}-1}}\left(\frac{2}{n}S_g(x_\circ) + a(x_\circ)\right)\epsilon^2\log(\frac{1}{\epsilon^2}) + O(\epsilon^2)\right)$$

Ce qui donne pour $\epsilon \longrightarrow 0^+$

$$J_\lambda\left(u_\epsilon\right) < \frac{2}{nK_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}$$

pourvu qu'il existe un point $x_\circ$ de M tel que

$$\frac{2}{n}S_g(x_\circ) + a(x_\circ) > 0$$

i.e.

$$S_g(x_\circ) > -3a(x_\circ).$$

■

Chapitre 3

Problème elliptique non-linéaire avec termes singuliers

3.1 Introduction

Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$. Soient $a \in L^r(M)$ et $b \in L^s(M)$ où $r > \frac{n}{2}$ et $s > \frac{n}{4}$ et f une fonction de classe C^∞ sur M strictement positive.
On considère l'équation suivante:

$$\Delta_g^2 u + div_g \left(a(x) \nabla_g u\right) + b(x) u = \lambda \left|u\right|^{q-2} u + f(x) \left|u\right|^{N-2} u \tag{3.1}$$

où $1 < q < 2$ et $N = \frac{2n}{n-4}$ l'exposent critique de Sobolev et λ un réel strictement positif. Dans cette section, on démontre l'existence d'une solution non triviale, en procédant par la technique variationnelle.
On considère sur $H_2^2(M)$ la fonctionnelle:

$$J_\lambda(u) = \frac{1}{2} \int_M \left(\Delta_g u\right)^2 - a(x) \left|\nabla_g u\right|^2 + b(x) u^2 dv(g) - \frac{\lambda}{q} \int_M \left|u\right|^q dv(g) - \frac{1}{N} \int_M f(x) \left|u\right|^N dv(g)$$

On pose:

$$\Phi_\lambda(u) := \langle \nabla J_\lambda(u),\ u\rangle$$

$$\Phi_\lambda(u) = \int_M (\Delta_g u)^2 - a(x)\left|\nabla_g u\right|^2 + b(x)u^2 dv(g) - \lambda \int_M |u|^q \, dv(g) - \int_M f(x)\,|u|^N \, dv(g)$$

On considère l'ensemble M_λ donné par:

$$M_\lambda = \left\{ u \in H_2^2(M) :\ \Phi_\lambda(u) = 0 \text{ et } \|u\| \geq \tau > 0 \right\}$$

On prendra les fonctions $a(x)$ et $b(x)$ de telle manière que :

$$\|u\|^2 = \int_M (\Delta_g u)^2 - a(x)\left|\nabla_g u\right|^2 + b(x)u^2 dv(g)$$

soit une norme équivalente à celle de $H_2^2(M)$.

Exemple 3.1 *Pour avoir une norme équivalente à celle de $H_2^2(M)$, on peut prendre par exemple $a(x)$ et $b(x)$ comme suit*

$$\|a\|_r < +\infty \ \text{et}\ \|b\|_s < +\infty \ \text{où}\ r > \frac{n}{2}\ \text{et}\ \ s > \frac{n}{4}$$

Définition 3.1 *On dit que l'opérateur $P_g : u \to P_g(u) = \Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u$ est coercif s'il existe $\Lambda > 0$ telle que pour tout $u \in H_2^2(M)$:*

$$\int_M P_g(u).udv(g) \geq \Lambda \|u\|^2_{H_2^2(M)}$$

Proposition 5 *$\|u\| = \left(\int_M (\Delta_g u)^2 - a(x)\left|\nabla_g u\right|^2 + b(x)u^2 dv(g)\right)^{\frac{1}{2}}$ est une norme équivalente à celle de $H_2^2(M)$ si et seulement si l'opérateur $P_g : u \to P_g(u) = \Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u$ est coercif.*

Preuve:

1. $(\Rightarrow)$

On a supposé que $\|.\|$soit une norme équivalente à celle de $H_2^2(M)$ i.e. ils existent deux constantes α et $\beta > 0$ telles que pour tout $u \in H_2^2(M)$

$$\alpha \|u\|_{H_2^2(M)} \leq \|u\| \leq \beta \|u\|_{H_2^2(M)}$$

Donc l'opérateur

$$P_g(u) := \Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u$$

est coercif.

2. $(\Leftarrow)$

Si on suppose que l'opérateur

$$P_g(u) := \Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u$$

est coercif, il existe $\Lambda > 0$ tel que pour tout $u \in H_2^2(M)$:

$$\int_M P_g(u)udv(g) = \int_M (\Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u)udv(g) \geq \Lambda \|u\|^2_{H_2^2(M)}$$

Comme M est compacte et $a \in L^r(M)$ et $b \in L^s(M)$ où $r > \frac{n}{2}$ et $s > \frac{n}{4}$

$$\int_M (\Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u)udv(g) = \int_M \left(\Delta_g u\right)^2 - a(x)\left|\nabla_g u\right|^2 + b(x)u^2 dv(g)$$

D'aprés l'inégalité de Hölder, on obtient

$$\left|\int_M (\Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u)udv(g)\right| \leq \|\Delta_g u\|_2^2 + \|a\|_{\frac{n}{2}} \|\nabla_g u\|^2_{2^*} + \|b\|_{\frac{n}{4}} \|u\|_N^2$$

D'aprés l'inégalité de Sobolev, on obtient :

$$\|\nabla_g u\|^2_{2^*} \leq \max((1+\eta)K(n,2)^2, A_\eta) \int_M \left|\nabla_g^2 u\right|^2 + \left|\nabla_g u\right|^2 dv(g)$$

Et

$$\|u\|_N^2 \leq \max((1+\varepsilon)K_\circ, B_\varepsilon)\, \|u\|_{H_2^2(M)}^2$$

D'aprés une égalité bien connue ([3] page 115)

$$\int_M \left|\nabla_g^2 u\right|^2 dv(g) = \int_M |\Delta_g u|^2 - R_{ij}\nabla^i u \nabla^j u dv(g)$$

Alors, il éxiste $\beta > 0$

$$\int_M \left|\nabla_g^2 u\right|^2 dv(g) \leq \int_M (\Delta_g u)^2 + \beta \left|\nabla_g u\right|^2 dv(g)$$

On obtient

$$\|\nabla_g u\|_{2^*}^2 \leq (\beta+1)\max((1+\eta)K(n,2)^2, A_\eta) \int_M (\Delta_g u)^2 + |\nabla_g u|^2 + u^2 dv(g)$$

On obtient

$$\int_M P_g(u)udv(g) \leq \|u\|_{H_2^2(M)}^2 + (\beta+1)\,\|a\|_{\frac{n}{2}} \max((1+\eta)K(n,2)^2, A_\eta)\,\|u\|_{H_2^2(M)}^2 +$$

$$\|b\|_{\frac{n}{4}} \max((1+\varepsilon)K_\circ, B_\varepsilon)\,\|u\|_{H_2^2(M)}^2$$

$$\int_M P_g(u)udv(g) \leq \|u\|_{H_2^2(M)}^2 \times$$

$$\underbrace{\max\left(1, \|b\|_{\frac{n}{4}} \max((1+\varepsilon)K_\circ, B_\varepsilon), (\beta+1)\,\|a\|_{\frac{n}{2}} \max((1+\varepsilon)K(n,2)^2, A_\varepsilon)\right)}_{>0}$$

D'où le résultat.

■

Lemme 3.1 *L'ensemble M_λ est non vide pour $\lambda \in (0,\, \lambda_\circ)$ où*

$$\lambda_\circ = \frac{\left(2^{q-2} - 2^{q-N}\right)\Lambda^{\frac{N-q}{N-2}}}{V(M)^{\left(1-\frac{q}{N}\right)} \left(\max_{x\in M} f(x)\right)^{\frac{2-q}{N-2}} \left(\max((1+\varepsilon)K_\circ, A_\varepsilon)\right)^{\frac{N-q}{N-2}}}$$

avec $1 < q < 2$.

Preuve:
(Même démonstration que le lemme **2.1**). ■

3.2 Etude de la fonctionnelle J_λ sur M_λ

Lemme 3.2 *Soit* (M, g) *une variété riemannienne compacte de dimension* $n \geq 5$. *Il existe* $A > 0$ *telque* $J_\lambda(u) \geq A > 0$ *pour tout* $u \in M_\lambda$ *et pour tout* $\lambda \in (0, \min(\lambda_\circ, \lambda_1))$ *où*

$$\lambda_1 = \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}}.$$

Preuve: (Même démonstration que le lemme **2.2**). ■

Lemme 3.3 *Soit* (M, g) *une variété riemannienne compacte de dimension* $n \geq 5$, *alors les deux assertions suivantes sont vraies:*

1. $\langle \nabla \Phi_\lambda(u), u \rangle < 0$ *pour tout* $u \in M_\lambda$ *et pour tout* $\lambda \in (0, \min(\lambda_\circ, \lambda_1))$.

2. *Les points critiques de* J_λ *sont les points de* M_λ.

Preuve:
(Même démonstration que le lemme **2.3**). ■

3.3 Existence d'une solution non triviale de l'équation (3.1) sur $\mathbf{M}_\lambda$

On va voir dans ce qui suit que la fonctionnelle d'énergie J_λ vérifie les conditions de Palais-Smale sur la contrainte M_λ.

Théorème 3.1 *Soit* (M,g) *une variété riemannienne compacte de dimension* $n \geq 5$. *Soit* $(u_m)_m$ *une suite dans* M_λ *telle que:*

$$\begin{cases} J_\lambda(u_m) \leq c \\ \nabla J_\lambda(u_m) - \mu_m \nabla\Phi_\lambda(u_m) \to 0 \end{cases}$$

Supposons que

$$c < \frac{2}{nK_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}}$$

Alors il existe une sous-suite de $(u_m)_m$ *convergente fortement dans* $H_2^2(M)$.

Preuve: Soit $(u_m)_m \subset M_\lambda$

$$J_\lambda(u_m) = \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\int_M |u_m|^q \, dv(g)$$

Dans un premier temps on montre que la suite $(u_m)_m$ est bornée dans $H_2^2(M)$.
Par le Lemme **3.2**, on obtient que:

$$J_\lambda(u_m) \geq \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|u_m\|^q$$

$$J_\lambda(u_m) \geq \|u_m\|^2\,(\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}) > 0$$

et comme $0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}}$ et $J_\lambda(u_m) \leq c$, alors

$$c \geq J_\lambda(u_m)$$

$$\geq [\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}]\,\|u_m\|^2 > 0$$

d'où

$$0 \leq \|u_m\|^2 \leq \frac{c}{\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}} < +\infty.$$

Donc $(u_m)_m$ est bornée dans $H_2^2(M)$. La réflexivité de l'espace $H_2^2(M)$ et la compacité de l'inclusion $H_2^2(M) \subset H_p^k(M)$ ($k = 0,1; p < N$), implique qu'il existe une sous-suite notée $(u_m)_m$ telle que :

1. $u_m \rightarrow u$ faiblement dans $H_2^2(M)$.

2. $u_m \rightarrow u$ fortement dans $L^p(M)$ où $p < N$.

3. $\nabla u_m \rightarrow \nabla u$ fortement dans $L^p(M)$ où $p < 2^* = \frac{2n}{n-2}$.

4. $u_m \rightarrow u$ presque partout dans M.

Comme $\frac{2s}{s-1} < N = \frac{2n}{n-4}$, on obtient:

$$\left| \int_M b(x) \left| u_m - u \right|^2 dv(g) \right| \leq \|b\|_s \|u_m - u\|_{\frac{2s}{s-1}}$$

D'après l'inégalité de Sobolev, on obtient

$$\left| \int_M b(x) \left| u_m - u \right|^2 dv(g) \right| \leq \|b\|_s \left(K_o + \epsilon\right) \|\Delta \left(u_m - u\right)\|_2^2 + A_\epsilon \|u_m - u\|_2^2$$

Comme

$$K_\circ = \frac{16}{(n-4)n(n^2-4)2^{n-1}\omega_n^{\frac{4}{n}}} < 1$$

Alors,

$$\left| \int_M b(x) \left| u_m - u \right|^2 dv(g) \right| \leq \|b\|_s \|\Delta \left(u_m - u\right)\|_2^2 + o(1)$$

De la même façon pour

$$\left| \int_M a(x) \left| \nabla (u_m - u) \right|^2 dv(g) \right| \leq \|a\|_r \|\nabla (u_m - u)\|_2^2 + o(1)$$

Car $\frac{2s}{s-1} < N = \frac{2n}{n-4}$.

D'après le lemme de Brézis-Lieb, on peut écrire

$$\int_M \left(\Delta_g u_m\right)^2 dv(g) = \int_M \left(\Delta_g u\right)^2 dv(g) + \int_M \left(\Delta_g (u_m - u)\right)^2 dv(g) + o(1)$$

et aussi

$$\int_M f(x)\left|u_m\right|^N dv(g) = \int_M f(x)\left|u\right|^N dv(g) + \int_M f(x)\left|u_m - u\right|^N dv(g) + o(1).$$

On va montrer que $u \in M_\lambda$.

Comme $u_m \to u$ faiblement dans $H_2^2(M)$ i.e. si pour tout $\phi \in (H_2^2(M))^*$,

$$\int_M \left(\Delta_g u_m \Delta_g \phi - a(x)\left\langle \nabla u_m, \nabla \phi\right\rangle_g + a u_m \phi\right) dv(g) =$$

$$\int_M \left(\Delta_g u \Delta_g \phi - a(x)\left\langle \nabla u, \nabla \phi\right\rangle_g + a u \phi\right) dv(g) + o(1)$$

En particulier pour $\phi = u$, on obtient,

$$\int_M \left(\Delta_g u_m \Delta_g u - a(x)\left\langle \nabla u_m, \nabla u\right\rangle_g + a u_m u\right) dv(g) = \left\|u\right\|^2 + o(1)$$

et aussi pour $\phi = u_m$,

$$\int_M \left(\Delta_g u_m \Delta_g u - a(x)\left\langle \nabla u_m, \nabla u\right\rangle_g + a u_m u\right) dv(g) = \left\|u_m\right\|^2 + o(1)$$

et puisque $(u_m)_m \subset M_\lambda$ i.e.

$$\int_M \left(\lambda\left|u_m\right|^{q-2} u_m u + f(x)\left|u_m\right|^{N-2} u_m u\right) dv(g) = \left\|u\right\|^2 + o(1)$$

mais quand $m \to +\infty$

$$\int_M \left(\lambda\left|u_m\right|^{q-2} u_m u + f(x)\left|u_m\right|^{N-2} u_m u\right) dv(g) \to \int_M \left(\lambda\left|u\right|^q + f(x)\left|u\right|^N\right) dv(g)$$

ce qui donne

$$\Phi_\lambda(u_m) = \Phi_\lambda(u) = \left\|u\right\|^2 - \lambda\int_M \left|u\right|^q dv(g) - \int_M f(x)\left|u\right|^N dv(g) = 0$$

ou encore

$$\|u\| + o(1) = \|u_m\| \geq \tau > 0$$

D'où $u \in M_\lambda$.

On va montrer que $\mu_m \to 0$ quand $m \to +\infty$

En testant avec u_m, on obtient

$$\langle \nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m), u_m \rangle = o(1)$$

$$= \underbrace{\langle \nabla J_\lambda(u_m), u_m \rangle}_{=0} - \mu_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle = o(1)$$

Donc,

$$\mu_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle = o(1)$$

D'après le Lemme **3.2**, on a que $\limsup_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle < 0$

et par consequent

$$\mu_m \to 0 \text{ quand } m \to +\infty.$$

On va montrer que $u_m \to u$ converge fortement dans $H_2^2(M)$, nous avons

$$J_\lambda(u_m) - J_\lambda(u)$$

$$= \frac{1}{2} \int_M (\Delta_g(u_m - u))^2 \, dv(g) - \frac{1}{N} \int_M f(x) \, |u_m - u|^N \, dv(g) + o(1). \tag{3.2}$$

Puisque $u_m - u \to 0$ faiblement dans $H_2^2(M)$, on teste avec $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$

$$\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u \rangle = o(1)$$

$$= \int_M (\Delta_g(u_m - u))^2 \, dv(g) - \int_M f(x) \, |u_m - u|^N \, dv(g) = o(1). \tag{3.3}$$

De sorte que

$$\int_M (\Delta_g(u_m-u))^2\,dv(g) = \int_M f(x)\,|u_m-u|^N\,dv(g) + o(1)$$

et en tenant compte de (3.2), on obtient

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2}\int_M (\Delta_g(u_m-u))^2\,dv(g) - \frac{1}{N}\int_M (\Delta_g(u_m-u))^2\,dv(g) + o(1)$$

i.e.

$$J_\lambda(u_m) - J_\lambda(u) = \frac{2}{n}\int_M (\Delta_g(u_m-u))^2\,dv(g)$$

Independament, d'après l'inégalité de Sobolev, on obtient pour tout $u \in H_2^2(M)$

$$\|u\|_N^2 \le (1+\varepsilon)K_\circ \int_M (\Delta_g u)^2 + |\nabla_g u|^2\,dv(g) + A_\varepsilon \int_M u^2 dv(g)$$

En testant l'inégalité de Sobolev par $u_m - u$, on obtient

$$\|u_m-u\|_N^2 \le (1+\varepsilon)K_\circ \int_M (\Delta_g(u_m-u))^2\,dv(g) + o(1) \tag{3.4}$$

Comme

$$\int_M f(x)\,|u_m-u|^N\,dv(g) \le \max_{x\in M} f(x) \int_M |u_m-u|^N\,dv(g)$$

en remplaçant dans (3.4), on obtient

$$\int_M f(x)\,|u_m-u|^N\,dv(g) \le (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta_g(u_m-u)\|_2^N + o(1)$$

et faisant appel à l'égalité (3.3),

$$o(1) \ge \|\Delta_g(u_m-u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta_g(u_m-u)\|_2^N + o(1)$$

$$\ge \|\Delta_g(u_m-u)\|_2^2 (1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta_g(u_m-u)\|_2^{N-2}) + o(1)$$

et par conséquent si

$$\limsup_{m\to+\infty}\|\Delta_g(u_m-u)\|_2^{N-2}<\frac{1}{(1+\varepsilon)^{\frac{n}{n-4}}K_\circ^{\frac{n}{n-4}}\max_{x\in M}f(x)}$$

on trouve que

$$\frac{2}{n}\int_M(\Delta_g(u_m-u))^2\,dv(g)<c$$

Comme

$$c<\frac{2}{nK_\circ^{\frac{n}{4}}(\max_{x\in M}f(x))^{\frac{n-4}{4}}}$$

alors

$$\int_M(\Delta_g(u_m-u))^2\,dv(g)<\frac{1}{K_\circ^{\frac{n}{4}}(\max_{x\in M}f(x))^{\frac{n-4}{4}}}$$

Par conséquent

$$o(1)\geq\|\Delta_g(u_m-u)\|_2^2\underbrace{(1-(1+\varepsilon)^{\frac{n}{n-4}}\max_{x\in M}f(x)K_\circ^{\frac{n}{n-4}}\|\Delta_g(u_m-u)\|_2^{N-2})}_{>0}+o(1)$$

ou encore

$$\|\Delta_g(u_m-u)\|_2^2=o(1)$$

i.e. $u_m\to u$ converge fortement dans $H_2^2(M)$. ■

Maintenant, on montre qu'il existe une suite de Palais-Smale au niveau c sur la contrainte M_λ.

Lemme 3.4 *Soit (M,g) une variété riemannienne compacte de dimension $n\geq 5$, pour tout $\lambda\in(0,\,\min(\lambda_\circ,\lambda_1):=\lambda^*)$, Alors il existe $(u_m,\,\mu_m)_m\in M_\lambda\times\mathbb{R}$ telle que:*

$$\begin{cases}\nabla J_\lambda(u_m)-\mu_m\nabla\Phi_\lambda(u_m)\to 0\ \textit{dans}\ (H_2^2(M))^*\\ J_\lambda(u_m)\ \textit{est bornée}\end{cases}$$

Preuve:

Comme la fonctionnelle J_λ est de classe C^1 au sens de Fréchet d'après les Lemmes **3.1;**

3.2 et **3.3**, on a montré J_λ est bornée dans M_λ.

D'après le lemme d'Ekeland, il existe $(u_m,\, \mu_m)_m \in M_\lambda \times \mathbb{R}$ telle que:

$$\begin{cases} \nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m) \to 0 \text{ dans } (H_2^2(M))^* \\ J_\lambda(u_m) \text{ est bornée} \end{cases}$$

■

Théorème 3.2 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$ et f une fonction strictement positive. Supposons que l'opérateur : $P_g(u) := \Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u$ est coercif et supposons que*

$$c < \frac{2}{nK_\circ^{\frac{n}{4}} \left(\max_{x\in M} f(x)\right)^{\frac{n}{4}-1}}$$

Alors il existe un $\lambda^* > 0$ telle que $\lambda \in (0,\, \lambda^*)$, l'équation (3.1) possède une solution faible non-triviale.

Preuve:

D'après les Lemmes **3.1; 3.2; 3.3** et **3.4**, on a démontré qu'il existe un $\varphi \in M_\lambda$ telle que:

$$J_\lambda(\varphi) = \max_{u \in M_\lambda} J_\lambda(u)$$

En appliquant le théorème des multiplicateurs de Lagrange, il existe $\mu \in R$ tel que pour tout $\varphi \in M_\lambda$:

$$\nabla J_\lambda(\varphi) = \mu \nabla \Phi_\lambda(\varphi) \tag{3.5}$$

En testant au point $\varphi \in M_\lambda$ l'équation (3.5), on obtient

$$\Phi_\lambda(\varphi) = \left\langle \nabla J_\lambda(\varphi), \varphi \right\rangle = \mu \left\langle \nabla \Phi_\lambda(\varphi), \varphi \right\rangle = 0$$

D'aprés le lemme **3.3** on obtient:

$$\nabla J_\lambda(\varphi) = 0$$

D'où pour tout $\varphi \in M_\lambda$:

$$\langle \nabla J_\lambda(\varphi), \varphi \rangle = 0$$

■

3.3.1 Application géométrique

On considère l'équation suivante :

$$\Delta_g^2 u + div_g \left(\frac{a(x)}{\rho^\sigma} \nabla_g u \right) + \frac{b(x)}{\rho^\mu} u = f(x) |u|^{N-2} u + \lambda |u|^{q-2} u \tag{3.6}$$

Où a et b deux fonctions de classe $C^\infty(M)$ et ρ la fonction distance (Définition **1.6**).
On considère sur $H_2^2(M)$ la fonctionnelle:

$$J_{\lambda,\sigma,\mu}(u) = \frac{1}{2} \int_M (\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\mu} u^2 dv(g) - \frac{1}{N} \int_M f(x) |u|^N dv(g) - \frac{\lambda}{q} \int_M |u|^q dv(g)$$

On pose :

$$\Phi_{\lambda,\sigma,\mu}(u) := \langle \nabla J_{\lambda,\sigma,\mu}(u),\ u \rangle$$

$$\Phi_{\lambda,\sigma,\mu}(u) = \int_M (\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\mu} u^2 dv(g) - \int_M f(x) |u|^N dv(g) - \lambda \int_M |u|^q dv(g)$$

Théorème 3.3 *Soient* $0 < \sigma < \frac{n}{r} < 2$ *et* $0 < \mu < \frac{n}{s} < 4$ *et onsuppose que*

$$\sup_{u \in H_2^2(M)} J_{\lambda,\sigma,\mu}(u) < \frac{2}{n K_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}}$$

Alors, il existe un $\lambda^* > 0$ *telle que* $\lambda \in (0,\ \lambda^*)$*, l'équation (3.6) possède une solution faible non triviale* $u_{\sigma,\delta} \in M_\lambda$*.*

Preuve: Si on pose $\tilde{a} := \frac{a(x)}{\rho^\sigma}$ et $\tilde{b} := \frac{b(x)}{\rho^\mu}$, si $\sigma \in (0,2)$ et $\mu \in (0,4)$, alors $\tilde{a} \in L^r(M)$, $\tilde{b} \in L^s(M)$ telle que $0 < \sigma < \frac{n}{r} < 2$ et $0 < \mu < \frac{n}{s} < 4$, alors ce théorème est un corollaire immédiat du théorème **3.2**. ■

3.4 Le cas critique $\sigma = 2$ et $\mu = 4$

Ce cas correpond à l'équation non-linéaire ci-dessous:

$$\Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^2}\nabla_g u\right) + \frac{b(x)}{\rho^4}u = f(x)\,|u|^{N-2}\,u + \lambda\,|u|^{q-2}\,u \tag{3.7}$$

Lemme 3.5 $\|u\| = (\int_M P_g(u).udv(g))^{\frac{1}{2}}$ *est une norme équivalente à celle de* $H_2^2(M)$ *si et seulement si l'opérateur* $P_g : u \to P_g(u) := \Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^2}\nabla_g u\right) + \frac{b(x)}{\rho^4}u$ *est coercif.*

Preuve: $(\Rightarrow)$

On a supposé que $\|.\|$soit une norme équivalente à celle de $H_2^2(M)$ i.e. ils existent deux constantes α et $\beta > 0$ telles que pour tout $u \in H_2^2(M)$

$$\alpha\,\|u\|_{H_2^2(M)} \leq \|u\| \leq \beta\,\|u\|_{H_2^2(M)}$$

Donc l'opérateur

$$P_g(u) := \Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^2}\nabla_g u\right) + \frac{b(x)}{\rho^4}u$$

est coercif.

$(\Leftarrow)$

Si on suppose que l'opérateur

$$P_g(u) := \Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^2}\nabla_g u\right) + \frac{b(x)}{\rho^4}u$$

est coercif, il existe $\Lambda > 0$ tel que pour tout $u \in H_2^2(M)$:

$$\int_M P_g(u)udv(g) = \int_M (\Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^2}\nabla_g u\right) + \frac{b(x)}{\rho^4}u)udv(g) \geq \Lambda\,\|u\|_{H_2^2(M)}^2$$

$$= \int_M (\Delta_g u)^2 - \frac{a(x)}{\rho^2}\,|\nabla_g u|^2 + \frac{b(x)}{\rho^4}u^2 dv(g)$$

Donc

$$\left|\int_M (\Delta_g u)^2 - \frac{a(x)}{\rho^2}|\nabla_g u|^2 + \frac{b(x)}{\rho^4}u^2 dv(g)\right| \leq$$

$$\|\Delta_g u\|_2^2 + \max_{x\in M}(|a(x)|)\int_M \rho^{-2}|\nabla_g u|^2 dv(g) + \max_{x\in M}(|b(x)|)\int_M \rho^{-4}u^2 dv(g).$$

D'aprés l'inégalité de Hardy-Sobolev, on obtient:

$$\int_M \rho^{-4}u^2 dv(g) \leq \max\left((1+\epsilon)K(n,2,-4)^2, A(\epsilon)\right)\|u\|^2_{H_2^2(M)}$$

et

$$\int_M \rho^{-2}|\nabla_g u|^2 dv(g) \leq \max\left(\eta + K(n,2,-2), B(\eta)\right)\int_M (\Delta_g u)^2 + |\nabla_g u|^2 dv(g)$$

D'aprés une égalité bien connue ([3] page 115)

$$\int_M \left|\nabla_g^2 u\right|^2 dv(g) = \int_M (\Delta_g u)^2 - R_{ij}\nabla^i u \nabla^j u dv(g)$$

Alors, il éxiste $\beta > 0$

$$\int_M \left|\nabla_g^2 u\right|^2 dv(g) \leq \max(1,\beta)\int_M (\Delta_g u)^2 + |\nabla_g u|^2 dv(g).$$

Alors, il éxiste $\xi > 0$ et $\zeta > 0$

$$\int_M \rho^{-2}|\nabla_g u|^2 dv(g) \leq \xi\|u\|^2_{H_2^2(M)}$$

et

$$\int_M \rho^{-4}u^2 dv(g) \leq \zeta\|u\|^2_{H_2^2(M)}$$

On obtient

$$\int_M P_g(u)u dv(g) \leq \|u\|^2_{H_2^2(M)} + \xi\max_{x\in M}(|a(x)|)\|u\|^2_{H_2^2(M)} + \zeta\max_{x\in M}(|b(x)|)\|u\|^2_{H_2^2(M)}$$

$$\int_M P_g(u)udv(g) \leq \underbrace{\max\left(1, \xi \max_{x\in M}(|a(x)|), \zeta \max_{x\in M}(|b(x)|)\right)}_{>0} \|u\|^2_{H_2^2(M)}.$$

D'où le résultat. ■

Théorème 3.4 *Soit (M,g) une variété riemannienne compacte de dimension $n \geq 5$. Soit $(u_m)_m$ une suite dans M_λ telle que:*

$$\begin{cases} J_{\lambda,\sigma,\mu}(u_m) \leq c_{\sigma,\mu} \\ \nabla J_{\lambda,\sigma,\mu}(u_m) - \mu_m \nabla \Phi_{\lambda,\sigma,\mu}(u_m) \to 0 \end{cases}$$

Supposons que

$$\begin{cases} c_{\sigma,\mu} < \frac{2}{nK_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}} \\ \frac{1}{2} + a^- K^2(n,1,2) + b^- K^2(n,2,4) > 0 \end{cases}$$

Alors l'équation (3.7) possède une solution dans $H_2^2(M)$.

Preuve:

Soit $(u_m)_m \subset M_{\lambda,\sigma,\mu}$:

$$J_{\lambda,\sigma,\mu}(u_m) = \frac{N-2}{2N} \|u_m\|^2 - \lambda \frac{N-q}{Nq} \int_M |u_m|^q \, dv(g)$$

Dans un premier temps on montre que

$$\lim_{(\sigma,\mu)\to(2^-,4^-)} \inf \Lambda_{\sigma,\mu} \nrightarrow 0$$

(On garde même démonstration que le théorème **5.3**).

Maintenant on montre que la suite $(u_m)_m$ est bornée dans $H_2^2(M)$.

D'aprés qui se précède on a déja démontré (Théorème **3.3**) que $(u_m)_m$ est bornée dans $H_2^2(M)$.

La réflexivité de l'espace $H_2^2(M)$ et la compacité de l'inclusion $H_2^2(M) \subset H_p^k(M)$ ($k = 0, 1$; $p < N$), implique qu'il existe une sous-suite notée $(u_m)_m$ telle que :

1. $u_m \to u$ faiblement dans $H_2^2(M)$.

2. $u_m \to u$ fortement dans $L^p(M)$ où $p < N$.

3. $\nabla u_m \to \nabla u$ fortement dans $L^p(M)$ où $p < 2^* = \frac{2n}{n-2}$.

4. $u_m \to u$ presque partout dans M.

D'après le lemme de Brézis-Lieb, on peut écrire

$$\int_M (\Delta_g u_m)^2 \, dv(g) = \int_M (\Delta_g u)^2 \, dv(g) + \int_M (\Delta_g (u_m - u))^2 \, dv(g) + o(1)$$

et aussi

$$\int_M f(x) \left| u_m \right|^N dv(g) = \int_M f(x) \left| u \right|^N dv(g) + \int_M f(x) \left| u_m - u \right|^N dv(g) + o(1)$$

Il existe deux suites $(\sigma_m)_m$ et $(\mu_m)_m$ qui convergent respectivement vers 2 et 4 telle que la suite de fonctions $(u_m)_m$ converge faiblement dans $H_2^2(M)$ et $L^2(M, \rho^{-4})$ et que la suite $(\nabla u_m)_m$converge faiblement dans $H_1^2(M)$ et $L^2(M, \rho^{-2})$.

Soit $0 < \delta < \delta(M)$, on a

$$\int_M \frac{b(x)}{\rho^{\mu_m}} u^2 dv(g) = \int_{B_P(\delta)} \frac{b(x)}{\rho^{\mu_m}} u^2 dv(g) + \int_{M - B_P(\delta)} \frac{b(x)}{\rho^{\mu_m}} u^2 dv(g)$$

et

$$\int_M \frac{a(x)}{\rho^{\sigma_m}} \left| \nabla u \right|^2 dv(g) = \int_{B_P(\delta)} \frac{a(x)}{\rho^{\sigma_m}} \left| \nabla u \right|^2 dv(g) + \int_{M - B_P(\delta)} \frac{a(x)}{\rho^{\sigma_m}} \left| \nabla u \right|^2 dv(g).$$

D'après le théorème de la convergence dominée de Lebesgue, on obtient que

$$\int_M \frac{a(x)}{\rho^{\sigma_m}} \left| \nabla u \right|^2 dv(g) = \int_M \frac{a(x)}{\rho^2} \left| \nabla u \right|^2 dv(g) + o(1) \quad \text{quand} \quad \sigma_m \to 2^-$$

et aussi

$$\int_M \frac{b(x)}{\rho^{\mu_m}} u^2 dv(g) = \int_M \frac{b(x)}{\rho^4} u^2 dv(g) + o(1) \quad \text{quand} \quad \mu_m \to 4^-$$

Comme $u_m \to u$ faiblement dans $H_2^2(M)$, alors $\nabla u_m \to \nabla u$ faiblement dans $L^2(M, \rho^{-2})$

et $u_m \to u$ faiblement dans $L^2(M,\rho^{-4})$ i.e pour toute $\varphi \in L^2(M)$:

$$\int_M \frac{a(x)}{\rho^2}\nabla u_m \nabla \varphi dv(g) = \int_M \frac{a(x)}{\rho^2}\nabla u \nabla \varphi dv(g) + o(1)$$

et

$$\int_M \frac{b(x)}{\rho^4} u_m \varphi dv(g) = \int_M \frac{b(x)}{\rho^4} u \varphi dv(g) + o(1)$$

Pour tout $\phi \in H_2^2(M)$, on a

$$\int_M \left(\Delta_g^2 u_m + div_g\left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m\right) + \frac{b(x)}{\rho^{\delta_m}} u_m\right)\phi dv(g) = \int_M \left(\lambda\left|u_m\right|^{q-2} u_m + f(x)\left|u_m\right|^{N-2} u_m\right)\phi dv(g)$$

On veut passer à la limite dans cette égalité. C'est immédiat pour la première partie, car on a la convergence faible dans $H_2^2(M)$ i.e pour tout $\phi \in L^2(M)$, on a

$$\int_M u_m \Delta_g^2 \phi dv(g) = \int_M u \Delta_g^2 \phi dv(g) + o(1) = \int_M \Delta_g \phi \Delta_g u dv(g) + o(1)$$

et pour la deuxième partie on a

$$\int_M \left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m - \frac{a(x)}{\rho^2}\nabla_g u\right)\phi dv(g) = \int_M \left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m + \frac{a(x)}{\rho^2}\left(\nabla_g u_m - \nabla_g u_m\right) - \frac{a(x)}{\rho^2}\nabla_g u\right)\phi dv(g)$$

Donc

$$\left|\int_M \left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m - \frac{a(x)}{\rho^2}\nabla_g u\right)\phi dv(g)\right| \leq$$

$$\left|\int_M \left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m - \frac{a(x)}{\rho^2}\nabla_g u_m\right)\phi dv(g)\right| + \left|\int_M \left(\frac{a(x)}{\rho^2}\nabla_g u_m - \frac{a(x)}{\rho^2}\nabla_g u\right)\phi dv(g)\right|$$

$$\leq \int_M \left|a(x)\phi\nabla_g u_m\right|\left|\frac{1}{\rho^{\sigma_m}} - \frac{1}{\rho^2}\right| dv(g) + \left|\int_M \frac{a(x)}{\rho^2}\nabla_g\left(u_m - u\right)\phi dv(g)\right|$$

La convergence faible dans $L^2(M,\rho^{-2})$ et le théorème de la convergence dominée de Lebesgue implique que le secont menbre converge vers 0.

et pour la troisième partie on a

$$\int_M \left(\frac{b(x)}{\rho^{\mu_m}} u_m - \frac{b(x)}{\rho^4} u \right) \phi dv(g) = \int_M \left(\frac{b(x)}{\rho^{\mu_m}} u_m - \frac{b(x)}{\rho^4} u_m + \frac{b(x)}{\rho^4} u_m - \frac{b(x)}{\rho^4} u \right) \phi dv(g)$$

Donc

$$\left| \int_M \left(\frac{b(x)}{\rho^{\mu_m}} u_m - \frac{b(x)}{\rho^4} u \right) \phi dv(g) \right|$$

$$\leq \int_M |b(x) \phi u_m| \left| \frac{1}{\rho^{\mu_m}} - \frac{1}{\rho^4} \right| dv(g) + \left| \int_M \frac{b(x)}{\rho^4} (u_m - u) \phi dv(g) \right|$$

La convergence faible dans $L^2(M, \rho^{-4})$ et le théorème de la convergence dominée de Lebesgue implique que le secont menbre converge vers 0.

D'après le théorème de la convergence dominée de Lebesgue, on peut écrire

$$\int_M \frac{a(x)}{\rho^{\sigma_m}} |\nabla u_m|^2 dv(g) = \int_M \frac{a(x)}{\rho^2} |\nabla u|^2 dv(g) + o(1) \quad \text{quand} \quad \sigma_m \to 2^-$$

et aussi

$$\int_M \frac{b(x)}{\rho^{\mu_m}} (u_m)^2 dv(g) = \int_M \frac{b(x)}{\rho^4} u^2 dv(g) + o(1) \quad \text{quand} \quad \mu_m \to 4^-$$

On va montrer que $u \in M_\lambda$.

Comme $u_m \to u$ faiblement dans $H_2^2(M)$ i.e. si pour tout $\phi \in (H_2^2(M))^*$,

$$\phi (u_m - u) = o(1)$$

En prend par exemple $\phi = \Delta_g^2 \varphi + div_g \left(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g \varphi \right) + \frac{b(x)}{\rho^{\mu_m}} \varphi$ où $\varphi \in L^2(M)$

$$\int_M \left(\Delta_g^2 \varphi + div_g \left(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g \varphi \right) + \frac{b(x)}{\rho^{\mu_m}} \varphi \right) (u_m - u) \; dv(g) = o(1)$$

En particulier pour $\varphi = u$, on obtient,

$$\int_M \left(\Delta_g u_m \Delta_g u - \frac{a(x)}{\rho^{\sigma_m}} \langle \nabla u_m, \nabla u \rangle_g + \frac{b(x)}{\rho^{\delta_m}} u_m \; u \right) dv(g) = \|u\|^2 + o(1)$$

et aussi pour $\varphi = u_m$,

$$\int_M \left(\Delta_g u_m \Delta_g u - \frac{a(x)}{\rho^{\sigma_m}} \langle \nabla u_m, \nabla u \rangle_g + \frac{b(x)}{\rho^{\mu_m}} u_m \, u \right) dv(g) = \|u_m\|^2 + o(1)$$

et puisque $(u_m)_m \subset M_\lambda$ i.e.

$$\int_M \left(\lambda \left| u_m \right|^{q-2} u_m u + f(x) \left| u_m \right|^{N-2} u_m u \right) dv(g) \ = \|u\|^2 + o(1)$$

mais quand $m \to +\infty$

$$\int_M \left(\lambda \left| u_m \right|^{q-2} u_m u + f(x) \left| u_m \right|^{N-2} u_m u \right) dv(g) \to \int_M \left(\lambda \left| u \right|^{q} + f(x) \left| u \right|^{N} \right) dv(g)$$

ce qui donne

$$\Phi_\lambda(u_m) = \Phi_\lambda(u) = \|u\|^2 - \lambda \int_M |u|^q \, dv(g) - \int_M f(x) \left| u \right|^N dv(g) = 0$$

ou encore

$$\|u\| + o(1) = \|u_m\| \geq \tau > 0$$

D'où $u \in M_\lambda$.

On va montrer que $\mu_m \to 0$ quand $m \to +\infty$

En testant avec u_m, on obtient

$$\langle \nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m), u_m \rangle = o(1)$$

$$= \underbrace{\langle \nabla J_\lambda(u_m), u_m \rangle}_{=0} - \mu_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle = o(1)$$

Donc,

$$\mu_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle = o(1)$$

D'après le Lemme **3.3**, on a que $\limsup_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle < 0$

et par consequent

$$\mu_m \to 0 \text{ quand } m \to +\infty.$$

On va montrer que $u_m \to u$ converge fortement dans $H_2^2(M)$, nous avons

$$J_\lambda(u_m) - J_\lambda(u)$$

$$= \frac{1}{2}\|\Delta_g(u_m - u)\|_2^2 - \frac{1}{N}\int_M f(x)\,|u_m - u|^N\,dv(g) + o(1) \tag{3.8}$$

Puisque $u_m - u \to 0$ faiblement dans $H_2^2(M)$, on teste avec $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$

$$\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u\rangle = o(1)$$

$$= \|\Delta_g(u_m - u)\|_2^2 - \int_M f(x)\,|u_m - u|^N\,dv(g) = o(1) \tag{3.9}$$

De sorte que

$$\|\Delta_g(u_m - u)\|_2^2 = \int_M f(x)\,|u_m - u|^N\,dv(g) + o(1)$$

et en tenant compte de (3.8), on obtient

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2}\|\Delta_g(u_m - u)\|_2^2 - \frac{1}{N}\|\Delta_g(u_m - u)\|_2^2 + o(1)$$

i.e.

$$J_\lambda(u_m) - J_\lambda(u) = \frac{2}{n}\|\Delta_g(u_m - u)\|_2^2$$

Independament, d'après l'inégalité de Sobolev, on obtient pout tout $u \in H_2^2(M)$

$$\|u\|_N^2 \le (1+\varepsilon)K_\circ \int_M (\Delta_g u)^2 + |\nabla_g u|^2\,dv(g) + A_\varepsilon \int_M u^2 dv(g)$$

En testant l'inégalité de Sobolev par $u_m - u$, on obtient:

$$\|u_m - u\|_N^2 \le (1+\varepsilon)K_\circ \int_M (\Delta_g(u_m - u))^2\,dv(g) + o(1) \tag{3.10}$$

Comme
$$\int_M f(x)\left|u_m-u\right|^N dv(g) \leq \max_{x\in M} f(x) \int_M \left|u_m-u\right|^N dv(g)$$
en remplaçant dans (3.9), on obtient:
$$\int_M f(x)\left|u_m-u\right|^N dv(g) \leq (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g(u_m-u)\right\|_2^N + o(1)$$
et faisant appel à l'égalité (3.8),
$$o(1) \geq \left\|\Delta_g\left(u_m-u\right)\right\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g(u_m-u)\right\|_2^N + o(1)$$
Alors
$$o(1) \geq \left\|\Delta_g\left(u_m-u\right)\right\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g\left(u_m-u\right)\right\|_2^N + o(1)$$
Comme$\|.\|$ est une norme équivalente à celle de $H_2^2(M)$, alors il éxiste une constante $\Lambda > 0$ telle que:
On obtient :
$$o(1) \geq \left\|\Delta_g\left(u_m-u\right)\right\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g\left(u_m-u\right)\right\|_2^N + o(1)$$
$$o(1) \geq \left\|\Delta_g\left(u_m-u\right)\right\|_2^2 \left(1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g\left(u_m-u\right)\right\|_2^{N-2}\right) + o(1)$$
et par conséquent si
$$\limsup_{m\to+\infty} \left\|\Delta_g\left(u_m-u\right)\right\|_2^{N-2} < \frac{1}{(1+\varepsilon)^{\frac{n}{n-4}} K_\circ^{\frac{n}{n-4}} \max_{x\in M} f(x)}$$
on trouve que
$$\frac{2}{n}\int_M \left(\Delta_g(u_m-u)\right)^2 dv(g) < c.$$

Comme

$$c < \frac{2}{n\ K_\circ^{\frac{n}{4}} (Max\ _{x\in M} f(x))^{\frac{n-4}{4}}}$$

alors

$$\int_M (\Delta_g(u_m - u))^2\, dv(g) < \frac{1}{K_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}}.$$

Par conséquent

$$o(1) \geq \|\Delta_g(u_m - u)\|_2^2 \underbrace{(1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^{N-2})}_{>0} + o(1)$$

ou encore

$$\|\Delta_g(u_m - u)\|_2^2 = o(1)$$

i.e. $u_m \to u$ converge fortement dans $H_2^2(M)$.

■

3.5 Fonctions tests

Pour vérifier l'hypothèse du théorème générique **3.2**, on considère les fonctions tests suivantes:

$$u_\epsilon(x) = (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{8}} \frac{\eta(\rho)}{((\rho\theta)^2 + \epsilon^2)^{\frac{n-4}{2}}}$$

avec

$$\theta = (1 + \|a\|_r + \|b\|_s)^{\frac{1}{n}}$$

où $f(x_\circ) = \max_{x\in M} f(x)$ et η est une fonction de classe C^∞ égale à 1 sur $B(x_\circ, \delta)$ et 0 sur $M - B(x_\circ, 2\delta)$ et $\rho = d(x_\circ, .)$ désigne la distance géodésique au point $x_\circ$.

3.5.1 Application aux variétés riemanniennes compactes de dimensions $n > 6$

Théorème 3.5 *Soit (M, g) une variété riemannienne compacte de dimension $n > 6$, si en un point $x_\circ$ où f atteint son maximum, la condition*

$$\left(\frac{n\,(n^2+4n-20)}{(n-2)(n-4)(n-6)\,(1+\|b\|_s+\|a\|_r)^{\frac{4}{n}}}-\frac{n-2}{(n-1)}\right)S_g(x_\circ)-\frac{3\Delta f(x_\circ)}{f(x_\circ)}>0$$

est vérifiée, alors l'équation (3.1) admet une solution u non triviale vérifiant

$$J_\lambda\,(u)<\frac{2}{nK_\circ^{\frac{n}{4}}\,(f(x_\circ))^{\frac{n}{4}-1}}$$

Preuve: Nous reprenons les caculs qui ont été faits dans [7]

$$\int_M f(x)\,|u_\epsilon(x)|^N\,dv(g)=\frac{2\theta^{-n}}{nK_\circ^{\frac{n}{4}}\,(f(x_\circ))^{\frac{n-4}{4}}}\left(\frac{n}{2}-(\frac{n\Delta f(x_\circ)}{4(n-2)f(x_\circ)}+\frac{nS_g(x_\circ)}{12(n-2)})\epsilon^2+o(\epsilon^2)\right)$$

Maintenant

$$\left|\frac{\partial u_\epsilon}{\partial r}\right|=|\nabla u_\epsilon|=\theta^{-2}(n-4)(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{8}}\frac{\rho}{((\frac{\rho}{\theta})^2+\epsilon^2)^{\frac{n-2}{2}}}$$

et donc

$$\int_M a(x)\,|\nabla u_\epsilon|^2\,dv(g)=\int_{B(x_\circ,\delta)}a(x)\,|\nabla u_\epsilon|^2\,dv(g)+\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)}a(x)\,|\nabla u_\epsilon|^2\,dv(g).$$

D'autre part $a\in L^r(M)$ avec $r>\frac{n}{2}$ donc

$$\int_M a(x)\,|\nabla u_\epsilon|^2\,dv(g)\le\|a\|_r\,\|\nabla u_\epsilon\|^2_{\frac{2r}{r-1}}\,.$$

On calcule à présent le terme

$$\left(\int_{B(x_\circ,\delta)} |\nabla u_\epsilon|^{\frac{2r}{r-1}}\, dv(g)\right)^{\frac{r-1}{r}} = \theta^{-4}(n-4)^2(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}}\times$$

$$\left(\int_0^\delta \frac{\rho^{\frac{2r}{r-1}+n-1}}{(\left(\frac{\theta}{\rho}\right)^2+\epsilon^2)^{\frac{(n-2)r}{r-1}}}\left(\int_{S(\rho)}\sqrt{|g(x)|}d\Omega\right)d\rho\right)^{\frac{r-1}{r}}.$$

Comme

$$\int_{S(\rho)}\sqrt{|g(x)|}d\Omega = \omega_{n-1}\left(1-\frac{S_g(x_\circ)}{6n}\rho^2+o(\rho^2)\right).$$

Alors

$$\left(\int_{B(x_\circ,\delta)} |\nabla u_\epsilon|^{\frac{2r}{r-1}}\, dv(g)\right)^{\frac{r-1}{r}} = (n-4)^2(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}}\times(\omega_{n-1})^{\frac{r-1}{r}}$$

$$\left(\int_0^\delta \frac{\rho^{\frac{2r}{r-1}+n-1}}{((\rho\theta)^2+\epsilon^2)^{\frac{(n-2)r}{r-1}}}d\rho\left(1-\frac{S_g(x_\circ)}{6n}\rho^2+o(\rho^2)\right)\right)^{\frac{r-1}{r}}$$

En faisant le changement de variable suivant

$$\left\langle x=(\frac{\rho\theta}{\epsilon})^2,\ d\rho=\frac{\epsilon dx}{2\theta\sqrt{x}}\ \text{ et }\ \rho=\frac{\epsilon}{\theta}\sqrt{x}\right\rangle.$$

On obtient

$$\int_{B(x_\circ,\delta)} a(x)\,|\nabla u_\epsilon|^2\, dv(g) \le \theta^{\frac{-nr}{r-1}}\,\|a\|_r\times\frac{(n-4)^2\times(\omega_{n-1})^{\frac{r-1}{r}}}{2^{\frac{r-1}{r}}}(\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}}\times\epsilon^{2-\frac{n}{r}}$$

$$\left(\left(I^{\frac{n-2}{2}+\frac{r}{r-1}}_{\frac{(n-2)r}{r-1}}\right)^{\frac{r-1}{r}}+o(\epsilon^2)\right).$$

On pose que

$$A=\frac{(n-4)^2\times[(n-4)n(n^2-4)]^{\frac{n-4}{4}}\times(\omega_{n-1})^{\frac{r-1}{r}}\times n\ K_\circ^{\frac{n}{4}}}{2^{\frac{r-1}{r}}}\times\left(I^{\frac{n-2}{2}+\frac{r}{r-1}}_{\frac{(n-2)r}{r-1}}\right)^{\frac{r-1}{r}}.$$

On obtient

$$\int_{B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g) \leq \frac{2}{n\ K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\theta^{-n\frac{r}{r-1}}(A\left\|a\right\|_r + o(\epsilon^2)) \times \epsilon^{2-\frac{n}{r}}.$$

Il nous reste à calculer l'intégrale $\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g)$.

Toutes les intégrales sont du type

$$\left|\int_{(\frac{\delta}{\epsilon})^2}^{(\frac{2\delta}{\epsilon})^2} \frac{x^q}{(x+1)^p}dx\right| \leq C\left(\frac{1}{\epsilon}\right)^{2(q-p+1)} = C\epsilon^{2(p-q-1)}$$

et comme $p-q=n-4\geq 3$, on obtient

$$\int_{(\frac{\delta}{\epsilon})^2}^{(\frac{2\delta}{\epsilon})^2} \frac{x^q}{(x+1)^p}dx = o(\epsilon^2)$$

et par conséquent

$$\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv(g) = o(\epsilon^2).$$

Alors,

$$\int_M a(x)\left|\nabla u_\epsilon\right|^2 dv(g) \leq \frac{2}{n\ K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\theta^{-n\frac{r}{r-1}}(A\left\|a\right\|_r + o(\epsilon^2)) \times \epsilon^{2-\frac{n}{r}}.$$

Maintenant on calcule

$$\int_M b(x)u_\epsilon^2 dv(g) = \int_{B(x_\circ,\delta)} b(x)u_\epsilon^2 dv(g) + \int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} b(x)u_\epsilon^2 dv(g).$$

D'autre part $b \in L^s(M)$ avec $s > \frac{n}{4}$ et donc

$$\int_M b(x)u_\epsilon^2 dv(g) \leq \left\|b\right\|_s \left\|u_\epsilon\right\|^2_{\frac{2s}{s-1}}.$$

Le premier terme devient

$$\|u_\epsilon\|^2_{\frac{2s}{s-1};B(x_\circ,\delta)} = (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}} \left(\int_0^\delta \frac{\rho^{n-1}}{((\rho\theta)^2+\epsilon^2)^{\frac{(n-4)s}{(s-1)}}} \left(\int_{S(\rho)} \sqrt{|g(x)|}d\Omega\right) d\rho\right)^{\frac{s-1}{s}}$$

et

$$\|u_\epsilon\|^2_{\frac{2s}{s-1};B(x_\circ,\delta)} = (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}} (\omega_{n-1})^{\frac{s-1}{s}} \times$$

$$\left(\int_0^\delta \frac{\rho^{n-1}}{((\rho\theta)^2+\epsilon^2)^{\frac{(n-4)s}{(s-1)}}} \left(1 - \frac{S_g(x_\circ)}{6n}\rho^2 + o(\rho^2)\right) d\rho\right)^{\frac{s-1}{s}}.$$

En faisant le changement de variable suivant

$$\left\langle x = (\frac{\rho\theta}{\epsilon})^2,\ d\rho = \frac{\epsilon dx}{2\theta\sqrt{x}} \quad \text{et} \quad \rho = \frac{\epsilon}{\theta}\sqrt{x}\right\rangle$$

on obtient

$$\|u_\epsilon\|^2_{\frac{2s}{s-1};B(x_\circ,\delta)} = (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}} \frac{(\omega_{n-1})^{\frac{s-1}{s}}}{\epsilon^{2(n-4)}} \times$$

$$\left(\frac{1}{2}\left(\frac{\epsilon}{\theta}\right)^n \int_0^{(\frac{\delta\theta}{\epsilon})^2} \frac{x^{\frac{n}{2}}}{(x+1)^{\frac{(n-4)s}{(s-1)}}}dx - \frac{S_g(x_\circ)}{12n}\left(\frac{\epsilon}{\theta}\right)^{n+2} \int_0^{(\frac{\delta\theta}{\epsilon})^2} \frac{x^{\frac{n}{2}+1}}{(x+1)^{\frac{(n-4)s}{(s-1)}}}dx + o(\epsilon^{n+2})\right)^{\frac{s-1}{s}}$$

Pour $\epsilon \to 0$ on a

$$\|u_\epsilon\|^2_{\frac{2s}{s-1};B(x_\circ,\delta)} = (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}} \frac{(\omega_{n-1})^{\frac{s-1}{s}}}{\epsilon^{2(n-4)}} \times (\frac{\epsilon^n}{2\theta^n})^{\frac{s-1}{s}}$$

$$\left(\int_0^{+\infty} \frac{x^{\frac{n}{2}}}{(x+1)^{\frac{(n-4)s}{(s-1)}}}dx - \frac{S_g(x_\circ)}{12n\theta^2}\epsilon^2 \int_0^{+\infty} \frac{x^{\frac{n}{2}+1}}{(x+1)^{\frac{(n-4)s}{(s-1)}}}dx + o(\epsilon^2)\right)^{\frac{s-1}{s}}.$$

On obteint

$$\|u_\epsilon\|^2_{\frac{2s}{s-1};B(x_\circ,\delta)} = (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)})^{\frac{n-4}{4}} \frac{(\omega_{n-1})^{\frac{s-1}{s}}}{\epsilon^{2(n-4)}} \times (\frac{\epsilon^n}{2\theta^n})^{\frac{s-1}{s}}$$

$$\left(\int_0^{+\infty}\frac{x^{\frac{n}{2}}}{(x+1)^{\frac{(n-4)s}{(s-1)}}}dx-\frac{S_g(x_\circ)}{12n\theta^2}\epsilon^2\int_0^{+\infty}\frac{x^{\frac{n}{2}+1}}{(x+1)^{\frac{(n-4)s}{(s-1)}}}dx+o(\epsilon^2)\right)^{\frac{s-1}{s}}$$

et alors

$$\|u_\epsilon\|^2_{\frac{2s}{s-1};B(x_\circ,\delta)}=(\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}}\times(\frac{\epsilon^n}{2\theta^n})^{\frac{s-1}{s}}\epsilon^{4-\frac{n}{s}}\times$$

$$\left[\left(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}}\right)^{\frac{s-1}{s}}-\frac{(s-1)S_g(x_\circ)}{12n\ s\theta^2}\left(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}}\right)^{-\frac{1}{s}}I^{\frac{n}{2}+1}_{\frac{(n-4)s}{(s-1)}}\epsilon^2+o(\epsilon^2)\right].$$

Comme le développement limité se fait jusqu'à l'ordre 2, on obtient

$$\int_M b(x)u_\epsilon^2dv(g)\leq\|b\|_s\,(\frac{(n-4)n(n^2-4)}{f(x_\circ)})^{\frac{n-4}{4}}\times(\frac{\omega_{n-1}}{2\theta^n})^{\frac{s-1}{s}}\epsilon^{4-\frac{n}{s}}\times$$

$$\left(\left(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}}\right)^{\frac{s-1}{s}}+o(\epsilon^2)\right).$$

On pose

$$B=\frac{n\ K_\circ^{\frac{n}{4}}\times((n-4)n(n^2-4))^{\frac{n-4}{4}}}{4}\times(\frac{\omega_{n-1}}{2\theta^n})^{\frac{s-1}{s}}\times\left(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}}\right)^{\frac{s-1}{s}}.$$

On obtient

$$\int_M b(x)u_\epsilon^2dv(g)\leq\frac{2}{n\ K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\theta^{-n\frac{s-1}{s}}(B\|b\|_s+o(\epsilon^2))\epsilon^{4-\frac{n}{s}}.$$

Pour le calcul de $\int_M(\Delta u_\epsilon)^2dv(g)$ on obtient

$$\frac{1}{2}\int_M(\Delta u_\epsilon)^2dv(g)=\frac{2\theta^{-n}}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(\frac{n}{4}-\frac{n^3+4n^2-20n}{24(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2+o(\epsilon^2)\right)$$

Récapitulant, nous avons

$$\frac{1}{2}\int_M\left((\Delta u_\epsilon)^2-a(x)\left|\nabla u_\epsilon\right|^2+b(x)u_\epsilon^2\right)dv(g)\leq\frac{2\theta^{-n}}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\times$$

$$\left(\frac{n}{4}+\|b\|_s B\theta^{\frac{-n}{r-1}}\epsilon^{4-\frac{n}{s}}+\|a\|_r A\theta^{\frac{-n}{s-1}}\epsilon^{2-\frac{n}{r}}-\frac{n^3+4n^2-20n}{24(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2+o(\epsilon^2)\right)$$

Tenant compte de l'expression de J_λ

$$J_\lambda(tu_\epsilon)=\frac{1}{2}t^2\|u_\epsilon\|^2-\frac{\lambda}{q}t^q\|u_\epsilon\|_q^q-\frac{1}{N}t^N\int_M f(x)\left|u_\epsilon(x)\right|^N dv(g)$$

où

$$\|u_\epsilon\|^2=\int_M\left|\Delta u_\epsilon\right|^2-a(x)\left|\nabla u_\epsilon\right|^2+b(x)u_\epsilon^2dv(g)$$

et $\lambda>0$, on obtient

$$J_\lambda(tu_\epsilon)\leq\frac{t^2}{2}\|u_\epsilon\|^2-\frac{t^N}{N}\int_M f(x)\left|u_\epsilon(x)\right|^N dv(g)$$

$$J_\lambda(tu_\epsilon)\leq\frac{2\theta^{-n}}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\times$$

$$\left\{\frac{t^2}{2}\left(\frac{n}{4}+\|b\|_s B\theta^{\frac{-n}{r-1}}\epsilon^{4-\frac{n}{s}}+\|a\|_r A\theta^{\frac{-n}{s-1}}\epsilon^{2-\frac{n}{r}}-\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2\right)-\right.$$

$$\left.\frac{t^N}{N}\left(1-\left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)}+\frac{S_g(x_\circ)}{6(n-1)}\right)\epsilon^2\right)+o(\epsilon^2)\right\}$$

Pour ϵ assez petit, nous avons

$$1+\|b\|_s B\theta^{\frac{-n}{r-1}}\epsilon^{4-\frac{n}{s}}+\|a\|_r A\theta^{\frac{-n}{s-1}}\epsilon^{2-\frac{n}{r}}\leq(1+\|a\|_r+\|b\|_s)^{\frac{4}{n}}$$

Et comme la fonction $\varphi(t)=\alpha\frac{t^2}{2}-\frac{t^N}{N}$ atteint son maximum au point $t^*=\alpha^{\frac{1}{N-2}}$ où

$$\varphi(t^*)=\frac{2}{n}\alpha^{\frac{n}{4}}$$

alors

$$J_\lambda(tu_\epsilon)\leq J_\lambda(t^*u_\epsilon)=\frac{2\theta^{-n}}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\times\left\{\left(1+\|b\|_s+\|a\|_r-\frac{(t^*)^2}{2}\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2\right)-\right.$$

$$\frac{(t^*)^N}{N}\left(1-\left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)}+\frac{S_g(x_\circ)}{6(n-1)}\right)\epsilon^2\right)+o(\epsilon^2)\Bigg\}.$$

Pour assurer

$$\sup_{t\geq 0} J_\lambda\left(tu_\epsilon\right) < \frac{2}{n\ K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}$$

on prend

$$\left(\frac{n\left(n^2+4n-20\right)\left(1+\|b\|_s+\|a\|_r\right)^{-\frac{4}{n}}}{3(n-2)(n-4)(n-6)}-\frac{n-2}{3(n-1)}\right)S_g(x_\circ)-\frac{\Delta f(x_\circ)}{f(x_\circ)}>0.$$

Ce qui achève la preuve. ■

3.5.2 Application aux variétés riemanniennes compactes de dimensions $n = 6$

Théorème 3.6 *Lorsque $n = 6$, s'il existe un point $x_\circ \in M$ où $S_g(x_\circ) > 0$ alors (3.1) admet une solution u non triviale.*

Preuve: Le developement de l'intégrale:

$$\int_M f(x)\left|u_\epsilon(x)\right|^N dv(g)=\frac{\theta^{-n}}{nK_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left[\frac{n}{2}-\left(\frac{n\Delta f(x_\circ)}{4(n-2)f(x_\circ)}+\frac{nS_g(x_\circ)}{12(n-2)}\right)\epsilon^2+o(\epsilon^2)\right]$$

De même pour

$$\int_M a(x)\left|\nabla u_\epsilon\right|^2 dv(g)\leq\frac{2}{nK_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\theta^{-n\frac{r}{r-1}}\left(A\left\|a\right\|_r+o(\epsilon^2)\right)\times\epsilon^{2-\frac{n}{r}}$$

et

$$\int_M b(x)u_\epsilon^2 dv(g)\leq\frac{2}{nK_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\theta^{-n\frac{s-1}{s}}\left(B\left\|b\right\|_s+o(\epsilon^2)\right)\epsilon^{4-\frac{n}{s}}.$$

Enfin

$$\int_M (\Delta u_\epsilon)^2 \, dv(g) = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \left(1 - \frac{(n-4)S_g(x_\circ)\theta^{-2}}{3n^2(n^2-4)I_n^{\frac{n}{2}-1}} \epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) + o(\epsilon^2)\right).$$

Tenant compte de l'expression de J_λ

$$J_\lambda(u_\epsilon) = \frac{1}{2}\|u_\epsilon\|^2 - \frac{\lambda}{q}\|u_\epsilon\|_q^q - \frac{1}{N}\int_M f(x)\,|u_\epsilon(x)|^N \, dv(g)$$

où

$$\|u_\epsilon\|^2 = \int_M |\Delta u_\epsilon|^2 - a(x)\,|\nabla u_\epsilon|^2 + b(x)u_\epsilon^2 dv(g)$$

et $\lambda > 0$, on trouve

$$J_\lambda(u_\epsilon) \leq \frac{1}{2}\|u_\epsilon\|^2 - \frac{1}{N}\int_M f(x)\,|u_\epsilon(x)|^N \, dv(g)$$

$$J_\lambda(u_\epsilon) \leq \frac{\theta^{-n}}{nK_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \times$$

$$\left[\frac{t^2}{2}\left(1 + \|a\|_r + \|b\|_s\right)^{1-\frac{4}{n}} - \frac{t^N}{2} - \frac{(n-4)S_g(x_\circ)}{12n(n^2-4)I_n^{\frac{n}{2}-1}} t^2\epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) + o(\epsilon^2)\right]$$

Pour assurer

$$\sup_{t\geq 0} J_\lambda(tu_\epsilon) < \frac{2}{nK_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}$$

Avec le même argument, on a

$$S_g(x_\circ) > 0$$

Ce qui achève la preuve. ■

Chapitre 4

Problème elliptique non-linéaire avec singularité au second membre

4.1 Introduction

Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$. Soient $a \in L^r(M)$, $b \in L^s(M)$ et $h \in L^d(M)$ où $r > \frac{n}{2}$, $s > \frac{n}{4}$ et $d > \frac{N}{N-q} := \alpha$, $2 < q < N$ et $\lambda > 0$ un paramètre real et f une fonction de classe C^∞ sur M strictement positive.

On considère l'équation suivante:

$$\Delta_g^2 u + div_g \left(a(x) \nabla_g u\right) + b(x) u = f(x) \left|u\right|^{N-2} u + \lambda h(x) \left|u\right|^{q-2} u \tag{4.1}$$

Pour tout $u \in H_2^2(M)$, on définit la fonctionnelle d'énergie J_λ, associée à l'équation (4.1), par:

$$J_\lambda(u) = \frac{1}{2} \int_M \left((\Delta_g u)^2 - a(x) \left|\nabla_g u\right|^2 + b(x) u^2 \right) dv(g) - \frac{\lambda}{q} \int_M h(x) \left|u\right|^q dv(g) - \frac{1}{N} \int_M f(x) \left|u\right|^N dv(g).$$

En remarquant que la fonctionnelle J_λ est de classe C^1 au sens de Fréchet et sa dérivée est donnée par:

$$\langle \nabla J_\lambda (u), v \rangle = \int_M (\Delta_g u.\Delta_g v - a(x)g(\nabla_g u, \nabla_g v) + b(x)uv)\, dv(g)$$

$$-\lambda \int_M h(x) |u|^{q-2} uvdv(g) - \int_M f(x) |u|^{N-2} uvdv(g),$$

Dans tout ce que suit on note par:

(h^1) $P_g : u \to P_g(u) := \Delta_g^2 u + div_g (a(x)\nabla_g u) + b(x)u$ est coercif $\Leftrightarrow \exists \Lambda > 0, \forall u \in H_2^2(M)$:

$$\langle P_g(u); u \rangle \geq \Lambda \|u\|^2_{H_2^2(M)}.$$

(h^2) La fonction h n'est pas nulle presque partout dans M.

(h^3) $\lambda \in (0, \lambda_1)$ où

$$\lambda_1 = \frac{q(N-2)}{2(N-q)} \Lambda^{\frac{q}{2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{-\frac{q}{2}} \|h\|_\alpha^{-1}.$$

Nous établissons les resultats suivants.

4.2 Etude de la géométrie de la fonctionnelle J_λ dans $H_2^2(M)$

Proposition 6

$$\|u\| = (\int_M u.P_g(u)dv(g))^{\frac{1}{2}}$$

est une norme équivalente à celle de $H_2^2(M)$ si et seulement si l'opérateur $P_g : u \to P_g(u) := \Delta_g^2 u + div_g (a(x)\nabla_g u) + b(x)u$ est coercif.

Preuve:

(Même démonstration que le lemme **2.1**). ■

Lemme 4.1 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$, alors* *la fonctionnelle J_λ verifie les conditions suivantes:*

1. *Il existe deux constantes $r, R > 0$ telle que $J_\lambda(u) \geq R > 0$ pour $\|u\| = r$.*

2. *Il existe $v \in H_2^2(M)$ on a $\|v\| > r$, et $J_\lambda(v) < 0$.*

Preuve:

1. La démonstration est basée sur le lemme du col

   Soit $u \in H_2^2(M)$ telle que$\|u\| = r > 0$; on obtient

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_M h(x)\,|u|^q\,dv(g) - \frac{1}{N}\int_M f(x)\,|u|^N\,dv(g)$$

   Par l'inégalité de Holder-Sobolev, on obtient

$$J_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\,\|h\|_\alpha\,\|u\|^q_{H_2^2(M)}$$

$$-\frac{1}{N}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}}\max_{x\in M}(f(x))\,\|u\|^N_{H_2^2(M)}.$$

   Comme l'opérateur $P_g(.)$ est coercif, il existe une constante $\Lambda > 0$, telle que:

$$J_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\Lambda^{-\frac{q}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\,\|h\|_\alpha\,\|u\|^q$$

$$-\frac{1}{N}\Lambda^{-\frac{N}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}}\max_{x\in M}(f(x))\,\|u\|^N$$

   En tenant compte de $\|u\| = r$, on obtient

$$J_\lambda(u) \geq r^2$$

$$\times\left(\frac{1}{2} - \lambda\frac{\Lambda^{-\frac{q}{2}}}{q}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\,\|h\|_\alpha\,r^{q-2} - \frac{\Lambda^{-\frac{N}{2}}}{N}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}}\max_{x\in M}(f(x))r^{N-2}\right)$$

   Alors, Il existe deux constantes r, $R > 0$, telle que pour tout $u \in H_2^2(M)$ et $\|u\| = r$, on a

$$J_\lambda(u) \geq R > 0.$$

2. Soit $t > 0$ and $u \in H_2^2(M) - \{0\}$; alors

$$J_\lambda(tu) = \frac{t^2}{2}\|u\|^2 - \frac{\lambda t^q}{q}\int_M h(x)\,|u|^q\,dv(g) - \frac{t^N}{N}\int_M f(x)\,|u|^N\,dv(g)$$

Comme $2 < q < N$, donne

$$J_\lambda(tu) \longrightarrow -\infty \quad \text{quand} \quad t \longrightarrow +\infty$$

Cela termine la démonstration de ce lemme.

■

En appliquant le lemme du col, on obtient qu'il existe une suite $(u_n)_n$ dans $H_2^2(M)$ telle que:

$$J(u_n) \longrightarrow c_\lambda \quad \text{et} \quad \nabla J(u_n) \longrightarrow 0 \quad \text{dans} \quad \left(H_2^2(M)\right)^*$$

et

$$\Gamma = \left\{\eta \in C^1\left([0;1]\,;\ H_2^2(M)\right) :\ \eta(0) = 0,\ \eta(1) = v\right\}$$

On pose

$$c_\lambda = \min_{\eta\in\Gamma}\max_{t\in[0,1]}\left(J_\lambda\left(\eta(t)\right)\right)$$

et

$$\lambda_1 = \frac{q(N-2)}{2(N-q)}\Lambda^{\frac{q}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{-\frac{q}{2}}\|h\|_\alpha^{-1}.$$

On a le lemme suivant:

Lemme 4.2 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$. On suppose que les assertions (h^1), (h^2) et (h^3) sont vérifiées, alors toute suite de Palais-Smale au niveau c_λ est bornée dans $H_2^2(M)$.*

Preuve: Soit $(u_m)_m$ dans $H_2^2(M)$ telle que:

$$J(u_m) \longrightarrow c_\lambda \text{ et } \nabla J(u_m) \longrightarrow 0 \text{ dans } \left(H_2^2(M)\right)^*$$

On obtient

$$J(u_m) - \frac{1}{N}\langle \nabla J(u_m), u_m \rangle = \left(\frac{1}{2} - \frac{1}{N}\right)\|u_m\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{N}\right)\int_M h(x)\,|u_m|^q\, dv(g)$$

Par les inégalités de Hölder et Sobolev, on obtient

$$J(u_m) - \frac{1}{N}\langle \nabla J(u_m), u_m \rangle = c_\lambda + o(1)$$

$$\geq \left(\frac{1}{2} - \frac{1}{N}\right)\|u_m\|^2 - \left(\frac{1}{q} - \frac{1}{N}\right)(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|h\|_\alpha \|u_m\|^q_{H_2^2(M)}$$

et en tenant compte de la coercitivité de l'opérateur P_g, il existe une constante $\Lambda > 0$ telle que:

$$c_\lambda + o(1) \geq \left(\frac{1}{2} - \frac{1}{N}\right)\|u_m\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{N}\right)\Lambda^{-\frac{q}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|h\|_\alpha \|u_m\|^q$$

On considère que $\|u_n\| \geq 1$, on obtient que

$$c_\lambda + o(1) \geq \left(\frac{N-2}{2} - \lambda\frac{N-q}{q}\Lambda^{-\frac{q}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|h\|_\alpha\right)\frac{\|u_m\|^q}{N}$$

Comme $0 < \lambda < \lambda_1 := \frac{N-2}{2N} - \lambda\frac{N-q}{qN}\Lambda^{-\frac{q}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|h\|_\alpha$,

$$\|u_m\| \leq \left(\frac{c_\lambda}{\frac{N-2}{2N} - \lambda\frac{N-q}{qN}\Lambda^{-\frac{q}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|h\|_\alpha}\right)^{\frac{1}{q}} + o(1)$$

Alors la suite $(u_m)_m$ est bornée dans $H_2^2(M)$. ■

Théorème 4.1 *Soit (M,g) une variété riemannienne compacte de dimension $n \geq 5$, on suppose que (h^1), (h^2) et (h^3) sont vérifiées et soit $(u_m)_m$ une suite de Palais-Smale au* *niveau c_λ.*

Alors, il existe une sous-suite notée encore $(u_m)_m$ convergente fortement dans $H_2^2(M)$.

Preuve:

(Même démonstration que le théorème **3.1**). ■

4.3 Application géométrique

On considère l'équation suivante :

$$\Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^\sigma}\nabla_g u\right) + \frac{b(x)}{\rho^\mu}u = f(x)\left|u\right|^{N-2}u + \lambda\frac{h(x)}{\rho^\beta}\left|u\right|^{q-2}u$$

Où a, b et h trois fonctions de classe $C^\infty(M)$ et $2 < q < N$ et $\lambda > 0$ un paremetre réel.

On considère sur $H_2^2(M)$ la fonctionnelle: J_λ: $H_2^2(M) \longrightarrow \mathbb{R}$, définie par:

$$J_\lambda(u) = \frac{1}{2}\int_M\left((\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma}\left|\nabla_g u\right|^2 + \frac{b(x)}{\rho^\mu}u^2\right)dv(g) - \frac{\lambda}{q}\int_M\frac{h(x)}{\rho^\beta}\left|u\right|^q dv(g) - \frac{1}{N}\int_M f(x)\left|u\right|^N dv(g)$$

Théorème 4.2 *Soient $0 < \sigma < \frac{n}{r} < 2$, $0 < \mu < \frac{n}{s} < 4$ et $0 < \beta < \frac{N}{d} < N - q$. On suppose que les assertions (h^1), (h^2) et (h^3) sont vérifiées et*

$$\sup_{u\in H_2^2(M)} J_\lambda^{\sigma,\mu,\beta}(u) < \frac{2}{nK_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}$$

Alors, l'equation possède une solution non triviale $u_{\sigma,\mu,\beta} \in H_2^2(M)$.

Preuve: On pose $\tilde{a}(x) := \frac{a(x)}{\rho^\sigma}$, $\tilde{b}(x) := \frac{b(x)}{\rho^\mu}$ et $\tilde{h}(x) := \frac{h(x)}{\rho^\beta}$, on remarque que $\tilde{a} \in L^r(M)$, $\tilde{b} \in L^s(M)$ et $\tilde{h} \in L^d(M)$ telle que $r > \frac{n}{2}$, $s > \frac{n}{4}$ et $d > \frac{N}{N-q}$.c.q.f.d ■

4.4 Le cas critique quand $\sigma = 2$, $\mu = 4$ et $\beta = \frac{n(2-q)}{2} + 2q$

Théorème 4.3 *Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$, on suppose que les assertions (h^1), (h^2) et (h^3) sont vérifiées.Soit $(u_m)_m := (u_{\sigma_m,\mu_m,\beta_m})_m$ une suite dans $H_2^2(M)$ telle que:*

$$\begin{cases} J_\lambda^{\sigma,\mu,\beta}(u_m) \to c_\lambda^{\sigma,\mu,\beta} \text{ pour tout } n \in \mathbb{N} \\ \nabla J_\lambda^{\sigma,\mu,\beta}(u_m) \to 0 \text{ faiblement dans } H_2^2(M) \end{cases}$$

et

$$\begin{cases} c_\lambda^{\sigma,\mu,\beta} < \frac{2}{nK_\circ^{\frac{n}{4}} (\max_{x \in M} f(x))^{\frac{n-4}{4}}} \\ \frac{1}{2} + a^- K^2(n,1,2) + b^- K^2(n,2,4) > 0 \end{cases}$$

Alors l'équation

$$\Delta_g^2 u + div_g \left(\frac{a(x)}{\rho^\sigma} \nabla_g u \right) + \frac{b(x)}{\rho^\mu} u = f(x) |u|^{N-2} u + \lambda \frac{h(x)}{\rho^\beta} |u|^{q-2} u$$

possède une solution non triviale $u_{\sigma,\mu,\beta} \in H_2^2(M)$.

Preuve: Soit $(u_m)_m \subset H_2^2(M)$, telle que:

$$J_\lambda^{\sigma,\mu,\beta}(u_m) = c_\lambda^{\sigma,\mu,\beta} + o(1) \text{ et } \nabla J_\lambda^{\sigma,\mu,\beta}(u_m) = o(1) \text{ dans } \left(H_2^2(M)\right)^*$$

On obtient:

$$J_\lambda^{\sigma,\mu,\beta}(u_m) - \frac{1}{N} \left\langle J_\lambda^{\sigma,\mu,\beta}(u_m), u_m \right\rangle = \left(\frac{1}{2} - \frac{1}{N} \right) \|u_m\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{N} \right) \int_M h(x) |u_m|^q \, dv(g)$$

Pat les inégalités de Hölder et Sobolev, on obtient

$$J_\lambda^{\sigma,\mu,\beta}(u_m) - \frac{1}{N} \left\langle \nabla J_\lambda^{\sigma,\mu,\beta}(u_m), u_m \right\rangle = c_\lambda^{\sigma,\mu,\beta} + o(1)$$

et

$$c_\lambda^{\sigma,\mu,\beta} + o(1) \geq \left(\frac{1}{2} - \frac{1}{N}\right) \|u_m\|^2 - \left(\frac{1}{q} - \frac{1}{N}\right) (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|h\|_\alpha \|u_m\|^q_{H_2^2(M)}$$

Comme (h^1) et (h^2) sont satisfaitent et $\|u_n\| \geq 1$, on a

$$\|u_m\| \leq \left[\left(\frac{N-2}{2} - \lambda \frac{N-q}{q} \Lambda^{-\frac{q}{2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|h\|_\alpha \right)^{-1} N c_\lambda^{\sigma,\mu,\beta} \right]^{\frac{1}{q}} + o(1)$$

Alors $(u_m)_m$ est bornée dans $H_2^2(M)$. ■

4.5 Fonctions tests

Pour vérifier l'hypothèse du théorème générique **4.2**, on considère les fonctions tests suivantes:

$$u_\epsilon(x) = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{8}} \frac{\eta(\rho)}{((\xi\theta)^2 + \epsilon^2)^{\frac{n-4}{2}}}$$

avec

$$\xi = (1 + \|a\|_r + \|b\|_s)^{\frac{1}{n}}$$

où $f(x_\circ) = \max_{x \in M} f(x)$ et η est une fonction de classe C^∞ égale à 1 sur $B(x_\circ, \delta)$ et 0 sur $M - B(x_\circ, 2\delta)$ et $\rho = d(x_\circ, .)$ désigne la distance géodésique au point $x_\circ$.

Pour normaliser les fonction, on considère la suite des fonctions suivates par

$$\varphi_\epsilon(t) := \frac{u_\epsilon(x)}{\|u_\epsilon\|_N} t \quad \text{pour} \quad t > 0$$

4.5.1 Application aux variétés riemanniennes compactes de dimensions $n > 6$

Theorem 7 *Soit (M, g) une variété riemannienne compacte de dimension $n > 6$. Soient $a \in L^r(M)$, $b \in L^s(M)$ et $h \in L^d(M)$, où $r > \frac{n}{2}$, $s > \frac{n}{4}$, $d > \frac{N}{N-q}$ et $2 < q < N$. On* *suppose que les assertions (h^1), (h^2) et (h^3) sont vérifiées et en un point $x_\circ$ où f atteint son maximum, la condition*

$$\left(\frac{n(n-2\sqrt{6}+2)(n+2\sqrt{6}+2)-(n-6)(n-4)^3(n+2)}{12(n^2-4)(n-4)^2(n-6)\left(1+\|a\|_r+\|b\|_s\right)^{\frac{4}{n}}}S_g(x_\circ)-\frac{(n-4)\Delta f(x_\circ)}{8(n-2)f(x_\circ)}\right)>0$$

est vérifiée, alors l'équation (4.1) admet une solution u non triviale vérifiant

$$J_\lambda(u)<\frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n}{4}-1}}$$

Preuve: Les mêmes calculs que la section précente nous permettons d'écrire:

$$J_\lambda(\varphi_\epsilon(t))\leq\frac{2}{nK_\circ^{\frac{n}{4}}\,(f(x_\circ))^{\frac{n-4}{4}}}\times$$

$$\left[1-\left(\frac{n(n-2\sqrt{6}+2)(n+2\sqrt{6}+2)-(n-6)(n-4)^3(n+2)}{12(n^2-4)(n-4)^2(n-6)\left(1+\|a\|_r+\|b\|_s\right)^{\frac{4}{n}}}S_g(x_\circ)-\frac{(n-4)\Delta f(x_\circ)}{8(n-2)f(x_\circ)}\right)\epsilon^2+o(\epsilon^2)\right].$$

Ce qui achève la preuve. ■

4.5.2 Application aux variétés riemanniennes compactes de dimensions $n=6$

Théorème 4.4 *Lorsque $n=6$, s'il existe un point $x_\circ\in M$ où $S_g(x_\circ)>0$ alors (4.1) admet une solution u non triviale.*

Preuve: La même manière que la section précédente, on obtient:

$$J_\lambda(\varphi_\epsilon(t))\leq\frac{\theta^{-n}}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\times$$

$$\left[\frac{t^2}{2}\left(1+\|a\|_r+\|b\|_s\right)^{1-\frac{4}{n}}-\frac{t^N}{2}-\frac{(n-4)S_g(x_\circ)}{12n(n^2-4)I_n^{\frac{n}{2}-1}}t^2\epsilon^2\log\left(\frac{1}{\epsilon^2}\right)+o(\epsilon^2)\right]$$

Pour assurer

$$\sup_{t\geq 0} J_\lambda\left(tu_\epsilon\right) < \frac{2}{n\ K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}$$

On doit prendre

$$S_g(x_\circ) > 0$$

Ce qui achève la preuve. ■

Chapitre 5

Multiplicité des solutions du problème de type Q-courbure singulière

5.1 Introduction

Soit (M, g) une variété riemannienne compacte de dimension $n \geq 5$. Soient a, b et f trois fonctions de classe C^∞ sur M avec f strictement positive , $0 < \sigma < 2$ et $0 < \beta < 4$.
On considère l'équation suivante:

$$\Delta_g^2 u + div_g \left(\frac{a(x)}{\rho^\sigma} \nabla_g u \right) + \frac{b(x)}{\rho^\beta} u = f(x) |u|^{N-2} u + \lambda |u|^{q-2} u \tag{5.1}$$

Pour tout $u \in H_2^2(M)$, on définit la fonctionnelle d'énergie J_λ, associée à l'équation (5.1), par:

$$J_\lambda(u) = \frac{1}{2} \int_M \left((\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\beta} u^2 \right) dv(g) - \frac{1}{N} \int_M f(x) |u|^N dv(g) - \frac{\lambda}{q} \int_M |u|^q dv(g)$$

On sait que les solutions de l'équation (5.1) sont des points critiques de la fonctionnelle J_λ.

En remarquant que la fonctionnelle J_λ n'est pas bornée inferieurement sur $H_2^2(M)$, par contre elle l'est sur une particulière variété dite de Nehari.

La variété de Nehari est définie par:

$$N_\lambda = \left\{ u \in H_2^2(M) \backslash \{0\} : \ \ \langle \nabla J_\lambda(u), \ u \rangle = 0 \right\}$$

Comme la fonctionnelle d'énergie J_λ est de classe C^1 au sens de Fréchet et sa dérivé est donnée par:

$$\Phi_\lambda(u) := \langle \nabla J_\lambda(u), \ u \rangle$$

$$\Phi_\lambda(u) = \int_M \left((\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\beta} u^2 \right) dv(g) - \lambda \int_M |u|^q \, dv(g) - \int_M f(x) |u|^N \, dv(g)$$

et aussi

$$\langle \nabla \Phi_\lambda(u), u \rangle = 2 \int_M \left((\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\beta} u^2 \right) dv(g) - \lambda q \int_M |u|^q \, dv(g) - N \int_M f(x) |u|^N \, dv(g)$$

Donc pour tout $u \in N_\lambda$, on obtient:

$$J_\lambda(u) = \frac{N-2}{2N} \|u\|^2 + \frac{\lambda (q-N)}{Nq} \int_M |u|^q \, dv(g).$$

On considère le problème de minimisation suivant :

$$\alpha_\lambda := \inf_{u \in N_\lambda} J_\lambda(u)$$

Nous établissons les résultats suivants.

5.2 Etude de la fonctionnelle J_λ sur N_λ

Lemme 5.1 *Pour tout* $\lambda \in (0,\ \lambda_\circ)$, *la fonctionnelle* J_λ *est bornée inferieurement et coercive sur* N_λ *où*

$$\lambda_\circ = \frac{(N-2)\, q\Lambda^{\frac{q}{2}}}{2\,(N-q)\, V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}}$$

avec $1 < q < 2$.

Preuve: Soit $u \in N_\lambda$, en utilisant (5.1) et l'inégalité de Sobolev, on obtient

$$J_\lambda(u) \geq \frac{N-2}{2N} \|u\|^2 - \lambda \frac{N-q}{Nq} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q_{H_2^2(M)}$$

Par la coercitivité de l'opérateur P_g, il existe une constante $\Lambda > 0$, telle que:

$$J_\lambda(u) \geq \frac{N-2}{2N} \|u\|^2 - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q$$

On considère les deux cas suivants:

Cas (i), si $u \in N_\lambda$ et $\|u\| \geq 1$.on obtient

$$J_\lambda(u) \geq \left[\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\right] \|u\|^q$$

et puisque

$$0 < \lambda < \frac{(N-2)\, q\Lambda^{\frac{q}{2}}}{2\,(N-q)\, V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}} := \lambda_\circ$$

dans ce cas là, on obtient que $u \in N_\lambda$:

$$J_\lambda(u) > 0$$

Cas (ii), si $u \in N_\lambda$ et $\|u\| < 1$,on obtient

$$J_\lambda(u) \geq \frac{N-2}{2N}\|u\|^2 - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}$$

Donc

$$J_\lambda(u) > -\lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}$$

Ainsi, J_λ est coercive et bornée inferieurement sur N_λ . ■

On considère la partition suivante:

$$\begin{cases} N_\lambda^+ = \{u \in N_\lambda : \ \langle \nabla\Phi_\lambda(u),\ u\rangle > 0\} \\ N_\lambda^- = \{u \in N_\lambda : \ \langle \nabla\Phi_\lambda(u),\ u\rangle < 0\} \\ N_\lambda^0 = \{u \in N_\lambda : \ \langle \nabla\Phi_\lambda(u),\ u\rangle = 0\} \end{cases}$$

Lemme 5.2 *Pour tout $\lambda \in (0,\ \lambda_\circ)$, si v est un minimum local de $J_\lambda$sur N_λ et $v \notin N_\lambda^0$. Alors, $\nabla J_\lambda(v) = 0$ sur $(H_2^2(M))^*$.*

Preuve: Si v est un minimum local de J_λ sur N_λ, c'est-à-dire

$$J_\lambda(v) = \min_{u \in N_\lambda} J_\lambda(u)$$

Par le théorème des multiplicateurs de Lagrange multipliers, il existe une constante $\mu \in \mathbb{R}$ telle que pour tout $\varphi \in N_\lambda$:

$$\langle \nabla J_\lambda(v), \varphi\rangle = \mu \langle \nabla\Phi_\lambda(v), \varphi\rangle = 0$$

Si $\mu = 0$, alors le lemme est prouvé. Sinon, on prend $\varphi = v$ et comme $v \in N_\lambda$:

$$\langle \nabla J_\lambda(v), v\rangle = \mu \langle \nabla\Phi_\lambda(v), v\rangle = 0$$

Alors,

$$\langle \nabla \Phi_\lambda(v), v \rangle = 0$$

Contraduction avec $v \notin N_\lambda^0$ ■

On a les lemmes suivants

Lemme 5.3 *Pour tout* $\lambda \in (0,\, \lambda_\circ)$, *on a* $N_\lambda^0 = \emptyset$.

Preuve: Raisonnons par l'absurde,on suppose que: $N_\lambda^0 \neq \emptyset$ pour tout$\lambda \in (0,\, \lambda_\circ)$.
Si $u \in N_\lambda^0$, on a:

$$\langle \nabla \Phi_\lambda(u), u \rangle = 2\left\| u \right\|^2 - \lambda q \left\| u \right\|_q^q - N \int_M f(x) \left| u \right|^N dv(g) = 0 \tag{5.2}$$

et

$$\Phi_\lambda(u) = \left\| u \right\|^2 - \lambda \left\| u \right\|_q^q - \int_M f(x) \left| u \right|^N dv(g) = 0 \tag{5.3}$$

Alors,

$$\left\| u \right\|^2 = \frac{N-q}{2-q} \int_M f(x) \left| u \right|^N dv(g) \tag{5.4}$$

(5.2), (5.3) et (5.4), donnent:

$$\lambda \left\| u \right\|_q^q = \frac{N-2}{2-q} \int_M f(x) \left| u \right|^N dv(g) \tag{5.5}$$

En utilisant l'inégalité de Sobolev et par la coercitivité de l'opérateur P_g, il existe une constante $\Lambda > 0$, telle que:

$$\int_M f(x) \left| u \right|^N dv(g) \leq \Lambda^{-\frac{N}{2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \max_{x \in M} f(x) \left\| u \right\|^N$$

Combinant (5.4) et (5.5), on trouve

$$\left\| u \right\| \geq \left[\frac{(N-q)\, \Lambda^{-\frac{N}{2}} \left((\max((1+\varepsilon)K_\circ, A_\varepsilon)\right)^{\frac{N}{2}} \max_{x \in M} f(x)}{(2-q)} \right]^{\frac{1}{2-N}}$$

On se donne la fonctionnelle $I_\lambda : N_\lambda \to \mathbb{R}$ définit par

$$I_\lambda(u) = \left[\left(\frac{N-q}{2-q}\right)^{\frac{q}{2}}\left(\frac{2-q}{N-2}\right)\right]^{\frac{2}{2-q}}\left(\frac{\|u\|^q}{\lambda\,\|u\|_q^q}\right)^{\frac{2}{q-2}} - \frac{1}{N}\int_M f(x)\,|u|^N\,dv(g)$$

Si $u \in N_\lambda^0$, alors

$$I_\lambda(u) = \left[\left(\frac{N-q}{2-q}\right)^{\frac{q}{2}}\left(\frac{2-q}{N-2}\right)\right]^{\frac{2}{2-q}}\left[\frac{\left(\frac{N-q}{2-q}\int_M f(x)\,|u|^N\,dv(g)\right)^{\frac{q}{2}}}{\frac{N-2}{2-q}\int_M f(x)\,|u|^N\,dv(g)}\right]^{\frac{2}{q-2}} - \frac{1}{N}\int_M f(x)\,|u|^N\,dv(g).$$

$$I_\lambda(u) = 0 \tag{5.6}$$

On pose

$$\theta = \left[\left(\frac{N-q}{2-q}\right)^{\frac{q}{2}}\left(\frac{2-q}{N-2}\right)\right]^{\frac{2}{2-q}}$$

En utilisant l'inégalité de Sobolev et par la coercitivité de l'opérateur P_g, il existe une constante $\Lambda > 0$, telle que:

$$I_\lambda(u) \geq \theta\left(\frac{\|u\|^q}{\lambda^{\frac{N-q}{Nq}}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\,\|u\|^q}\right)^{\frac{2}{q-2}} -$$

$$\frac{\Lambda^{-\frac{N}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}}\max_{x\in M} f(x)}{N}\,\|u\|^N.$$

Donc,

$$I_\lambda(u) \geq \left(\frac{\Lambda^{\frac{q}{2}}\left(\frac{N-q}{2-q}\right)^{\frac{q}{2}}\left(\frac{2-q}{N-2}\right)\left(\frac{Nq}{N-q}\right)}{\lambda V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}}\right)^{\frac{2}{q-2}} -$$

$$\frac{1}{N}\left(\left(\frac{N-q}{2-q}\right)\Lambda^{-\frac{N}{2}}\left((\max((1+\varepsilon)K_\circ, A_\varepsilon)\right)^{\frac{N}{2}}\max_{x\in M} f(x)\right)^{\frac{2}{2-N}}.$$

Ce qui implique que pour λ suffisamant petit, on obtient $I_\lambda(u) > 0$ pour tout $u \in N_\lambda^0$. On obtient une contradiction avec (5.6). Alors, on déduit qu'il existe $\lambda_\circ > 0$, tq pour tout $\lambda \in (0,\, \lambda_\circ)$, on a $N_\lambda^0 = \emptyset$. ■

D'après le lemme **5.3**, on a $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ pour tout $0 < \lambda < \lambda_\circ$.

Maintenant on définit:

$$\alpha_\lambda = \inf_{u \in N_\lambda} J_\lambda(u),\ \alpha_\lambda^+ = \inf_{u \in N_\lambda^+} J_\lambda(u) \text{ et } \alpha_\lambda^- = \inf_{u \in N_\lambda^-} J_\lambda(u)$$

Pour tout $\lambda \in (0, \lambda^* = \min(\lambda_\circ; \lambda_1))$, alors $N_\lambda^+ \neq \emptyset$ et $N_\lambda^- \neq \emptyset$ où

$$\lambda_1 := \frac{\frac{N-2}{N-q}\left[\frac{\Lambda}{\max((1+\varepsilon)K_\circ, A_\varepsilon)}\right]^{\frac{N-q}{N-2}}}{V(M)^{1-\frac{q}{N}}\left(\frac{(N-q)}{(2-q)}\max_{x \in M} f(x)\right)^{\frac{2-q}{N-2}}}$$

Lemme 5.4 *Si* $0 < \lambda < \lambda^*$, *on a*

1. $\alpha_\lambda^+ = \inf_{u \in N_\lambda^+} J_\lambda(u) < 0.$

2. $\alpha_\lambda \leq \alpha_\lambda^+ < 0.$

Preuve:

1. Soit $u \in N_\lambda^+$, on a

$$J_\lambda(u) = \frac{N-2}{2N}\|u\|^2 - \frac{\lambda(N-q)}{Nq}\|u\|_q^q$$

et

$$\langle \nabla\Phi_\lambda(u), u\rangle = 2\|u\|^2 - \lambda q\|u\|_q^q - N\int_M f(x)\,|u|^N\, dv(g) > 0$$

On obtient,

$$J_\lambda(u) \leq \frac{\lambda(N-q)}{2N}\|u\|_q^q - \frac{\lambda(N-q)}{Nq}\|u\|_q^q$$

Alors,

$$J_\lambda(u) \leq \frac{\lambda(N-q)}{N}\left(\frac{1}{2} - \frac{1}{q}\right)\|u\|_q^q < 0$$

D'où,

$$J_\lambda(u) < 0$$

i.e.,

$$\inf_{u\in N_\lambda^+} J_\lambda(u) < 0.$$

2. Soit $u \in N_\lambda^+$, on obtient

$$J_\lambda(u) = \frac{N-2}{2N}\|u\|^2 - \frac{\lambda(N-q)}{Nq}\|u\|_q^q$$

$$J_\lambda(u) < \frac{N-2}{N}\left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 < 0$$

On conclut que $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

■

Lemme 5.5 *Si* $0 < \lambda < \lambda^*$, *alors*

$$\alpha_\lambda^- = \inf_{u\in N_\lambda^-} J_\lambda(u) > 0$$

Preuve: On considère $u \in N_\lambda^-$, on obtient:

$$J_\lambda(u) = \frac{N-2}{2N}\|u\|^2 - \frac{\lambda(N-q)}{Nq}\|u\|_q^q$$

et aussi

$$\langle \nabla\Phi_\lambda(u), u\rangle = 2\|u\|^2 - \lambda q\|u\|_q^q - N\int_M f(x)|u|^N dv(g) < 0 \tag{5.7}$$

On a,

$$\|u\|^2 > \frac{\lambda(N-q)}{(N-2)}\|u\|_q^q \tag{5.8}$$

En utilisant l'inégalité de Sobolev et par la coercitivité de l'opérateur P_g, il existe une constante $\Lambda > 0$, telle que:

$$J_\lambda(u) \geq \frac{N-2}{2N}\|u\|^2 - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|u\|^q$$

On a les deux cas suivants:

Cas (i), si $u \in N_\lambda$ et $\|u\| \geq 1$, on a

$$J_\lambda(u) \geq \left[\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\right]\|u\|^q$$

comme

$$0 < \lambda < \frac{(N-2)\,q\Lambda^{\frac{q}{2}}}{2\,(N-q)\,V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}} := \lambda_\circ$$

on obtient

$$J_\lambda(u) > 0.$$

Cas (ii), si $u \in N_\lambda$ et $\|u\| < 1$,

En utilisant l'inégalité de Sobolev et par la coercitivité de l'opérateur P_g, il existe une constante $\Lambda > 0$, on obtient

$$0 < \xi \leq \|u\| < 1$$

telle que

$$\xi = \left[\frac{(2-q)\,\Lambda^{\frac{N}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{-\frac{N}{2}}}{(N-q)\max_{x\in M} f(x)}\right]^{\frac{1}{N-2}}$$

Alors,

$$J_\lambda(u) \geq \frac{N-2}{2N}\xi^2 - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}$$

Si on prend

$$0 < \lambda < \frac{\frac{(N-2)}{2(N-q)}\xi^2}{\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}} := \lambda_2$$

Alors, pour tout $0 < \lambda < \min(\lambda_0; \lambda_1; \lambda_2)$, on a

$$J_\lambda(u) > 0$$

i.e.,

$$\inf_{u \in N_\lambda^-} J_\lambda(u) > 0$$

■

5.3 Existence des points critiques de J_λ sur N_λ^+ et N_λ^-

Théorème 5.1 *Soit $(u_m^+)_m \subset N_\lambda^+$ une suite minimisante telle que*

$$\begin{cases} J_\lambda(u_m^+) = \alpha_\lambda^+ + o(1) \\ \nabla J_\lambda(u_m^+) = o(1), \;\; dans \;\; (H_2^2(M))^* \end{cases}$$

et

$$\left|\alpha_\lambda^+\right| < \frac{2}{nK_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}}$$

Alors pour tout $\lambda \in (0,\ \lambda^*)$, $(u_m^+)_m$ une sous-suite qui converge fortement vers un point $u^+ \in N_\lambda^+$ dans $H_2^2(M)$, en plus $J_\lambda(u^+) = \alpha_\lambda^+ < 0$ et u^+ est une solution non-triviale de l'équation (5.1).

Preuve: Soit $(u_m^+)_m \subset N_\lambda^+$ une suite de Palais-Smale de J_λ telle que

$$J_\lambda(u_m^+) = \alpha_\lambda^+ + o(1) \text{ et } \nabla J_\lambda(u_m) = o(1), \text{ dans } \left(H_2^2(M)\right)^*$$

On obtient,

$$-\alpha_\lambda^+ + o(1) \le J_\lambda(u_m^+) - \frac{1}{q}\left\langle \nabla J_\lambda(u_m^+); u_m^+ \right\rangle \le \alpha_\lambda^+ + o(1).$$

Alors,

$$-\alpha_\lambda^+ + o(1) \leq -\frac{(N-2)(N-q)}{2Nq} \left\| u_m^+ \right\|^2 \leq \alpha_\lambda^+ + o(1)$$

et par conséquent:

$$\left\| u_m^+ \right\|^2 \leq \frac{2Nq}{(N-2)(N-q)} \alpha_\lambda^+ + o(1)$$

Donc $(u_m^+)_m$ est bornée dans $H_2^2(M)$. La réflexivité de l'espace $H_2^2(M)$ et la compacité de l'inclusion $H_2^2(M) \subset H_p^k(M)$ ($k = 0, 1;\ p < N$), implique qu'il existe une sous-suite notée $(u_m^+)_m$ telle que :

(1). $u_m^+ \to u^+$ faiblement dans $H_2^2(M)$.

(2). $u_m^+ \to u^+$ fortement dans $L^p(M)$ pour $1 < p < N = \frac{2n}{n-4}$.

(3). $\nabla u_m^+ \to \nabla u^+$ fortement dans $L^q(M)$ pour $1 < q < 2^* = \frac{2n}{n-2}$.

(4). $u_m^+ \to u^+$ p.p dans M.

On applique le théorème **1.7,** on obtient

(5). $u_m^+ \to u^+$ fortement dans $L^2(M, \rho^{-\mu})$ pour $0 < \mu < 4$.

(6). $u_m^+ \to u^+$ fortement dans $L^2(M, \rho^{-\sigma})$ pour $0 < \sigma < 2$.

· On va montrer que u^+ est une solution faible non triviale de l'équation (5.1).

Par l'inégalité de Sobolev, il existe une constante $c > 0$ telle que:

$$\left\| \left| u_m^+ \right|^{N-2} u_m^+ \right\|_{\frac{N}{N-1}} = \left\| u_m^+ \right\|_N^{N-1} \leq c \left\| u_m^+ \right\|_{H_2^2(M)}^{N-1} < +\infty$$

et

$$\left\| \left| u_m^+ \right|^{q-2} u_m^+ \right\|_{\frac{N}{q-1}} = \left\| u_m^+ \right\|_N^{q-1} < +\infty$$

et comme $u^+ \in H_2^2(M) \subset L^N(M) \subset L^{\frac{N}{N+1-q}}(M)$, on a pour toute $\varphi \in H_2^2(M)$:

$$\int_M \left| u_m^+ \right|^{q-2} u_m^+ . \varphi dv(g) = \int_M \left| u^+ \right|^{q-2} u^+ . \varphi dv(g) + o(1)$$

On a

$$\left\langle \nabla J_\lambda(u_m^+), \varphi \right\rangle = \int_M \left(\Delta_g^2 u_m^+ + div_g \left(a(x) \nabla_g u_m^+ \right) + b(x) u_m^+ \right) \varphi dv(g) -$$

$$\left(\lambda \int_M \left|u_m^+\right|^{q-2} u_m^+ \varphi dv(g) + \int_M f(x) \left|u_m^+\right|^{N-2} u_m^+ \varphi dv(g)\right)$$

et on trouve pour tout $\varphi \in H_2^2(M)$:

$$\left\langle \nabla J_\lambda(u_m^+), \varphi \right\rangle = \left\langle \nabla J_\lambda(u^+), \varphi \right\rangle + o(1)$$

Quand $m \to +\infty$, on trouve:

$$\left\langle \nabla J_\lambda(u^+), \varphi \right\rangle = 0.$$

Comme on a montré que $u^+ \in N_\lambda$, on va montrer que $u^+ \in N_\lambda^+$.

On suppose que $u^+ \in N_\lambda^-$, on a:

$$\left\langle \nabla J_\lambda(u^+), u^+ \right\rangle = 0 \text{ et } \left\langle \nabla \Phi_\lambda(u^+), u^+ \right\rangle < 0$$

On obtient

$$J_\lambda(u^+) > 0$$

Contradiction avec le lemme **5.5**.

On obtient

$$J_\lambda(u^+) = \alpha_\lambda^+ < 0.$$

et u^+ est une solution faible de notre équation (5.1).

On pose

$$w_m := u_m^+ - u^+$$

Alors la suite $(w_m)_m \to 0$ faiblement dans $H_2^2(M)$.

On applique le lemme de Brézis-Lieb à la suite $(w_m)_m$, on trouve

$$\left\|\Delta_g u_m^+\right\|_2^2 - \left\|\Delta_g u^+\right\|_2^2 = \left\|\Delta_g w_m\right\|_2^2 + o(1) \tag{5.9}$$

et

$$\int_M f(x) \left(\left|u_m^+\right|^N - \left|u^+\right|^N\right) dv(g) = \int_M f(x) \left|w_m\right|^N dv(g) + o(1) \tag{5.10}$$

En utilisant les relations (5.9) et (5.10) et le fait que $u_m^+ \to u^+$ fortement dans $L^2(M, \rho^{-\mu})$ et $u_m^+ \to u^+$ fortement dans $L^2(M, \rho^{-\sigma})$, on obtient

$$J_\lambda(u_m^+) = J_\lambda(u_m^+) - J_\lambda(u^+) + J_\lambda(u^+)$$

$$= \frac{1}{2}\left\|\Delta_g\left(u_m^+ - u^+\right)\right\|_2^2 - \frac{1}{N}\int_M f(x)\left|u_m^+ - u^+\right|^N dv(g) + J_\lambda(u^+) + o(1)$$

Comme $u_m^+ - u^+ \to 0$ faiblement dans $H_2^2(M)$, on teste par $\nabla J_\lambda(u_m^+) - \nabla J_\lambda(u^+)$

$$\left\langle \nabla J_\lambda(u_m^+) - \nabla J_\lambda(u^+), u_m^+ - u^+\right\rangle = o(1)$$

$$= \left\|\Delta_g\left(u_m^+ - u^+\right)\right\|_2^2 - \int_M f(x)\left|u_m^+ - u^+\right|^N dv(g) = o(1) \tag{5.11}$$

Alors,

$$\left\|\Delta_g\left(u_m^+ - u^+\right)\right\|_2^2 = \int_M f(x)\left|u_m^+ - u^+\right|^N dv(g) + o(1)$$

Comme u^+ est une solution faible de notre équation (5.1), on trouve

$$J_\lambda(u_m^+) = \left(\frac{1}{2} - \frac{1}{N}\right)\int_M f(x)\left|u_m^+ - u^+\right|^N dv(g) +$$

$$\left(\frac{1}{2} - \frac{1}{N}\right)\int_M f(x)\left|u^+\right|^N dv(g) + \lambda\left(\frac{1}{2} - \frac{1}{q}\right)\left\|u^+\right\|_q^q + o(1)$$

En appliquant le lemme de Brézis-Lieb et comme $1 < q < 2$, on obtient

$$J_\lambda(u_m^+) \leq \left(\frac{1}{2} - \frac{1}{N}\right)\int_M f(x)\left|u_m^+\right|^N dv(g) + o(1)$$

$$= \frac{2}{n}\int_M f(x)\left|u_m^+\right|^N dv(g) + o(1)$$

Comme

$$\left|\alpha_\lambda^+\right| < \frac{2}{n}\limsup_{m\to+\infty}\int_M f(x)\left|u_m^+\right|^N dv(g)$$

On écrit

$$\left|u_m^+\right|^N - \left|u_m^+\right|^{N-2}(u_m^+ - u^+)^2 = 2\left|u_m^+\right|^{N-1}u^+ - \left|u_m^+\right|^{N-2}\left|u^+\right|^2$$

Comme $u_m^+ \to u^+$ faiblement dans $H_2^2(M)$ et $0 < N-2 < N$, on a

$$\int_M f(x)\left|u_m^+\right|^{N-2}\left|u^+\right|^2 dv(g) = \int_M f(x)\left|u^+\right|^N dv(g) + o(1) \tag{5.12}$$

En combinant les relations (5.11) et (5.12), on trouve

$$\int_M f(x)\left(\left|u_m^+\right|^N - \left|u^+\right|^N\right)dv(g) = \int_M f(x)\left|u_m^+\right|^{N-2}(u_m^+ - u^+)^2 dv(g) + o(1)$$

$$\leq \left(\int_M f(x)\left|u_m^+\right|^N dv(g)\right)^{1-\frac{2}{N}}\left(\int_M f(x)\left|u_m^+ - u^+\right|^N dv(g)\right)^{\frac{2}{N}}$$

De et en utilisant l'inégalité de Sobolev, on obtient

$$\left\|\Delta_g\left(u_m^+ - u^+\right)\right\|_2^2 \leq \left(\int_M f(x)\left|u_m^+\right|^N dv(g)\right)^{1-\frac{2}{N}}\left(\int_M f(x)\left|u_m^+ - u^+\right|^N dv(g)\right)^{\frac{2}{N}}$$

$$\leq \left(\int_M f(x)\left|u_m^+\right|^N dv(g)\right)^{1-\frac{2}{N}}\left(\max_{x\in\in M} f(x)\right)^{\frac{2}{N}}\left\|u_m^+ - u^+\right\|_N^2$$

$$\leq (1+\varepsilon)K_\circ\left(\max_{x\in\in M} f(x)\right)^{\frac{2}{N}}\left(\int_M f(x)\left|u_m^+\right|^N dv(g)\right)^{1-\frac{2}{N}}\left\|\Delta_g(u_m^+ - u^+)\right\|_2^2 + o(1)$$

Donc

$$\left\|\Delta_g\left(u_m^+ - u^+\right)\right\|_2^2\left(1 - (1+\varepsilon)K_\circ\left(\max_{x\in\in M} f(x)\right)^{\frac{2}{N}}\left(\int_M f(x)\left|u_m^+\right|^N dv(g)\right)^{1-\frac{2}{N}}\right) \leq o(1)$$

Et par conséquent si

$$\limsup_{m\to+\infty}\int_M f(x)\left|u_m^+\right|^N dv(g) < \frac{1}{(1+\varepsilon)^{\frac{n}{n-4}}K_\circ^{\frac{n}{n-4}}\max_{x\in M} f(x)} \tag{5.13}$$

Alors,

$$\left\|\Delta_g(u_m^+ - u^+)\right\|_2^2 = o(1)$$

i.e. $u_m^+ \to u^+$ fortement dans $H_2^2(M)$. ■

Théorème 5.2 *Soit* $(u_m^-)_m \subset N_\lambda^-$ *une suite minimisante telle que*

$$\begin{cases} J_\lambda(u_m^-) = \alpha_\lambda^- + o(1) \\ \nabla J_\lambda(u_m^-) = o(1), \ \ dans \ \ (H_2^2(M))^* \end{cases}$$

et

$$\alpha_\lambda^- < \frac{2}{nK_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}}$$

Alors pour tout $\lambda \in (0, \lambda^*)$, $(u_m^-)_m$ *une sous-suite qui converge fortement vers un point* $u^- \in N_\lambda^-$ *dans* $H_2^2(M)$, *en plus* $J_\lambda(u^-) = \alpha_\lambda^- > 0$ *et* u^- *est une solution non-triviale de l'équation (5.1).*

Preuve: Même démonstration que le théorème précédent. ■

Remarque 5.1 *Par les théorèmes **5.1** et **5.2**, on en déduit que* u^+ *et* u^- *sont deux solutions non-triviales distinctes de l'équation (5.1) car comme* $u^+ \in N_\lambda^+$ *et* $u^- \in N_\lambda^-$ *et* $N_\lambda^- \cap N_\lambda^+ = \emptyset$.

5.4 Le cas critique $\sigma = 2$ et $\beta = 4$

Ce cas correpond à l'équation non-linéaire ci-dessous:

$$\Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^2}\nabla_g u\right) + \frac{b(x)}{\rho^4}u = f(x)\left|u\right|^{N-2}u + \lambda\left|u\right|^{q-2}u$$

Théorème 5.3 *Soient* $\lambda \in (0,\ \lambda^{*})$ *et* $\left(u^{+}_{m,\sigma,\beta}\right)_{m\in\mathbb{N}} \subset N^{+}_{\lambda,\sigma,\beta}$ *telle que*

$$\begin{cases} J_{\lambda,\sigma,\beta}(u^{+}_{m,\sigma,\beta}) = c^{+}_{\lambda,\sigma,\beta} + o(1) \\ \nabla J_{\lambda,\sigma,\beta}(u^{+}_{m,\sigma,\beta}) = o(1),\ \ dans\ \ \left(H^{2}_{2}(M)\right)^{*} \end{cases}$$

Supposons que

$$\begin{cases} \left|c^{+}_{\lambda,\sigma,\beta}\right| < \frac{2}{nK_{\circ}^{\frac{n}{4}}\left(\max_{x\in M} f(x)\right)^{\frac{n-4}{4}}} \\ \frac{1}{2} + a^{-}K^{2}(n,1,2) + b^{-}K^{2}(n,2,4) > 0 \end{cases}$$

Alors l'équation

$$\Delta^{2}_{g}u + div_{g}\left(\frac{a(x)}{\rho^{2}}\nabla_{g}u\right) + \frac{b(x)}{\rho^{4}}u = f(x)\left|u\right|^{N-2}u + \lambda\left|u\right|^{q-2}u$$

possède une solution non triviale $u^{+} \in N^{+}_{\lambda,\sigma,\beta}$ *dans* $H^{2}_{2}(M)$*.*

Preuve: Soit $(u^{+}_{m})_{m} = \left(u^{+}_{\sigma_{m};\beta_{m}}\right)_{m\in\mathbb{N}} \subset N^{+}_{\lambda,\sigma,\mu}$

$$J_{\lambda,\sigma,\beta}(u^{+}_{m}) = \frac{N-2}{2N}\left\|u^{+}_{m}\right\|^{2} - \lambda\frac{N-q}{Nq}\int_{M}\left|u^{+}_{m}\right|^{q}dv(g)$$

· Dans un premier temps on montre que:

$$\lim_{(\sigma,\beta)\to(2^{-},4^{-})}\inf \Lambda_{\sigma,\beta} \nrightarrow 0$$

Comme l'opérateur $u \to P_{g}(u) := \Delta^{2}_{g}u + div_{g}\left(\frac{a(x)}{\rho^{\sigma}}\nabla_{g}u\right) + \frac{b(x)}{\rho^{\beta}}u$ est coercif et si on note par $\lambda_{1,\sigma,\beta}$ sa première valeur propre, alors

$$\lambda_{1,\sigma,\beta} > 0$$

Soient $\delta, \delta' \in (0, d)$ telle que: $\delta < \delta'$ où d est le rayan d'injectivité de M $(2\delta < d)$.
On note par $u_{\sigma,\beta}$ la première fonction propre associée à $\lambda_{1,\sigma,\beta}$ c-à-d:

$$\int_M (\Delta_g u_{\sigma,\beta})^2 + \frac{a(x)}{\rho^\sigma} |\nabla_g u_{\sigma,\beta}|^2 + \frac{b(x)}{\rho^\beta} u_{\sigma,\beta}^2 dv(g) = \lambda_{1,\sigma,\beta} \|u_{\sigma,\beta}\|_2^2$$

On prend la fontion propre $u_{\sigma,\beta}$ associée à $\lambda_{1,\sigma,\beta}$ normalisée, alors

$$\int_M (\Delta_g u_{\sigma,\beta})^2 + \frac{a(x)}{\rho^\sigma} |\nabla_g u_{\sigma,\beta}|^2 + \frac{b(x)}{\rho^\beta} u_{\sigma,\beta}^2 dv(g) = \lambda_{1,\sigma,\beta}$$

On a:

$$\int_{B(P,\delta)} (\Delta_g u_{\sigma,\beta})^2 + \frac{a(x)}{\rho^\sigma} |\nabla_g u_{\sigma,\beta}|^2 + \frac{b(x)}{\rho^\beta} u_{\sigma,\beta}^2 dv(g) +$$

$$\int_{M-B(P,\delta)} (\Delta_g u_{\sigma,\beta})^2 + \frac{a(x)}{\rho^\sigma} |\nabla_g u_{\sigma,\beta}|^2 + \frac{b(x)}{\rho^\beta} u_{\sigma,\beta}^2 dv(g) = \lambda_{1,\sigma,\beta}$$

On considère ici η est une fonction de classe C^∞ égale à 1 sur $B(P,\delta)$ et 0 sur $M - B(P, 2\delta)$.
Par l'absurde, on suppose que

$$\lim_{(\sigma,\beta)\to(2^-,4^-)} \inf \lambda_{1,\sigma,\beta} \to 0$$

On considère la carte normale $\exp_P^{-1} : B(P,\delta') \to B(P,\delta')$, pour tout $Q \in B(P,\delta')$, on note par

$$\rho(Q) = d(P,Q) = |x|$$

la distance géodésique au point P.
Si on pose $\tilde{u}_{\sigma,\beta}(x) = u_{\sigma,\beta} \circ \exp_P^{-1}(x)$, on obtient une fonction bien définie sur $\mathbb{R}^n$ à support dans

$$\{x \in \mathbb{R}^n : |x| < 1\}.$$

Comme on est dans un système de coordonnées géodésique centré en P, on peut trouver $\epsilon > 0$ suffisamment petit pour estimer la mesure riemannienne suivante:

$$(1-\epsilon)^{n-1}\,dx \le dv(g) \le (1+\epsilon)^{n-1}\,dx$$

Si on définit

$$v_{\sigma,\beta} = \eta \circ u_{\sigma,\beta}$$

On obtient

$$\int_{B(P,\delta)} \frac{u_{\sigma,\beta}^2}{\rho^\beta} dv(g) \le \int_{B(P,\delta')} \frac{v_{\sigma,\beta}^2}{\rho^\beta} dv(g) = \int_{\mathbb{R}^n} \frac{|\tilde{v}_{\sigma,\beta}|^2}{|x|^\beta} \sqrt{|g|} dx$$

$$\le (1+\epsilon)^{n-1} \int_{\mathbb{R}^n} \frac{|\tilde{v}_{\sigma,\beta}|^2}{|x|^\beta} dx$$

Par l'inégalité de Hardy-Sobolev deux fois, on trouve

$$\int_{\mathbb{R}^n} \frac{|\tilde{v}_{\sigma,\beta}|^2}{|x|^\beta} \sqrt{|g|} dx \le (1+\epsilon)^{n-1}\, C \int_{\mathbb{R}^n} \frac{\left|\tilde{\nabla}\tilde{v}_{\sigma,\beta}\right|^2}{|x|^{\beta-2}} dx$$

$$\le (1+\epsilon)^{n-1}\, C \int_{\mathbb{R}^n} \left|\tilde{\nabla}\left|\tilde{\nabla}\tilde{v}_{\sigma,\beta}\right|\right|^2 dx$$

Donc,

$$\int_{\mathbb{R}^n} \left|\tilde{\nabla}\left|\tilde{\nabla}\tilde{v}_{\sigma,\beta}\right|\right|^2 dx \le \int_{\mathbb{R}^n} \left|\tilde{\nabla}^2\tilde{v}_{\sigma,\beta}\right|^2 dx = \int_{\mathbb{R}^n} \left(\tilde{\Delta}\tilde{v}_{\sigma,\beta}\right)^2 dx$$

On trouve

$$\int_{B(P,\delta)} \frac{u_{\sigma,\beta}^2}{\rho^\beta} dv(g) \le$$

$$\left(\frac{1+\epsilon}{1-\epsilon}\right)^{n-1} C \int_{\mathbb{R}^n} \left(\tilde{\Delta}\tilde{v}_{\sigma,\beta}\right)^2 \sqrt{|g|} dx = \left(\frac{1+\epsilon}{1-\epsilon}\right)^{n-1} C \int_{B(P,\delta')} (\Delta v_{\sigma,\beta})^2\, dv(g)$$

Comme C est une constante universelle, on obtient

$$\left(\int_{B(P,\delta)} \frac{u_{\sigma,\beta}^2}{\rho^\beta} dv(g)\right)^{\frac{1}{2}} \le K_{\delta'}(n,2,\beta) \int_{B(P,\delta')} (\Delta v_{\sigma,\beta})^2\, dv(g) \tag{5.14}$$

avec $K_{\delta'}(n,2,\beta)$ est la bonne constante pour (5.14) et que $K_{\delta'}(n,2,\beta) \to K(n,2,\beta)$ quand $\delta' \to 0$.

De même façon, on trouve

$$\int_{B(P,\frac{\delta}{2})} \frac{|\nabla_g u_{\sigma,\beta}|^2}{\rho^\sigma} dv(g) \leq K_{\delta'}(n,1,\sigma) \int_{B(P,\delta')} (\Delta v_{\sigma,\beta})^2 \, dv(g) \tag{5.15}$$

avec $K_\delta(n,1,\sigma) \to K(n,1,\sigma)$ quand $\delta' \to 0$.

Les relations (5.14) et (5.15), donnent:

$$2\lambda_{\sigma,\beta} \geq \left(\frac{1}{2} + a^- K_{\delta'}(n,1,\sigma) + b^- K_{\delta'}(n,2,\beta)\right) \int_{B(P,\delta)} (\Delta_g u_{\sigma,\beta})^2 \, dv(g)$$

$$+\frac{1}{2}\int_{B(P,\delta)} (\Delta_g u_{\sigma,\beta})^2 \, dv(g) + \left(a^- K_{\delta'}(n,1,\sigma) + b^- K_{\delta'}(n,2,\beta)\right) \int_{B(P,\delta)-B(P,\delta')} (\Delta_g u_{\sigma,\beta})^2 \, dv(g) \tag{5.16}$$

où

$$a^- = \min(a(x);0) \quad \text{et} \quad b^- = \min(b(x);0)$$

Comme

$$\int_{B(P,\delta)-B(P,\delta')} (\Delta_g v_{\sigma,\beta})^2 \, dv(g) = \int_{B(P,\delta)-B(P,\delta')} (\Delta_g \eta)^2 (u_{\sigma,\beta})^2 \, dv(g) + 4\int_{B(P,\delta)-B(P,\delta')} \langle \nabla\eta, \nabla u_{\sigma,\beta}\rangle^2 \, dv(g)$$

$$+ \int_{B(P,\delta)-B(P,\delta')} \eta^2 (\Delta_g u_{\sigma,\beta})^2 \, dv(g) - 4\int_{B(P,\delta)-B(P,\delta')} \langle \nabla\eta, \nabla u_{\sigma,\beta}\rangle \, u_{\sigma,\beta} \Delta_g \eta dv(g)$$

$$+2\int_{B(P,\delta)-B(P,\delta')} \eta u_{\sigma,\beta} \Delta_g \eta \Delta_g u_{\sigma,\beta} dv(g) - \int_{B(P,\delta)-B(P,\delta')} \eta \langle \nabla\eta, \nabla u_{\sigma,\beta}\rangle \Delta_g u_{\sigma,\beta} dv(g)$$

Donc, il existe $c > 0$ tel que

$$|\nabla\eta| \leq c\delta' \quad \text{et} \quad |\Delta\eta| \leq c\,(\delta')^2$$

D'après l'inégalité de Hölder, on a:

$$\int_{B(P,\delta)-B(P,\delta')} (\Delta_g v_{\sigma,\beta})^2 \, dv(g) \leq \int_{B(P,\delta)-B(P,\delta')} (\Delta_g u_{\sigma,\beta})^2 \, dv(g) + c\delta' \int_{B(P,\delta)-B(P,\delta')} u_{\sigma,\beta}^2 dv(g)$$

$$+4c\,(\delta')^2 \int_{B(P,\delta)-B(P,\delta')} |\nabla u_{\sigma,\beta}|^2 \, dv(g)$$

$$+4c\,(\delta')^3 \left[\int_{B(P,\delta)-B(P,\delta')} u_{\sigma,\beta}^2 dv(g)\right]^{\frac{1}{2}} \left[\int_{B(P,\delta)-B(P,\delta')} |\nabla u_{\sigma,\beta}|^2 \, dv(g)\right]^{\frac{1}{2}}$$

$$+2c\,(\delta')^2 \left[\int_{B(P,\delta)-B(P,\delta')} u_{\sigma,\beta}^2 dv(g)\right]^{\frac{1}{2}} \left[\int_{B(P,\delta)-B(P,\delta')} (\Delta_g u_{\sigma,\beta})^2 \, dv(g)\right]^{\frac{1}{2}}$$

$$+4c\delta' \left[\int_{B(P,\delta)-B(P,\delta')} (\Delta_g u_{\sigma,\beta})^2 \, dv(g)\right]^{\frac{1}{2}} \left[\int_{B(P,\delta)-B(P,\delta')} |\nabla u_{\sigma,\beta}|^2 \, dv(g)\right]^{\frac{1}{2}}$$

et comme c est une constante universelle, on trouve:

$$\int_{B(P,\delta)-B(P,\delta')} (\Delta_g v_{\sigma,\beta})^2 \, dv(g) \leq \int_{B(P,\delta)-B(P,\delta')} (\Delta_g u_{\sigma,\beta})^2 \, dv(g) + O(\delta') \tag{5.17}$$

Si on prend δ' assez proche de δ tel que

$$\frac{1}{2} \int_{B(P,\delta)} (\Delta_g u_{\sigma,\beta})^2 \, dv(g) + \left(a^- K_{\delta'}(n,1,\sigma) + b^- K_{\delta'}(n,2,\beta)\right) \int_{B(P,\delta)-B(P,\delta')} (\Delta_g u_{\sigma,\beta})^2 \, dv(g) \geq 0$$

D'après (5.16), on en conclut que

$$\lambda_{\sigma,\beta} \geq \left(\frac{1}{2} + a^- K_{\delta'}(n,1,\sigma) + b^- K_{\delta'}(n,2,\beta)\right) \int_{B(P,\delta)} (\Delta_g u_{\sigma,\beta})^2 \, dv(g) \geq 0$$

Et si on prend $\frac{1}{2} + a^- K_{\delta'}(n,1,\sigma) + b^- K_{\delta'}(n,2,\beta) > 0$, on obtient

$$\int_{B(P,\delta')} (\Delta_g v_{\sigma,\beta})^2 \, dv(g) = \int_{B(P,\delta')} |\nabla v_{\sigma,\beta}|^2 \, dv(g) = 0$$

Donc $v_{\sigma,\beta} = \eta \circ u_{\sigma,\beta}$ est nulle presque partout dans M, alors $u_{\sigma,\beta} = o$ p.p dans $B(P,\delta)$ et P est un point arbitraire de M, on conclut que:

$$u_{\sigma,\beta} = o \ \text{ p.p dans } M.$$

contraduction avec $\|u_{\sigma,\beta}\|_2^2 = 1$.

· Maintenant, on montre que la suite $(u_m^+)_m$ est bornée dans $H_2^2(M)$.

$$J_\lambda(u_m) = c_\lambda \text{ et } \nabla J_{\lambda,\sigma,\mu}(u_m^+) = o(1), \text{ dans } \left(H_2^2(M)\right)^*$$

On obtient,

$$-c_\lambda + o(1) \leq J_\lambda(u_m^+) - \frac{1}{q}\left\langle \nabla J_\lambda(u_m^+); u_m^+ \right\rangle \leq c_\lambda + o(1)$$

Alors,

$$-c_\lambda + o(1) \leq \left(\frac{N-2}{2N} - \frac{N-2}{Nq}\right) \left\|u_m^+\right\|_{\sigma,\mu}^2 \leq c_\lambda + o(1)$$

D'où

$$\left(\frac{N-2}{Nq} - \frac{N-2}{2N}\right)^{-1} c_\lambda + o(1) \leq \left\|u_m^+\right\|_{\sigma,\mu}^2 \leq -\left(\frac{N-2}{Nq} - \frac{N-2}{2N}\right)^{-1} c_\lambda + o(1)$$

Où

$$\left\|u_m^+\right\|_{\sigma,\mu}^2 = \int_M \left(\Delta_g u_{m,\sigma,\mu}^+\right)^2 + \frac{a(x)}{\rho^\sigma}\left|\nabla_g u_{m,\sigma,\mu}^+\right|^2 + \frac{b(x)}{\rho^\mu}\left(u_{m,\sigma,\mu}^+\right)^2 dv(g)$$

Donc $(u_m^+)_m$ est bornée dans $H_2^2(M)$.

D'après la réflexivité de l'espace $H_2^2(M)$ et la compacité de l'inclusion $H_2^2(M) \subset H_p^k(M)$ ($k = 0,1$; $p < N$), implique qu'il existe une sous-suite notée $(u_m^+)_m$ telle que :

1. $u_m^+ \to u^+$ faiblement dans $H_2^2(M)$.

2. $u_m^+ \to u^+$ fortement dans $L^p(M)$ où $p < N$.

3. $\nabla u_m^+ \to \nabla u^+$ fortement dans $L^p(M)$ où $p < 2^* = \frac{2n}{n-2}$.

4. $u_m^+ \to u^+$ presque partout dans M.

Il existe deux suites $(\sigma_m)_m$ et $(\beta_m)_m$ qui convergent respectivement vers 2 et 4 telle que la suite de fonctions $(u_m^+)_m$ converge faiblement dans $H_2^2(M)$ et $L^2(M,\rho^{-4})$ et que la suite $(\nabla u_m^+)_m$converge faiblement dans $H_1^2(M)$ et $L^2(M,\rho^{-2})$.

Soit $0<\delta<\delta(M)$, on a

$$\int_M \frac{b(x)}{\rho^{\beta_m}}\left(u^+\right)^2 dv(g)=\int_{B(P,\delta)}\frac{b(x)}{\rho^{\beta_m}}\left(u^+\right)^2 dv(g)+\int_{M-B(P,\delta)}\frac{b(x)}{\rho^{\beta_m}}\left(u^+\right)^2 dv(g)$$

et

$$\int_M \frac{a(x)}{\rho^{\sigma_m}}\left|\nabla u^+\right|^2 dv(g)=\int_{B(P,\delta)}\frac{a(x)}{\rho^{\sigma_m}}\left|\nabla u^+\right|^2 dv(g)+\int_{M-B(P,\delta)}\frac{a(x)}{\rho^{\sigma_m}}\left|\nabla u^+\right|^2 dv(g)$$

D'après le théorème de la convergence dominée de Lebesgue, on obtient que

$$\int_M \frac{a(x)}{\rho^{\sigma_m}}\left|\nabla u^+\right|^2 dv(g)=\int_M \frac{a(x)}{\rho^{2}}\left|\nabla u^+\right|^2 dv(g)+o(1)\ \text{ quand }\ \sigma_m\to 2^-$$

et aussi

$$\int_M \frac{b(x)}{\rho^{\beta_m}}\left(u^+\right)^2 dv(g)=\int_M \frac{b(x)}{\rho^{4}}\left(u^+\right)^2 dv(g)+o(1)\ \text{ quand }\ \beta_m\to 4^-$$

Comme $u_m^+\to u^+$ faiblement dans $H_2^2(M)$, alors $\nabla u_m^+\to\nabla u^+$ faiblement dans $L^2(M,\rho^{-2})$ et $u_m^+\to u^+$ faiblement dans $L^2(M,\rho^{-4})$ c-à-d pour toute $\varphi\in L^2(M)$:

$$\int_M \frac{a(x)}{\rho^2}\nabla u_m^+\nabla\varphi dv(g)=\int_M \frac{a(x)}{\rho^2}\nabla u^+\nabla\varphi dv(g)+o(1)$$

et

$$\int_M \frac{b(x)}{\rho^4}u_m^+\varphi dv(g)=\int_M \frac{b(x)}{\rho^4}u^+\varphi dv(g)+o(1)$$

On pose $w_m:=u_m^+-u^+$, alors $w_m\to 0$ faiblement dans $H_2^2(M)$.

D'après le lemme de Brézis-Lieb, on peut écrire aussi

$$\left\|\Delta_g u_m^+\right\|_2^2-\left\|\Delta_g u^+\right\|_2^2=\left\|\Delta_g w_m\right\|_2^2+o(1)$$

et

$$\int_M f(x)\left(\left|u_m^+\right|^N-\left|u^+\right|^N\right)dv(g)=\int_M f(x)\left|w_m\right|^N dv(g)+o(1)$$

Comme $u_m^+ - u^+ \to 0$ faiblement dans $H_2^2(M)$, alors, $\nabla u_m^+ \to \nabla u^+$ faiblement dans $L^2(M,\rho^{-2})$ et $u_m^+ \to u^+$ faiblement dans $L^2(M,\rho^{-4})$ c-à-d pour tout $\phi \in L^2(M)$

$$\int_M w_m \Delta_g^2 \phi dv(g)=o(1)$$

alors, pour tout $\phi \in L^2(M)$, on a

$$\int_M u_m^+\Delta_g^2\phi dv(g)=\int_M u^+\Delta_g^2\phi dv(g)+o(1)=\int_M \Delta_g\phi\Delta_g u^+ dv(g)+o(1)$$

et pour la deuxième partie on a

$$\int_M\left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m^+-\frac{a(x)}{\rho^2}\nabla_g u^+\right)\phi dv(g)=$$

$$\int_M\left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m^+ +\frac{a(x)}{\rho^2}\left(\nabla_g u_m^+-\nabla_g u_m^+\right)-\frac{a(x)}{\rho^2}\nabla_g u^+\right)\phi dv(g)$$

Donc

$$\left|\int_M\left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m^+-\frac{a(x)}{\rho^2}\nabla_g u^+\right)\phi dv(g)\right|\leq$$

$$\left|\int_M\left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m^+-\frac{a(x)}{\rho^2}\nabla_g u_m^+\right)\phi dv(g)\right|+\left|\int_M\left(\frac{a(x)}{\rho^2}\nabla_g u_m^+-\frac{a(x)}{\rho^2}\nabla_g u^+\right)\phi dv(g)\right|$$

$$\leq\int_M\left|a(x)\phi\nabla_g u_m^+\right|\left|\frac{1}{\rho^{\sigma_m}}-\frac{1}{\rho^2}\right|dv(g)+\left|\int_M\frac{a(x)}{\rho^2}\nabla_g\left(u_m^+-u^+\right)\phi dv(g)\right|$$

La convergence faible dans $L^2(M,\rho^{-2})$ et le théorème de la convergence dominée de Lebesgue implique que le secont menbre converge vers 0.

Et pour la troisième partie on a

$$\int_M\left(\frac{b(x)}{\rho^{\beta_m}}u_m-\frac{b(x)}{\rho^4}u\right)\phi dv(g)=\int_M\left(\frac{b(x)}{\rho^{\beta_m}}u_m-\frac{b(x)}{\rho^4}u_m+\frac{b(x)}{\rho^4}u_m-\frac{b(x)}{\rho^4}u\right)\phi dv(g)$$

Donc

$$\left|\int_M \left(\frac{b(x)}{\rho^{\beta_m}}u_m^+ - \frac{b(x)}{\rho^4}u^+\right)\phi dv(g)\right|$$

$$\leq \int_M \left|b(x)\phi u_m^+\right| \left|\frac{1}{\rho^{\beta_m}} - \frac{1}{\rho^4}\right| dv(g) + \left|\int_M \frac{b(x)}{\rho^4}\left(u_m^+ - u^+\right)\phi dv(g)\right|$$

La convergence faible dans $L^2(M, \rho^{-4})$ et le théorème de la convergence dominée de Lebesgue implique que le secont menbre converge vers 0. ■

Théorème 5.4 *Soient* $\lambda \in (0, \lambda^*)$ *et* $\left(u_{m,\sigma,\beta}^-\right)_{m\in\mathbb{N}} \subset N_{\lambda,\sigma,\beta}^-$ *telle que*

$$\begin{cases} J_{\lambda,\sigma,\beta}(u_{m,\sigma,\beta}^-) = c_{\lambda,\sigma,\beta}^- + o(1) \\ \nabla J_{\lambda,\sigma,\beta}(u_{m,\sigma,\beta}^-) = o(1), \ \ dans \ \ (H_2^2(M))^* \end{cases}$$

Supposons que

$$\begin{cases} c_{\lambda,\sigma,\beta}^- < \frac{2}{nK_\circ^{\frac{n}{4}}(\max_{x\in M} f(x))^{\frac{n-4}{4}}} \\ \frac{1}{2} + a^- K^2(n,1,2) + b^- K^2(n,2,4) > 0 \end{cases}$$

Alors l'équation

$$\Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^2}\nabla_g u\right) + \frac{b(x)}{\rho^4}u = f(x)\left|u\right|^{N-2}u + \lambda\left|u\right|^{q-2}u$$

possède une solution non triviale $u^- \in N_{\lambda,\sigma,\beta}^-$ *dans* $H_2^2(M)$.

Preuve: Même démonstration que le théorème précédent. ■

5.5 Fonctions tests

Pour vérifier l'hypothèse du théorème générique **5.1**, on considère les fonctions tests suivantes:

$$u_\epsilon(x) = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{8}}\frac{\eta(\rho)}{\left((\rho\theta)^2 + \epsilon^2\right)^{\frac{n-4}{2}}}$$

avec

$$\theta = \left(1 + \left\|\frac{a}{\rho^\sigma}\right\|_r + \left\|\frac{b}{\rho^\beta}\right\|_s\right)^{\frac{1}{n}}$$

où $f(x_\circ) = \max_{x\in M} f(x)$ et η est une fonction de classe C^∞ égale à 1 sur $B(x_\circ, \delta)$ et 0 sur $M - B(x_\circ, 2\delta)$ où $\rho = d(x_\circ, .)$ désigne la distance géodésique au point $x_\circ$.

5.5.1 Application aux variétés riemanniennes compactes de dimensions $n > 6$

Théorème 5.5 *Soit (M, g) une variété riemannienne compacte de dimension $n > 6$, si en un point $x_\circ$ où f atteint son maximum, la condition*

$$\left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-1)}\right) < 0 \quad et \quad S_g(x_\circ) > 0.$$

est vérifiée, alors l'équation (5.1) admet une solution u non triviale vérifiant

$$J_\lambda(u) < \frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n}{4}-1}}$$

Preuve: Nous reprenons les mêmes calculs qui ont été faits dans le chapitre (**3**)

$$\int_M f(x)\,|u_\epsilon(x)|^N\,dv(g) = \frac{2\theta^{-n}}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(\frac{n}{2} - (\frac{n\Delta f(x_\circ)}{4(n-2)f(x_\circ)} + \frac{nS_g(x_\circ)}{12(n-2)})\epsilon^2 + o(\epsilon^2)\right)$$

et

$$\int_M \frac{a(x)}{\rho^\sigma}\,|\nabla u_\epsilon|^2\,dv(g) \leq \frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\theta^{-n\frac{r}{r-1}}\left[A\left\|\frac{a}{\rho^\sigma}\right\|_r + o(\epsilon^2)\right] \times \epsilon^{2-\frac{n}{r}}$$

et

$$\int_M \frac{b(x)}{\rho^\beta}u_\epsilon^2 dv(g) \leq \frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\theta^{-n\frac{s-1}{s}}\left[B\left\|\frac{b}{\rho^\beta}\right\|_s + o(\epsilon^2)\right]\epsilon^{4-\frac{n}{s}}$$

et aussi

$$\int_M (\Delta u_\epsilon)^2\,dv(g) = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1 - \frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2 + o(\epsilon^2)\right)$$

Tenant compte de l'expression de J_λ

$$J_\lambda(tu_\epsilon) \leq J_0(tu_\epsilon) = \frac{t^2}{2}\|u_\epsilon\|^2 - \frac{t^N}{N}\int_M f(x)\,|u_\epsilon(x)|^N\,dv(g)$$

Donc

$$J_\lambda(tu_\epsilon) \leq \frac{2\theta^{-n}}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\times$$

$$\left\{\frac{t^2}{2}\left(\frac{n}{4} + \left\|\frac{b(x)}{\rho^\beta}\right\|_s B\theta^{\frac{-n}{r-1}}\epsilon^{4-\frac{n}{s}} + \left\|\frac{a(x)}{\rho^\sigma}\right\|_r A\theta^{\frac{-n}{s-1}}\epsilon^{2-\frac{n}{r}} - \frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2\right) - \right.$$

$$\left.\frac{t^N}{N}\left(1 - \left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-1)}\right)\epsilon^2\right) + o(\epsilon^2)\right\}$$

Pour ϵ assez petit, on prend

$$1 + \left\|\frac{b}{\rho^\beta}\right\|_s B\theta^{\frac{-n}{r-1}}\epsilon^{4-\frac{n}{s}} + \left\|\frac{a}{\rho^\sigma}\right\|_r A\theta^{\frac{-n}{s-1}}\epsilon^{2-\frac{n}{r}} \leq \left(1 + \left\|\frac{a}{\rho^\sigma}\right\|_r + \left\|\frac{b}{\rho^\beta}\right\|_s\right)^{\frac{4}{n}}$$

Et comme la fonction $\varphi(t) = \alpha\frac{t^2}{2} - \frac{t^N}{N}$ atteint son maximum au point $t^* = \alpha^{\frac{1}{N-2}}$ où

$$\varphi(t^*) = \frac{2}{n}\alpha^{\frac{n}{4}}$$

Alors,

$$J_\lambda(tu_\epsilon) \leq J_\lambda(t^*u_\epsilon) = \frac{2\theta^{-n}}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \times \left\{1 + \left\|\frac{a}{\rho^\sigma}\right\|_r + \left\|\frac{b}{\rho^\beta}\right\|_s - \right.$$

$$\left.\frac{(t^*)^2}{2}\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2 - \frac{(t^*)^N}{N}\left(1 - \left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-1)}\right)\epsilon^2\right) + o(\epsilon^2)\right\}$$

Pour ϵ assez petit, nous avons

$$1 + \theta^{-\frac{s}{s-1}}\epsilon^{4-\frac{n}{s}}B\left\|\frac{b}{\rho^\beta}\right\|_s + \theta^{-\frac{r}{r-1}}\epsilon^{2-\frac{n}{r}}A\left\|\frac{a}{\rho^\sigma}\right\|_r \leq \theta^{\frac{4}{n}}$$

Et comme la fonction $\varphi(t) = \alpha\frac{t^2}{2} - \frac{t^N}{N}$ atteint son maximum au point $t^* = \alpha^{\frac{1}{N-2}}$ où

$$\varphi(t^*) = \frac{2}{n}\alpha^{\frac{n}{4}}$$

alors

$$J_\lambda(tu_\epsilon) \leq \frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \times$$

$$\left\{1 + \left[\left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-1)}\right)\frac{(t^*)^N}{N} - \frac{(n^2+4n-20)\,S_g(x_\circ)}{6(n^2-4)(n-6)}\frac{(t^*)^2}{2}\right]\epsilon^2 + o(\epsilon^2)\right\}.$$

Pour assurer

$$\sup_{t\geq 0} J_\lambda(tu_\epsilon) < \frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}$$

on prend

$$\left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-1)}\right) < 0 \;\text{ et }\; S_g(x_\circ) > 0.$$

Ce qui achève la preuve. ■

5.5.2 Application aux variétés riemanniennes compactes de dimensions $n = 6$

Théorème 5.6 *Lorsque $n = 6$, s'il existe un point $x_\circ \in M$ où $S_g(x_\circ) > 0$ alors (5.1) admet une solution u non triviale.*

Preuve: Le developement de l'intégrale:

$$\int_M f(x)\,|u_\epsilon(x)|^N\,dv(g) = \frac{\theta^{-n}}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left[\frac{n}{2} - \left(\frac{n\Delta f(x_\circ)}{4(n-2)f(x_\circ)} + \frac{nS_g(x_\circ)}{12(n-2)}\right)\epsilon^2 + o(\epsilon^2)\right]$$

De même pour

$$\int_M \frac{a(x)}{\rho^\sigma}\,|\nabla u_\epsilon|^2\,dv(g) \leq \frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\theta^{-n\frac{r}{r-1}}(A\left\|\frac{a}{\rho^\sigma}\right\|_r + o(\epsilon^2)) \times \epsilon^{2-\frac{n}{r}}$$

et

$$\int_M \frac{b(x)}{\rho^\beta} u_\epsilon^2 dv(g) \le \frac{2}{nK_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \theta^{-n\left(\frac{s-1}{s}\right)} (B \left\| \frac{b}{\rho^\beta} \right\|_s + o(\epsilon^2)) \epsilon^{4-\frac{n}{s}}$$

Pour le même calcul, on obtient

$$\int_M (\Delta u_\epsilon)^2 \, dv(g) = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \left(1 - \frac{(n-4) S_g(x_\circ) \theta^{-2}}{3n^2(n^2-4) I_n^{\frac{n}{2}-1}} \epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) + o(\epsilon^2) \right)$$

Pour ϵ assez petit, nous avons

$$1 + \theta^{-\frac{s}{s-1}} \epsilon^{4-\frac{n}{s}} B \left\| \frac{b}{\rho^\beta} \right\|_s + \theta^{-\frac{r}{r-1}} \epsilon^{2-\frac{n}{r}} A \left\| \frac{a}{\rho^\sigma} \right\|_r \le \theta^{\frac{4}{n}}$$

Tenant compte de l'expression de J_λ

$$J_\lambda(u_\epsilon) = \frac{1}{2} \|u_\epsilon\|^2 - \frac{\lambda}{q} \|u_\epsilon\|_q^q - \frac{1}{N} \int_M f(x) |u_\epsilon(x)|^N \, dv(g)$$

où

$$\|u_\epsilon\|^2 = \int_M |\Delta u_\epsilon|^2 - a(x) |\nabla u_\epsilon|^2 + b(x) u_\epsilon^2 dv(g)$$

et $\lambda > 0$, on obtient

$$J_\lambda(u_\epsilon) \le \frac{1}{2} \|u_\epsilon\|^2 - \frac{1}{N} \int_M f(x) |u_\epsilon(x)|^N \, dv(g)$$

$$J_\lambda(tu_\epsilon) \le \frac{\theta^{-n}}{nK_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \times$$

$$\left[\frac{t^2}{2} \theta^{1-\frac{4}{n}} - \frac{t^N}{N} - \frac{(n-4) S_g(x_\circ)}{12n(n^2-4) I_n^{\frac{n}{2}-1}} t^2 \epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) + o(\epsilon^2) \right]$$

Pour assurer

$$\sup_{t \ge 0} J_\lambda(tu_\epsilon) < \frac{2}{nK_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}$$

Avec le même argument, on a

$$S_g(x_\circ) > 0$$

Ce qui achève la preuve. ■

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Nonlinear Differential Equations and Applications NoDEA

Nonlinear elliptic fourth order equations existence and multiplicity results

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Abstract. This paper deals with the existence of solutions to a class of fourth order nonlinear elliptic equations. The technique used relies on critical points theory. The solutions appeared as critical points of a functional restricted to a suitable manifold. In the case of constant coefficients we obtain the existence of three distinct solutions.

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1. Introduction

Let (M, g) be a Riemannian compact smooth n-manifold, $n \geq 5$, with metric g and scalar curvature S_g, we let $H_2^2(M)$ be the standard Sobolev space which is the completion of the space

$$C_2^2(M) = \{u \in C^\infty(M) \colon \|u\|_{2,2} < +\infty\}$$

with respect to the norm $\|u\|_{2,2} = \sum_{l=0}^{2} \|\nabla^l u\|_2$.

In this paper, we investigate solutions of a class of fourth order elliptic equations, on compact n-dimensional Riemannian manifolds, of the form

$$\Delta^2 u + \nabla^i(a(x)\nabla_i u) + b(x)u = f(x)|u|^{N-2}u + \lambda|u|^{q-2}u \tag{1.1}$$

where a, b, and f are smooth functions on M, $N = \frac{2n}{n-4}$ is the critical exponent, $1 < q < 2$ a real number, $\lambda > 0$ a real parameter.

Consideration for such problem comes from conformal geometry: indeed, in 1983, Paneitz [11] introduced a conformal fourth order operator defined on 4-dimensional Riemannian manifolds which was generalized by Branson [6] to higher dimensions.

$$PB_g(u) = \Delta^2 u + div\left(-\frac{(n-2)^2+4}{2(n-1)(n-2)}S_g.g + \frac{4}{n-2}Ric\right)du + \frac{n-4}{2}Q^n u$$

where $\Delta u = -div\,(\nabla u)$, S_g is the scalar curvature, Ric is the Ricci curvature of g, d is the de Rham differential and where

$$Q^n = \frac{1}{2(n-1)}\Delta S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}S_g^2 - \frac{2}{(n-2)^2}|Ric|^2$$

is associated to the notion of Q-curvature.

We refer to a Paneitz–Branson type operator as an operator of the form

$$P_g u = \Delta^2 u + \nabla^i(a(x)\nabla_i u) + b(x)u.$$

Equation (1.1) is a perturbation of the equation

$$\Delta^2 u + \nabla^i(a(x)\nabla_i u) + b(x)u = f(x)|u|^{N-2}u. \tag{1.2}$$

Since the embedding $H_2^2 \hookrightarrow H_N^k$, $(k=0,1)$ fails to be compact, as known, one encounters serious difficulties in solving equations like (1.1).

Since 1990 many results have been established for precise functions a, h and f. Edmunds et al. ([10]) proved for $n \geq 8$ that if $\lambda \in (0, \lambda_1)$, with λ_1 is the first eigenvalue of Δ^2 on the euclidean open ball B, the problem

$$\begin{cases} \Delta^2 u - \lambda u = u|u|^{\frac{8}{n-4}} & \text{in } B \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B \end{cases}$$

has a non trivial solution.

In 1995, Van der Vorst [12] obtained the same results as Edmunds, Fortunato, Jannelli. when applied to the problem

$$\begin{cases} \Delta^2 u - \lambda u = u|u|^{\frac{8}{n-4}} & \text{in } \Omega \\ u = \Delta u = 0 \ \text{ on} & \partial\Omega \end{cases}$$

where Ω is an open bounded set of R^n and moreover he showed that the solution is positive.

In [8] Caraffa studied the equation

$$\Delta^2 u + \nabla^i(a(x)\nabla_i u) + b(x)u = f(x)|u|^{N-2}u \tag{1.3}$$

in the case $f(x) =$ constant; and in the particular case where the functions $a(x)$ and $b(x)$ are precise constants she obtained the existence of positive regular solutions. Djadli, Z. et al [9] have studied Paneitz type operators and they gave many interesting applications.

Let f be a C^∞ function on M, $f^- = -\inf(f,0)$, $f^+ = \sup(f,0)$ and for a, f, C^∞-functions on M, we let

$$\lambda_{a,f} = \inf_{u\in A} \frac{\int_M (\Delta u)^2 dv_g - \int_M a|\nabla u|^2 dv_g}{\int_M u^2 dvg}$$

where $A = \left\{ u \in H_2^2,\, u \geq 0,\, u \not\equiv 0 \text{ s. t. } \int_M f^- u dv_g = 0 \right\}$, and

$$\lambda_{a,f} = +\infty \quad \text{if } A = \phi$$

the second author establish the following:

Theorem 1. [4] *Let a, b be C^∞ functions on M with b negative. For every C^∞ function, f on M with $\int_M f^- dv_g > 0$, there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^- dv_g}$ such that if f satisfies the following conditions*

(1) $|b(x)| < \lambda_{a,f}$ *for any* $x \in M$

(2) $\frac{\sup f^+}{\int f^- dv_g} < C$

(3) $\sup_M f > 0$,

then the subcritical equation

$$\Delta^2 u + \nabla^i (a \nabla_i u) + bu = f|u|^{q-2}u, \qquad q \in]2, N[\tag{1.4}$$

has at least two distinct solutions u and v satisfying $F_q(u) < 0 < F_q(v)$ and of class $C^{4,\theta}$, for some $\theta \in (0,1)$, where F_q denotes the energy functional associated to Eq. (1.4).

Theorem 2. [4] *Let a, b be C^∞ functions on M with b negative. For every C^∞ function f on M with $\int_M f^- dv_g > 0$ there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^- dv_g}$ such that if f satisfies the following conditions*

(1) $|b(x)| < \lambda_{a,f}$ *for any* $x \in M$

(2) $\frac{\sup f^+}{\int f^- dv_g} < C$

the critical equation

$$\Delta^2 u + \nabla^i (a \nabla_i u) + bu = f|u|^{N-2}u$$

has a solution of class $C^{4,\theta}$, for some $\theta \in (0,1)$, with negative energy.

Let

$$\Delta^2 u + \nabla^i (a(x) \nabla_i u) + b(x)u = f(x)|u|^{N-2}u + \lambda |u|^{q-2}u + \epsilon g(x) \tag{1.5}$$

where a, b, f and g are smooth functions on M, $N = \frac{2n}{n-4}$ is the critical exponent, $2 < q < N$ a real number, $\lambda > 0$ a real parameter and $\epsilon > 0$ any small real number the second author obtain

Theorem 3. [5] *Let (M, g) be a compact Riemannian n-manifold, $n \geq 6$, a, b, f, g be smooth real functions on M with*

(i) *$f(x) > 0$ and $g(x) > 0$ everywhere on M*

(ii) *the operator $P(u) = \Delta^2 u + \nabla^i (a(x) \nabla_i u) + b(x)u$ is coercive*

(iii) *if $n > 6$, we suppose $-\frac{\Delta f(x_o)}{2f(x_o)} + C_1(n) S_g(x_o) + C_2(n) a(x_o) > 0$ and if $n = 6$, we suppose that $\frac{4}{3n} S_g(x_o) + \frac{1}{(n-4)} a(x_o) > 0$, where $C_1(n) = \frac{5n^2(n-7)+52(n-1)}{6n(n+2)(5n-6)}$ and $C_2(n) = \frac{8(n-1)}{(n+2)(n-6)}$*

Then Eq. (1.5) has at least two distinct solutions in H_2^2.

Our main results in this paper state as follows

Theorem 4. *Let (M, g) be an n-dimensional compact Riemannian manifold with $n \geqslant 6$ and f a positive smooth function on M. Assume that the operator $P(u) = \Delta^2 u + \nabla^i (a(x) \nabla_i u) + b(x)u$ is coercive.*

If $n>6$ and at the point x_o where the function f achieves its maximum the following condition is satisfied

$$\frac{n\left(n^2+4n-20\right)}{2\left(n-6\right)\left(n^2-4\right)}S_g\left(x_o\right)+\frac{n\left(n-1\right)}{\left(n-6\right)\left(n^2-4\right)}a\left(x_o\right)-\frac{n}{8\left(n-2\right)}\frac{\Delta f(x_o)}{f\left(x_o\right)}>0$$

then there exists a $\Lambda>0$ such that for any $\lambda\in(0,\Lambda)$, Eq. (1.1) have a non trivial solution of class $C^{4,\theta}(M)$, $\theta\in(0,1)$.

If $n=6$ and the condition $S_g(x_o)>-3a(x_o)$ is satisfied, then Eq. (1.1) has a solution u of class $C^{4,\theta}(M)$ provided that $\lambda\in(0,\Lambda)$.

In case of constant coefficients, Eq. (1.1) reduces to

$$\Delta^2u+\alpha\Delta u+\beta u=f(x)|u|^{N-2}u+\lambda|u|^{q-2}u \tag{1.6}$$

where α and β are real constants; we get the existence of three solutions.

Theorem 5. *Suppose all the conditions of Theorem 4 are satisfied and moreover $\frac{\alpha^2}{4}>\beta$ with $\alpha>0$. Then Eq. (1.6) has beside a positive u^+ and a negative u^- smooth solutions a third solution w distinct from u^+ and u^-.*

The technique relies on critical points theory. To find solutions we use a method developed in [1]. The solutions appeared as critical points restricted to a suitable manifold. In the case of constant coefficients we obtain the existence of two solutions.

Consider the functionals J_λ, J_λ^+ and J_λ^- defined on H_2^2 by

$$\begin{aligned}J_\lambda(u)&=\frac{1}{2}\left(\|\Delta u\|_2^2-\int_M a(x)|\nabla u|^2dv_g+\int_M b(x)u^2dv_g\right)\\&\quad-\frac{\lambda}{q}\int_M|u|^qdv_g-\frac{1}{N}\int_M f(x)|u|^Ndv_g,\end{aligned} \tag{1.7}$$

$$\begin{aligned}J_\lambda^+(u)&=\frac{1}{2}\left(\|\Delta u\|_2^2-\int_M a(x)|\nabla u|^2dv_g+\int_M b(x)u^2dv_g\right)\\&\quad-\frac{\lambda}{q}\int_M\left(u^+\right)^qdv_g-\frac{1}{N}\int_M f(x)\left(u^+\right)^Ndv_g\end{aligned} \tag{1.8}$$

and

$$\begin{aligned}J_\lambda^-(u)&=\frac{1}{2}\left(\|\Delta u\|_2^2-\int_M a(x)|\nabla u|^2dv_g+\int_M b(x)u^2dv_g\right)\\&\quad-\frac{\lambda}{q}\int_M|u^-|^qdv_g-\frac{1}{N}\int_M f(x)|u^-|^Ndv_g\end{aligned} \tag{1.9}$$

where $u^+=\max\left(u,0\right)$ and $u^-=\min\left(u,0\right)$.

Let

$$Q_\lambda\left(u\right)=\left\langle\nabla J_\lambda\left(u\right),u\right\rangle$$

and

$$Q_\lambda^\pm\left(u\right)=\left\langle\nabla J_\lambda^\pm\left(u\right),u\right\rangle$$

where $\left\langle\nabla J_\lambda\left(u\right),v\right\rangle$ denotes the value of $\nabla J_\lambda\left(u\right)$ at v.

We consider also the set

$$M_\lambda = \left\{ u \in H_2^2 : Q_\lambda(u) = 0 \text{ and } \|u\| \geq \rho > 0 \right\}$$

and

$$M_\lambda^\pm = \left\{ u \in H_2^2 : Q_\lambda^\pm(u) = 0 \text{ and } \|u\| \geq \rho > 0 \right\}.$$

Along this paper the functions a and b are taken such that

$$\|u\|^2 = \|\Delta u\|_2^2 - \int_M a(x)|\nabla u|^2 dv_g + \int_M b(x)u^2 dv_g$$

is a norm on $H_2^2(M)$ equivalent to the usual one: for example by letting $\max_{x\in M} a(x) < 0$ and $\min_{x\in M} b(x) > 0$ which is equivalent to assume that the operator $P_g(u) = \Delta^2 u + a(x)\Delta u + b(x)u$ is coercive.

First we ensure that the manifold M_λ (respectively $M_\lambda^\pm$) is not empty.

Lemma 1. *There is a real $\lambda_o > 0$ such that the set M_λ is non empty for any $\lambda \in (0, \lambda_o)$.*

Proof. For $u \in H_2^2(M)$ with $\|u\|_{2,2} \geq \rho > 0$ and $t > 0$,

$$Q_\lambda(tu) = t^2\|u\|^2 - \lambda t^q \|u\|_q^q - t^N \int_M f(x)|u|^N dv_g.$$

Put

$$\alpha(t) = \|u\|^2 - t^{N-2} \int_M f(x)|u|^N dv_g$$

and

$$\beta(t) = \lambda t^{q-2}\|u\|_q^q.$$

The Sobolev inequality leads to

$$\alpha(t) \geq \|u\|^2 - \max_{x\in M} f(x) \left(\max\left((1+\epsilon) K_o, A_\epsilon\right)\right)^N \|u\|_{H_2^2(M)}^N t^{N-2}$$

where

$$\frac{1}{K_\circ} = \inf{}_{u\in H_2^2(\mathbb{R}^n)-\{0\}} \frac{\|\Delta u\|_2^2}{\|u\|_N^2}$$

is the best constant in the Sobolev's embedding $H_2^2(R^n) \subset L^N(R^n)$ (see Aubin [3]) and A_ϵ is a positive constant depending on ϵ, and since the norms $\|.\|$ and $\|.\|_{H_2^2(M)}$ are equivalent, there is $\Lambda > 0$ such that

$$\alpha(t) \geq \|u\|^2 - \Lambda^{-\frac{N}{2}} \max_{x\in M} f(x) \left(\max\left((1+\epsilon) K_o, A_\epsilon\right)\right)^N \|u\|^N t^{N-2}. \tag{1.10}$$

Combining the Hölder and the Sobolev inequalities and taking account of the equivalence of the norms $\|.\|$ and $\|.\|_{H_2^2(M)}$, we get

$$\beta(t) \leq \lambda V(M)^{1-\frac{2}{N}} \left(\max\left((1+\epsilon) K_o, A\right)\right)^{\frac{n}{2}} \|u\|^q t^{q-2}. \tag{1.11}$$

Let $\alpha_1(t)$ and $\beta_1(t)$ denote respectively right hand sides of inequalities (1.10) and (1.11); $\alpha_1(t)$ vanishes for

$$t_o = \frac{1}{\|u\| \left(\max_{x\in M} f(x)\right)^{\frac{1}{N-2}} \left(\max\left((1+\epsilon) K_o, A_\epsilon\right)\right)^{\frac{N}{2(N-2)}}}. \tag{1.12}$$

So if we choose

$$\|u\| = \frac{1}{\left(\max_{x\in M} f(x)\right)^{\frac{1}{N-2}} \left(\max\left((1+\epsilon) K_o, A_\epsilon\right)\right)^{\frac{N}{2(N-2)}}}$$

$t_o = 1$.

Now since $\alpha_1(t)$ and $\beta_1(t)$ are both decreasing functions, we get

$$\begin{aligned}\min_{t\in\left(0,\frac{1}{2}\right)} \alpha_1(t) = \alpha_1\left(\frac{1}{2}\right) &= \frac{\left(1-2^{2-N}\right)}{\left(\max_{x\in M} f(x)\right)^{\frac{2}{N-2}} \left(\max\left((1+\epsilon) K_o, A_\epsilon\right)\right)^{\frac{N}{(N-2)}}} \\ &\geq \left(1-2^{2-N}\right)\rho^2 > 0\end{aligned}$$

and

$$\min_{t\in\left(0,\frac{1}{2}\right)} \beta_1(t) = \beta_1\left(\frac{1}{2}\right) = \frac{2^{2-q}\lambda V(M)^{\left(1-\frac{2}{N}\right)}}{\left(\max_{x\in M} f(x)\right)^{\frac{q}{N-2}} \left(\max\left((1+\epsilon) K_o, A_\epsilon\right)\right)^{\frac{q-2+N}{N-2}}} > 0.$$

The equation $Q_\lambda(u) = 0$ admits a solution if $\min_{t\in\left(0,\frac{1}{2}\right)} \alpha_1(t) \geq \min_{t\in\left(0,\frac{1}{2}\right)} \beta_1(t)$ that is to say if

$$0 < \lambda < \lambda_o = \frac{\left(2^{q-2} - 2^{q-N}\right) V(M)^{\left(1-\frac{2}{N}\right)}}{\left(\max_{x\in M} f(x)\right)^{\frac{q-2}{N-2}} \left(\max\left((1+\epsilon) K_o, A_\epsilon\right)\right)^{\frac{q-2}{N-2}}}.$$

Indeed, putting

$$\gamma(t) = \beta(t) - \alpha(t)$$

we get

$$\gamma(t) \leq \beta_1(t) - \alpha_1(t)$$

and

$$\gamma\left(\frac{1}{2}\right) \leq \beta_1\left(\frac{1}{2}\right) - \alpha_1\left(\frac{1}{2}\right) \leq 0.$$

If $\gamma\left(\frac{1}{2}\right) = 0$,

$$Q_\lambda\left(\frac{1}{2}u\right) = \frac{1}{4}\gamma\left(\frac{1}{2}\right) = 0$$

in case $\gamma\left(\frac{1}{2}\right) < 0$, the continuous function changes sign in the interval $\left(0, \frac{1}{2}\right)$, then there is $t_1 \in \left(0, \frac{1}{2}\right)$ such that

$$Q_\lambda(t_1 u) = t_1^2 \gamma(t_1) = 0.$$

Hence the set M_λ is nonempty for any $\lambda \in (0, \lambda_o)$.

Since $Q_\lambda^{\pm}(u) \geqslant Q_\lambda(u)$ the same calculations lead to the same conclusion. $\square$

2. Palais–Smale conditions

Lemma 2. *There is $A>0$ such that $J_\lambda(u)\geq A>0$ (resp. $J^{\pm}_\lambda(u)\geq A>0$) for any $u\in M_\lambda$ with $\lambda\in(0,\min(\lambda_o,\lambda_1))$ where $\lambda_1=\frac{\frac{(N-2)\Lambda^{-\frac{q}{2}}}{2(N-q)}}{V(M)^{1-\frac{2}{N}}(\max((1+\epsilon)K_o,A_\epsilon))^{\frac{q}{2}}\rho^{q-2}}$ and λ_o is as in Lemma 1.1.*

Proof. Let $u\in M_\lambda$, then

$$\langle\nabla J_\lambda,u\rangle=\|\Delta u\|_2^2-\int_M a(x)|\nabla u|^2dv_g+\int_M b(x)u^2dv_g$$
$$-\lambda\int_M |u|^q dv_g-\int_M f(x)|u|^N dv_g=0$$

so

$$\|u\|^2=\|\Delta u\|_2^2-\int_M a(x)|\nabla u|^2dv_g+\int_M b(x)u^2dv_g$$
$$=\lambda\int_M |u|^q dv_g+\int_M f(x)|u|^N dv_g$$

hence on M_λ the functional J_λ writes

$$J_\lambda(u)=\frac{N-2}{2N}\|u\|^2-\lambda\frac{N-q}{Nq}\int_M|u|^q dv_g.$$

Combining Hölder and Sobolev inequalities we obtain

$$J_\lambda(u)\geq\frac{N-2}{2N}\|u\|^2-\lambda\frac{N-q}{Nq}V(M)^{1-\frac{2}{N}}\left(\max\left((1+\epsilon)K_o,A_\epsilon\right)\right)^{\frac{q}{2}}\|u\|^q_{H_2^2(M)}$$

and still taking account of the equivalence of the norms $\|.\|_{H_2^2(M)}$ and $\|.\|$, we obtain

$$J_\lambda(u)\geq\left(\frac{N-2}{2N}-\lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{2}{N}}\left(\max\left((1+\epsilon)K_o,A_\epsilon\right)\right)^{\frac{q}{2}}\rho^{q-2}\right)\|u\|^2$$

where $\Lambda>0$ is a constant.

Hence if

$$0<\lambda<\frac{\frac{N-2}{2(N-q)}\Lambda^{-\frac{q}{2}}}{V(M)^{1-\frac{2}{N}}\left(\max\left((1+\epsilon)K_o,A_\epsilon\right)\right)^{\frac{q}{2}}\rho^{q-2}}$$

then

$$J_\lambda(u)>0$$

for any $u\in M_\lambda$.

Since $J^{\pm}_\lambda(u)\geqslant J_\lambda(u)$ we have the conclusion for $J^{\pm}_\lambda(u)$. □

Lemma 3. *Let (M,g) be a n-Riemannian manifold, $n\geq 5$. The following assertions are true*

(i) $(\nabla Q_\lambda(u),u)\leq -A<0$ *(resp.* $\left(\nabla Q^{\pm}_\lambda(u),u\right)\leq -A<0$*), for $u\in M_\lambda$ and any $\lambda\in(0,\min(\lambda_o,\lambda_1))$*

(ii) *The critical points of J_λ (resp. $J^{\pm}_\lambda$) are points of M_λ (resp. $M^{\pm}_\lambda$).*

Proof. (i) Let $u \in M_\lambda$, then

$$\|u\|^2 = \lambda \int_M |u|^q dv_g + \int_M f(x)|u|^N dv_g$$

and

$$\begin{aligned}\langle \nabla Q_\lambda(u), u\rangle &= 2\|u\|^2 - \lambda q \int_M |u|^q dv_g - N \int_M f(x)|u|^N dv_g \\ &= 2\|u\|^2 - \lambda q \int_M |u|^q dv_g - N \left(\|u\|^2 - \lambda \int_M |u|^q dv_g\right) \\ &= (2-N)\|u\|^2 + \lambda (N-q)\|u\|_q^q.\end{aligned}$$

The combination of Hölder and Sobolev inequalities allows us to write

$$\begin{aligned}\langle \nabla Q_\lambda(u), u\rangle \leq & (2-N)\|u\|^2 + \lambda (N-q) V(M)^{1-\frac{2}{N}} \left(\max (1+\epsilon) K_o, A_\epsilon\right)^{\frac{q}{2}} \\ & \times \|u\|^q_{H_2^2(M)}\end{aligned}$$

and since the norms $\|.\|$ and $\|.\|_{H_2^2(M)}$ are equivalent, we get

$$\begin{aligned}\langle \nabla Q_\lambda(u), u\rangle \leq & \left((2-N) + \lambda (N-q) V(M)^{1-\frac{2}{N}} \Lambda^{q-2} \left(\max (1+\epsilon) K_o, A_\epsilon\right)^{\frac{q}{2}} \rho^{q-2}\right) \\ & \times \|u\|^2.\end{aligned}$$

Hence if

$$0 < \lambda < \lambda_1 = \frac{\frac{N-2}{2(N-q)} \Lambda^{-\frac{q}{2}}}{V(M)^{1-\frac{2}{N}} \left(\max (1+\epsilon) K_o, A_\epsilon\right)^{\frac{q}{2}} \rho^{q-2}}$$

then for any $u \in M_\lambda$.

$$\langle \nabla Q_\lambda(u), u\rangle < 0$$

(ii) By the Lagrange multiplicators theorem we get the existence of a real number μ such that for any $u \in M_\lambda$

$$\nabla J_\lambda(u) = \mu \nabla Q_\lambda(u)$$

and by testing at the point $u \in M_\lambda$, we obtain

$$Q_\lambda(u) = \langle \nabla J_\lambda(u), u\rangle = \mu \langle \nabla Q_\lambda(u), u\rangle$$

and since $\langle \nabla Q_\lambda(u), u\rangle < 0$, we get necessarily that $\mu = 0$;

Hence for any $u \in M_\lambda$

$$\nabla J_\lambda(u) = 0.$$

The same computations are carried to conclude for $\left\langle \nabla Q_\lambda^\pm(u), u\right\rangle$ and $J_\lambda^\pm$. □

Lemma 4. *Let* $(u_m)_m$ *be a sequence in* M_λ *(resp.* $M_\lambda^\pm$*) such that*

$$J_\lambda(u_m) \leq c \text{ (resp. } J_\lambda^\pm(u_m) \leq c)$$

and

$$\nabla J_\lambda(u_m) - \mu_m \nabla Q_\lambda(u_m) \to 0 (\text{resp. } \nabla J_\lambda^\pm(u_m) - \mu_m \nabla Q_\lambda^\pm(u_m) \to 0).$$

Suppose that

$$c < \frac{2}{nK_o^{\frac{n}{4}} \max_{x\in M} f(x)^{\frac{n}{4}-1}}$$

then there is a subsequence of $(u_m)_m$ converging strongly in $H_2^2(M)$.

Proof. Let $(u_m)_m \subset M_\lambda$

$$J_\lambda(u_m) = \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\int_M |u_m|^q dv(g)$$

We have

$$J_\lambda(u_m) \geq \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|u_m\|^q$$

$$J_\lambda(u_m) \geq \|u_m\|^2\left(\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \times \|u_m\|^{q-2}\right)$$

with $0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{-\frac{q}{2}}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|u\|^{q-2}}$.

On the other hand, we have

$$c \geq J_\lambda(u_m)$$
$$\geq \left[\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|u_m\|^{q-2}\right]\|u_m\|^2 > 0$$

hence

$$0 \leq \|u_m\|^2 \leq \frac{c}{\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\rho^{q-2}} < +\infty.$$

So $(u_m)_m$ is bounded in $H_2^2(M)$. Since $H_2^2(M)$ is reflexive and the embedding $H_2^2(M) \subset H_p^k(M)$ $(k = 0, 1;\ p < N)$ is compact and we have

- $u_m \to u$ weakly in $H_2^2(M)$.
- $u_m \to u$ strongly in $H_p^k(M)$; $p < N$.
- $u_m \to u$ a.e. in M.

The Brezis–Lieb lemma [7] allows us to write

$$\int_M |\Delta u_m|^2 dv_g = \int_M |\Delta u|^2 dv_g + \int_M |\Delta(u_m - u)|^2 dvg + o(1)$$

and also

$$\int_M f(x)|u_m|^N dv_g = \int_M f(x)|u|^N dv_g + \int_M f(x)|u_m - u|^N dv_g + o(1).$$

We claim that $u \in M_\lambda$. Indeed since

$u_m \to u$ weakly in $H_2^2(M)$, we have for any $\phi \in H_2^2(M)$,

$$\int_M (\Delta u_m\Delta\phi - a(x)\langle\nabla u_m, \nabla\phi\rangle + b(x)u_m\phi)\, dv_g$$
$$= \int_M (\Delta u\Delta\phi - a(x)\langle\nabla u, \nabla\phi\rangle + b(x)u\phi)\, dv_g + o(1)$$

and in particular if we let $\phi = u$,

$$\int_M (\Delta u_m \Delta u - a(x) \langle \nabla u_m, \nabla u \rangle + b(x) u_m u) \, dv_g = \|u\|^2 + o(1)$$

and also if we put $\phi = u_m$, we obtain

$$\int_M \left(\Delta u_m \Delta u_m - a(x) \langle \nabla u_m, \nabla u_m \rangle + b(x) u_m^2\right) dv_g = \|u_m\|^2 + o(1).$$

Now since $(u_m)_m$ belongs to M_λ, we get

$$\int_M \left(\lambda |u_m|^{q-2} u_m u + f(x) |u_m|^{N-2} u_m u\right) dv(g) \; = \|u\|^2 + o(1)$$

and by letting $m \to +\infty$

$$\int_M \left(\lambda |u_m|^{q-2} u_m u + f(x) |u_m|^{N-2} u_m u\right) dv(g) \to \int_M \left(\lambda |u|^q + f(x) |u|^N\right) dv(g).$$

Hence

$$\Phi_\lambda(u_m) = \Phi_\lambda(u) = \|u\|^2 - \lambda \int_M |u|^q dv(g) - \int_M f(x) |u|^N dv(g) = 0$$

and we have

$$\|u\| + o(1) = \|u_m\| \geq \rho.$$

Consequently $u \in M_\lambda$.

Also we claim that $\mu_m \to 0$ as $n \to +\infty$ in fact testing with u_m, we get

$$\begin{aligned} &\langle \nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m), u_m \rangle = o(1) \\ &\quad = \underbrace{\langle \nabla J_\lambda(u_m), u_m \rangle}_{=0} - \mu_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle = o(1). \end{aligned}$$

hence

$$\mu_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle = o(1)$$

and by Lemma 3, we have

$$\limsup_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle < 0$$

so $\mu_m \to 0$ as $m \to +\infty$.

We are going to show now that $u_m \to u$ converges strongly in $H_2^2(M)$. First we have

$$\begin{aligned} &J_\lambda(u_m) - J_\lambda(u) \\ &\quad = \frac{1}{2} \int_M (\Delta(u_m - u))^2 \, dv_g - \frac{1}{N} \int_M f(x) |u_m - u|^N dv_g + o(1) \end{aligned} \tag{2.1}$$

and since $u_m - u \to 0$ converges weakly in $H_2^2(M)$, by testing $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$, we get

$$\begin{aligned} &\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u \rangle = o(1) \\ &\quad = \int_M (\Delta(u_m - u))^2 \, dv_g - \int_M f(x) |u_m - u|^N dv_g = o(1) \end{aligned}$$

that is to say

$$\int_M (\Delta_g(u_m-u))^2\,dv_g = \int_M f(x)|u_m-u|^N dv_g + o(1) \tag{2.2}$$

hence taking account of (2.1), we obtain

$$J_\lambda(u_m) - J_\lambda(u) = \frac{2}{n}\int_M (\Delta(u_m-u))^2\,dv_g + o\,(1).$$

The Sobolev inequality allows us to write

$$\|u_m-u\|_N^2 \le (1+\varepsilon)K_\circ \int_M (\Delta(u_m-u))^2\,dv_g + o(1)$$

so

$$\int_M f(x)|u_m-u|^N dv_g \le (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta(u_m-u)\|_2^N + o(1). \tag{2.3}$$

Taking account of equality (2.2), one writes

$$\begin{aligned} o(1) &\ge \|\Delta(u_m-u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta(u_m-u\|_2^N + o(1) \\ &\ge \|\Delta(u_m-u)\|_2^2 \left(1-(1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta(u_m-u)\|_2^{N-2}\right) + o(1). \end{aligned}$$

Hence if

$$\limsup_m \|\Delta(u_m-u)\|_2^{N-2} < \frac{1}{((1+\varepsilon)K_o)^{\frac{n}{n-4}} \max_{x\in M} f(x)}$$

we get

$$\frac{2}{n}\int_M |\Delta_g(u_m-u)|^2 dv(g) < c.$$

Since

$$c < \frac{2}{n\ K_\circ^{\frac{n}{4}} (\max\ _{x\in M} f(x))^{\frac{n-4}{4}}}$$

it follows that

$$\int_M (\Delta(u_m-u))^2\,dv_g < \frac{1}{K_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}}.$$

Consequently

$$o(1) \ge \|\Delta(u_m-u)\|_2^2 \underbrace{(1-(1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta(u_m-u)\|_2^{N-2})}_{>0} + o(1)$$

which shows that

$$\|\Delta(u_m-u)\|_2^2 = o(1)$$

and $u_m \to u$ converges strongly in $H_2^2(M)$. □

Theorem 6. *Let (M, g) be an n-dimensional compact Riemannian manifold with $n \geqslant 6$ and f be a smooth positive function. Assume that the operator $P_g(u) = \Delta^2 u + \bigtriangledown^i (a(x) \bigtriangledown_i u) + b(x)u$ is coercive and*

$$c < \frac{2}{nK_o^{\frac{n}{4}} \max_{x \in M} f(x)^{\frac{n}{4}-1}}$$

Then there exists a $\Lambda > 0$ such that for any $\lambda \in (0, \Lambda)$, Eq. (1.1) have a non trivial weak solution.

Proof. According to Lemmas 2, 3 and 4, we infer the existence of $v \in M_\lambda$ such that $J_\lambda(v) = \min_{u \in M_\lambda} J_\lambda$. So there is a real μ such that

$$\bigtriangledown J_\lambda(v) = \mu \bigtriangledown Q_\lambda(v)$$

and multiplying by v and taking account of Lemma 3 we obtain that $\mu = 0$. Hence v is a non solution of Eq. (1.1). □

3. Multiplicity of solutions in case of constant coefficients

When P_g has constant coefficients, we set

$$J_\lambda^+(u) = \frac{1}{2}\left(\|\Delta u\|_2^2 - \alpha \int_M |\nabla u|^2 dv_g + \beta \int_M u^2 dv_g\right) - \frac{\lambda}{q}\int_M \left(u^+\right)^q dv_g$$
$$-\frac{1}{N}\int_M f(x)\left(u^+\right)^N dv_g$$

where

$$u^+ = \max(u, 0)$$

Critical points of J_λ^+ are solutions to

$$\Delta^2 u + \alpha \Delta u + \beta u = \lambda\left(\left(u^+\right)^{q-2} + f\left(u^+\right)^{N-2}\right) u^+. \tag{3.1}$$

Similar arguments as the ones used in the precedent sections give that J_λ^+ has a critical point u. Standard arguments show that u is of class $C^{4,\theta}$ with $\theta \in (0, 1)$. If $\alpha^2 - 4\beta > 0$, we let $x_1 = \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2}$ and $x_2 = \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2}$ and moreover if $\alpha > 0$, then x_1, $x_2 > 0$ and

$$(\Delta + x_1)(\Delta + x_2) u = \Delta^2 u + \alpha \Delta u + \beta u \geq 0.$$

Applying the maximum principle twice, we obtain that u is a positive solution of class $C^{4,\theta}$, where $\theta \in (0, 1)$ of the equation

$$\Delta^2 u + \alpha \Delta u + \beta u = \lambda\left(u^{q-1} + f u^{N-1}\right).$$

and standard regularity results give that u is smooth.

In the same manner if we set

$$J_\lambda^-(u) = \frac{1}{2}\left(\|\Delta u\|_2^2 - \alpha \int_M |\nabla u|^2 dv_g + \beta \int_M u^2 dv_g\right) - \frac{\lambda}{q}\int_M |u^-|^q dv_g$$
$$-\frac{1}{N}\int_M f(x)|u^-|^N dv_g$$

where

$$u^- = \min(u, 0)$$

then the critical points of J_λ^- are solutions to

$$\Delta^2 u + \alpha\Delta u + \beta u = \lambda(|u^-|^{q-2} + f|u^-|^{N-2})u^-.$$

By the same argument as above we get that u^- is a negative smooth solution. Similar arguments as the ones we used for J_λ give that J_λ^+ and J_λ^- have critical points M_λ^+ and M_λ^- respectively where

$$M_\lambda^\pm = \left\{u \in H_2^2 : Q_\lambda^\pm(u) = 0 \text{ and } \|u\| \geq \rho > 0\right\}$$

and

$$Q_\lambda^\pm(u) = \langle \bigtriangledown J_\lambda^\pm(u), u\rangle.$$

Summarizing, we get

Theorem 7. *Let (M, g) be an n-dimensional compact Riemannian manifold with $n \geqslant 6$. Assume that the operator $P_g(u) = \Delta^2 u + \bigtriangledown^i(a(x)\bigtriangledown_i u) + b(x)u$ is coercive and*

$$c < \frac{2}{nK_o^{\frac{n}{4}} \max_{x\in M} f(x)^{\frac{n}{4}-1}}.$$

If moreover $\frac{\alpha^2}{4} > \beta$ with $\alpha > 0$. Then Eq. (1.6) has two distinct smooth solutions; one positive and the other negative.

Lemma 5. *For any $\lambda > 0$, sufficiently small, J_λ has two local minima.*

Proof. We follow closely the proof of Lemma 8 in [2]. As a consequence of Lemmas 2, 3 and 5, we infer the existence of $v_1 \in M_\lambda^+$ and a $v_2 \in M_\lambda^-$ such that

$$J_\lambda^+(v_1) = \min_{u\in M_\lambda^+} J_\lambda^+(u)$$

and

$$J_\lambda^-(v_2) = \min_{u\in M_\lambda^-} J_\lambda^-(u).$$

Note that v_1 and v_2 are respectively smooth positive and negative solutions of Eq. (1.6). Indeed by Lagrange mutiplicators theorem we get that

$$\bigtriangledown J_\lambda^+(v_1) = \mu \bigtriangledown Q_\lambda^+(v_1)$$

and multiplying by v_1 we deduce that

$$\mu = 0$$

Hence v_1 is a solution of (3.1) and as in Sect. 3 we get that v_1 positive, hence a positive solution of Eq. (1.6). v_2 is actually a negative solution of (1.6). We claim that v_1 and v_2 are local minima of J_λ if it is not the case let $w_n \in M_\lambda$ such that $w_n \to v_1$ in H_2^2 as $n \to +\infty$ and

$$J_\lambda(w_n) < J_\lambda^+(v_1) \tag{3.2}$$

We can choose w_n as

$$J_\lambda(w_n) = \inf_{u \in B_n \cap M_\lambda} J_\lambda(u) \tag{3.3}$$

where $B_n = \left\{u \in \|u - v_1\|_{H_1^2} \le \frac{1}{n}\right\}$. There exist parameters λ_n and μ_n such that

$$\triangledown J_\lambda(w_n) = \lambda_n \triangledown Q_\lambda(w_n) + \mu_n \left(\Delta^2 w_n + \alpha \Delta w_n + \beta w_n\right) \tag{3.4}$$

with $\mu_n \le 0$. Taking the inner product of the latter equality with w_n, we get

$$\lambda_n \langle \triangledown Q_\lambda(w_n), w_n \rangle + \mu_n \|w_n\|^2_{H_1^2} = 0$$

and we infer that $\lambda_n \le 0$.

Equation reads as

$$\Delta^2 w_n + \alpha \Delta w_n + \beta w_n = \frac{(-\lambda_n - \mu_n)}{(1 - \lambda_n - \mu_n)} f |w_n|^{N-2} w_n.$$

By standard methods, w_n is of class $C^{4,\theta}$, $0 < \theta < 1$. Hence w_n goes to v_1 in the C^2 topology, then $w_n > 0$. So (3.3) is a contradiction with (3.2). Hence v_1 and v_2 are respectively positive and negative solution of Eq. (1.6) of minimal positive energy. □

Next we prove

Theorem 8. *Let (M, g) be an n-dimensional compact Riemannian manifold with $n \geqslant 6$. Assume that the operator $P(u) = \Delta^2 u + \triangledown^i (a(x) \triangledown_i u) + b(x)u$ is coercive and*

$$c < \frac{2}{n K_o^{\frac{n}{4}} \max_{x \in M} f(x)^{\frac{n}{4}-1}}.$$

If moreover $\frac{\alpha^2}{4} > \beta$ with $\alpha > 0$. Then Eq. (1.6) has third solution w distinct of u^+ and u^-.

Proof. We can suppose that the minima of J_λ are realized by u^+ and u^-. The geometric conditions of the Mountain pass theorem are satisfied. If Γ denotes the set of paths $\gamma : [0, 1] \to M_\lambda$ such that $\gamma(0) = u^-$ and $\gamma(1) = u^+$. Let $c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} (J_\lambda(\gamma))$). By Lemma 4, we infer that c_λ is a critical level of the function J_λ with critical value w and by Lemma 1.3 $w \in M_\lambda$. Hence w is solution of Eq. (1.6) different from u^+ and u^-. □

4. Test functions

In this section we give the proof of Theorems 4 and 5.

Let $(y^1, \ldots, y^n)$ be normal coordinates centred at the point $x_\circ$ where the function attains its maximum and $S(r)$ be the geodesic sphere centred at x_o and of radius r ($r < d$ the injectivity radius). Denote by $d\sigma$ the volume element of the $(n-1)$-dimensional unit S^{n-1}.

Put

$$G(r) = \frac{1}{w_{n-1}} \int \int_{S(r)} \sqrt{|g(x)|} d\sigma$$

where w_{n-1} denotes the area of S^{n-1} and $|g(x)|$ the determinant of the metric g. An expansion of $G(r)$ in a neighborhood of $r=0$ writes as

$$G(r)=1-\frac{S_g(x_\circ)}{6n}r^2+o(r^2)$$

where $S_g(x_\circ)$ denotes the scalar curvature of M at the point $x_\circ$.

Let $B(x_\circ,\delta)$ be the ball centred at $x_\circ$ and of radius δ with $0<2\delta<d$ and let η be a smooth function equals to 1 on $B(x_o,\delta)$ and equals to 0 on $M-B(x_o,2\delta)$.

Put

$$u_\epsilon(x)=\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{8}}\frac{\eta(r)}{(r^2+\epsilon^2)^{\frac{n-4}{2}}}$$

where

$$f(x_\circ)=\max_{x\in M}f(x)$$

and $r=d(x_\circ,.)$ is geodesic distance to the point $x_\circ$.

We let, for $p-q>1$,

$$I_p^q=\int_0^{+\infty}\frac{t^q}{(1+t)^p}dt$$

which fulfills

$$I_{p+1}^q=\frac{p-q-1}{p}I_p^q\quad\text{and}\quad I_{p+1}^{q+1}=\frac{q+1}{p-q-1}I_{p+1}^q.$$

In the case where the dimension of the manifold $n>6$, we have

Theorem 9. *Let (M,g) be an n-dimensional compact Riemannian manifold with $n>6$. If at the point x_o where the function f achieves its maximum*

$$\frac{n\left(n^2+4n-20\right)}{2\left(n-6\right)\left(n^2-4\right)}S_g\left(x_o\right)+\frac{n\left(n-1\right)}{\left(n-6\right)\left(n^2-4\right)}a\left(x_o\right)-\frac{n}{8\left(n-2\right)}\frac{\Delta f(x_o)}{f\left(x_o\right)}>0$$

Eq. (1.1) have a non trivial solution of class $C^{4,\theta}(M)$, $\theta\in(0,1)$.

Proof. As in [8], we get

$$\int_M f(x)|u_\epsilon(x)|^N dv(g)=\frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1-\left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)}+\frac{S_g(x_\circ)}{6(n-2)}\right)\epsilon^2+o(\epsilon^2)\right)$$

and also

$$\int_M a(x)|\nabla u_\epsilon|^2dv(g)=\frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(\frac{4(n-1)a(x_\circ)}{(n^2-4)(n-6)}\epsilon^2+o(\epsilon^2)\right).$$

The computations give

$$\int_M b(x)u_\epsilon^2dv(g)=o(\epsilon^2)$$

and

$$\int_M |\Delta u_\epsilon|^2 dv(g) = \frac{1}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \left(1 - \frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_\circ)\epsilon^2 + o(\epsilon^2)\right).$$

Summarizing we obtain

$$\int_M |\Delta u_\epsilon|^2 - a(x)|\nabla u_\epsilon|^2 + b(x)u_\epsilon^2 dv(g) = \frac{1}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}$$
$$\left(1 - \left(\frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_\circ) + \frac{4(n-1)}{(n^2-4)(n-6)} a(x_\circ)\right)\epsilon^2 + o(\epsilon^2)\right).$$

Taking in mind that

$$J_\lambda(u_\epsilon) = \frac{1}{2}\|u_\epsilon\|^2 - \frac{\lambda}{q}\|u_\epsilon\|_q^q - \frac{1}{N}\int_M f(x)|u_\epsilon(x)|^N dv(g)$$

where

$$\|u_\epsilon\|^2 = \int_M |\Delta u_\epsilon|^2 - a(x)|\nabla u_\epsilon|^2 + b(x)u_\epsilon^2 dv(g)$$

and since $\lambda > 0$, we get

$$\begin{aligned} J_\lambda(u_\epsilon) &\leq \frac{1}{2}\|u_\epsilon\|^2 - \frac{1}{N}\int_M f(x)|u_\epsilon(x)|^N dv(g) \\ &\leq \frac{1}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \left[\frac{2}{n} - \left(\frac{n^2+4n-20}{(n^2-4)(n-6)} S_g(x_\circ)\right.\right. \\ &\quad \left.\left. + \frac{2(n-1)}{(n^2-4)(n-6)} a(x_\circ) - \frac{1}{4(n-2)} \frac{\Delta f(x_\circ)}{f(x_\circ)}\right)\epsilon^2 + o(\epsilon^2)\right] \\ &\leq \frac{2}{n\ K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \left[1 - \left(\frac{(n^2+4n-20)n}{2(n^2-4)(n-6)} S_g(x_\circ)\right.\right. \\ &\quad \left.\left. + \frac{(n-1)\ n}{(n^2-4)(n-6)} a(x_\circ) - \frac{n}{8(n-2)} \frac{\Delta f(x_\circ)}{f(x_\circ)}\right)\epsilon^2 + o(\epsilon^2)\right]. \end{aligned}$$

So the condition

$$J_\lambda(u_\epsilon) < \frac{2}{n\ K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}$$

is fulfilled if

$$\left(\frac{(n^2+4n-20)n}{2(n^2-4)(n-6)} S_g(x_\circ) + \frac{(n-1)\ n}{(n^2-4)(n-6)} a(x_\circ) - \frac{n}{8(n-2)} \frac{\Delta f(x_\circ)}{f(x_\circ)}\right) > 0.$$

In the case $n = 6$, the same calculations as in case $n > 6$ lead to

$$\begin{aligned} \int_M f(x)|u_\epsilon(x|^N dv(g) = \frac{1}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} &\left(1 - \left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-2)}\right)\right. \\ &\left.\times\ \epsilon^2 + o(\epsilon^2)\right). \end{aligned} \tag{4.1}$$

Also the same computations as in [8] with minor modifications allow us to write

$$\int_M a(x)|\nabla u_\epsilon|^2 dv(g) = (n-4)^2 \left(\frac{(n-4)n(n^2-4)}{f(x_\circ)}\right)^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \left(a(x_\circ)\epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) + O(\epsilon^2)\right)$$

and

$$\int_M |\Delta u_\epsilon|^2 dv(g) = (n-4)^2 \left(\frac{(n-4)n(n^2-4)}{f(x_\circ)}\right)^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \times \left(\frac{n(n+2)(n-2)}{(n-4)} I_n^{\frac{n}{2}-1} - \frac{2}{n} S_g(x_\circ)\epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) + O(\epsilon^2)\right).$$

Consequently

$$\begin{aligned}\int_M (\Delta u_\epsilon)^2 - a(x)|\nabla u_\epsilon|^2 + b(x)u_\epsilon^2 dv(g) &= (n-4)^2 \left(\frac{(n-4)n(n^2-4)}{f(x_\circ)}\right)^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \\ &\quad\times \left[\frac{n(n+2)(n-2)}{(n-4)} I_n^{\frac{n}{2}-1} - \left(\frac{2}{n} S_g(x_\circ) + a(x_\circ)\right)\epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) + O(\epsilon^2)\right] \\ &= \frac{1}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \left(1 - \frac{(n-4)}{n(n^2-4) I_n^{\frac{n}{2}-1}} \left(\frac{2}{n} S_g(x_\circ) + a(x_\circ)\right)\epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) + O(\epsilon^2)\right)\end{aligned}$$

and taking account of (4.1), we obtain

$$\begin{aligned}J_\lambda(u_\epsilon) &\le \frac{1}{2}\|u_\epsilon\|^2 - \frac{1}{N}\int_M f(x)|u_\epsilon(x)|^N dv(g) \\ &\le \frac{1}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \left(\frac{1}{2} - \frac{1}{N} - \frac{(n-4)\, t_o^2}{2n(n^2-4) I_n^{\frac{n}{2}-1}} \left(\frac{2}{n} S_g(x_\circ) + a(x_\circ)\right) \times \epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) + O(\epsilon^2)\right).\end{aligned}$$

So if in the point x_o where the maximum of the function f is achieved, the condition $\frac{2}{n}S_g(x_\circ) + a(x_\circ) > 0$ i.e. since $n = 6$, $S_g(x_\circ) > -3a(x_\circ)$ is fulfilled, we get for ϵ sufficiently small

$$J_\lambda(u_\epsilon) < \frac{2}{nK_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}.$$

□

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EXISTENCE OF SOLUTIONS TO SINGULAR FOURTH-ORDER ELLIPTIC EQUATIONS

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ABSTRACT. Using a method developed by Ambrosetti et al [1, 2] we prove the existence of weak non trivial solutions to fourth-order elliptic equations with singularities and with critical Sobolev growth.

1. INTRODUCTION

Fourth-order elliptic equations have been widely studied, because of their importance in the analysis on manifolds particularly those involving the Paneitz-Branson operators; see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 16]. Different techniques have been used for solving fourth-order equations, as example the variational method which was developed by Yamabe to solve the problem of the prescribed scalar curvature. Let (M, g) a compact smooth Riemannian manifold of dimension $n \geq 5$ with a metric g. We denote by $H_2^2(M)$ the standard Sobolev space which is the completed of the space $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{2,2} = \sum_{k=0}^{k=2} \|\nabla^k \varphi\|_2 .$$

$H_2^2(M)$ will be endowed with the suitable equivalent norm

$$\|u\|_{H_2^2(M)} = \Big(\int_M ((\Delta_g u)^2 + |\nabla_g u|^2 + u^2) dv_g \Big)^{1/2}.$$

In 1979, Vaugon [17] proved the existence of a positive value λ and a non trivial solution $u \in C^4(M)$ to the equation

$$\Delta_g^2 u - \operatorname{div}_g(a(x)\nabla_g u) + b(x)u = \lambda f(t, x)$$

where a, b are smooth functions on M and $f(t, x)$ is odd and increasing function in t fulfilling the inequality

$$|f(t, x)| < a + b|t|^{\frac{n+4}{n-4}}.$$

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Edminds, Fortunato and Jannelli [14] showed that all the solutions in $\mathbb{R}^n$ to the equation

$$\Delta^2 u = u^{\frac{n+4}{n-4}}$$

are positive, symmetric, radial and decreasing functions of the form

$$u_\epsilon(x) = \frac{((n-4)n(n^2-4)\epsilon^4)^{\frac{n-4}{8}}}{(r^2+\epsilon^2)^{\frac{n-4}{2}}}.$$

In 1995, Van Der Vorst [15] obtained the same results for the problem

$$\begin{gathered} \Delta^2 u - \lambda u = u|u|^{\frac{8}{n-4}} \quad \text{in } \Omega, \\ \Delta u = u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a bounded domain of $\mathbb{R}^n$.

In 1996, Bernis, Garcia-Azorero and Peral [9] obtained the existence at least of two positive solutions to the problem

$$\begin{gathered} \Delta^2 u - \lambda u|u|^{q-2} = u|u|^{\frac{8}{n-4}} \text{ in } \Omega, \\ \Delta u = u = 0 \text{ on } \partial\Omega, \end{gathered}$$

where Ω is bounded domain of $\mathbb{R}^n$, $1 < q < 2$ and $\lambda > 0$ in some interval. In 2001, Caraffa [12] obtained the existence of a non trivial solution of class $C^{4,\alpha}$, $\alpha \in (0,1)$ for the equation

$$\Delta_g^2 u - \nabla^\alpha(a(x)\nabla_\alpha u) + b(x)u = \lambda f(x)|u|^{N-2}u$$

with $\lambda > 0$, first for f a constant and next for a positive function f on M.

Recently the first author [4] showed the existence of at least two distinct non trivial solutions in the subcritical case and a non trivial solution in the critical case for the equation

$$\Delta_g^2 u - \nabla^\alpha(a(x)\nabla_\alpha u) + b(x)u = f(x)|u|^{N-2}u$$

where f is a changing sign smooth function and a and b are smooth functions. In [6] the same author proved the existence of at least two non trivial solutions to

$$\Delta_g^2 u - \nabla^\alpha(a(x)\nabla_\alpha u) + b(x)u = f(x)|u|^{N-2}u + |u|^{q-2}u + \varepsilon g(x)$$

where a, b, f, g are smooth functions on M with $f > 0$, $2 < q < N$, $\lambda > 0$ and $\epsilon > 0$ small enough. Let S_g denote the scalar curvature of M. In 2011, the authors proved the following result

Theorem 1.1 ([8]). *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 6$ and a, b, f smooth functions on M, $\lambda \in (0,\lambda_*)$ for some specified $\lambda_* > 0$, $1 < q < 2$ such that*

1. *$f(x) > 0$ on M.*
2. *At the point x_0 where f attains its maximum, we suppose that for $n = 6$, $S_g(x_0) + 3a(x_0) > 0$, and for $n > 6$*

$$\Big(\frac{(n^2+4n-20)}{2(n+2)(n-6)}S_g(x_0) + \frac{(n-1)}{(n+2)(n-6)}a(x_0) - \frac{1}{8}\frac{\Delta f(x_0)}{f(x_0)}\Big) > 0.$$

Then the equation

$$\Delta_g^2 u + \operatorname{div}_g(a(x)\nabla_g u) + b(x)u = \lambda|u|^{q-2}u + f(x)|u|^{N-2}u$$

admits a non trivial solution of class $C^{4,\alpha}(M)$, $\alpha \in (0,1)$.

Recently Madani [14] studied the Yamabe problem with singularities when the metric g admits a finite number of points with singularities and is smooth outside these points. More precisely, let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$, we denote by T^*M the cotangent space of M. The space $H_2^p(M, T^*M \otimes T^*M)$ is the set of sections s (2-covariant tensors) such that in normal coordinates the components s_{ij} of s are in H_2^p the complement of the space $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\varphi\|_{2,p} = \sum_{k=0}^{k=2} \|\nabla^k \varphi\|_p$.

Solving the singular Yamabe problem is equivalent to finding a positive solution $u \in H_2^p(M)$ of the equation

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = k|u|^{N-2}u, \tag{1.1}$$

where S_g is the scalar curvature of the g and k is a real constant. The Christoffels symbols belong to $H_1^p(M)$, the Riemannian curvature tensor, the Ricci tensor Ric_g and scalar curvature S_g are in $L^p(M)$, hence equation 1.1 is the singular Yamabe equation.

Under the assumptions that g is a metric in the Sobolev space $H_2^p(M, T^*M \otimes T^*M)$ with $p > n/2$ and that there exist a point $P \in M$ and $\delta > 0$ such that g is smooth in the ball $B_p(\delta)$, Madani [14] proved the existence of a metric $\overline{g} = u^{N-2}g$ conformal to g such that $u \in H_2^p(M)$, $u > 0$ and the scalar curvature $S_{\overline{g}}$ of $\overline{g}$ is constant if (M,g) is not conformal to the round sphere.

The author in [7] considered fourth-order elliptic equations, with singularities, of the form

$$\Delta^2 u - \nabla^i(a(x)\nabla_i u) + b(x)u = f|u|^{N-2}u \tag{1.2}$$

where the functions a and b are in $L^s(M)$, $s > \frac{n}{2}$ and in $L^p(M)$, $p > \frac{n}{4}$ respectively, $N = \frac{2n}{n-4}$ is the Sobolev critical exponent in the embedding $H_2^2(M) \hookrightarrow L^N(M)$. He established the following results. Let (M,g) be a compact n-dimensional Riemannian manifold, $n \geq 6$, $a \in L^s(M)$, $b \in L^p(M)$, with $s > \frac{n}{2}$, $p > \frac{n}{4}$, $f \in C^\infty(M)$ a positive function and $x_0 \in M$ such that $f(x_0) = \max_{x \in M} f(x)$.

Theorem 1.2. *For $n \geq 10$, or $n = 8, 9$ and $2 < p < 5$, $\frac{9}{4} < s < 11$ or $n = 7$, $\frac{7}{2} < s < 9$ and $\frac{7}{4} < p < 9$ we suppose that*

$$\frac{n^2+4n-20}{6(n-6)(n^2-4)} S_g(x_0) - \frac{n-4}{2n(n-2)} \frac{\Delta f(x_0)}{f(x_0)} > 0.$$

For $n = 6$ and $\frac{3}{2} < p < 2$, $3 < s < 4$, we assume that

$$S_g(x_0) > 0.$$

Then (1.2) has a non trivial weak solution u in $H_2^2(M)$. Moreover if $a \in H_1^s(M)$, then $u \in C^{0,\beta}(M)$, for some $\beta \in (0, 1 - \frac{n}{4p})$.

In this article, we extend results obtained in Theorem 1.1 to the case of singular elliptic fourth order, more precisely we are concerned with the following problem: Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$. Let $a \in L^r(M)$, $b \in L^s(M)$ where $r > \frac{n}{2}$, $s > \frac{n}{4}$ and f a positive C^∞-function on M; we look for non trivial solution of the equation

$$\Delta_g^2 u + \operatorname{div}_g(a(x)\nabla_g u) + b(x)u = \lambda|u|^{q-2}u + f(x)|u|^{N-2}u \tag{1.3}$$

where $1 < q < 2$ and $N = \frac{2n}{n-4}$ is the critical Sobolev exponent and $\lambda > 0$ a real number.

In case the $\lambda = 0$ and

$$a = \frac{4}{n-2} R_{ic_g} - \frac{(n-2)^2+4}{2(n-1)(n-2)} S_g.g, \quad b = \frac{n-4}{2} Q_g^n,$$

where

$$Q_g^n = \frac{1}{2(n-1)} \Delta S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |Ric_g|^2$$

suppose that g is a metric in the Sobolev space $H_4^p(M, T^*M \otimes T^*M)$ with $p > \frac{n}{4}$, then the Ricci Ric_g curvature and the scalar curvature S_g are in the Sobolev spaces $H_2^p(M, T^*M \otimes T^*M)$ and $H_2^p(M)$ respectively, hence $b \in L^s(M)$ with $s > \frac{n}{4}$ and by Sobolev embedding $a \in L^r(M)$ with $r > \frac{n}{2}$. In this latter case the equation

$$\Delta_g^2 u + \operatorname{div}_g(a(x)\nabla_g u) + b(x)u = f(x)|u|^{N-2}u \tag{1.4}$$

is called singular Q-curvature equation.For more general coefficients $a \in L^r(M)$ with $r > \frac{n}{2}$ and $b \in L^s(M)$ with $s > \frac{n}{4}$, the equation (1.4) is called singular Q-curvature type equation. To solve equation (1.3), we use a method developed in [1] and [2] which resumes to study the variations of functional associated to equation 1.3 on the manifold M_λ defined in section 2. Serious difficulties appear compared with the smooth case: considering the equation (4.3) in section 4, we need a Hardy-Sobolev inequality and Releich-Kondrakov embedding on a manifolds. In the case of the singular Yamabe equation theses latters were established in [14] and in the case of singular Q-curvature type equations by the first author in [7]. In the sharp cases (see section 5) the Hardy Sobolev inequality and the Releich-Kondrakov embedding are no more valid so we need an additional assumption with some tricks combined with the Lebesgue dominated convergence theorem.

Denote by P_g the operator defined in the weak sense on $H_2^2(M)$ by $P_g(u) = \Delta^2 u + \operatorname{div}(a\nabla u) + bu$. P_g is called coercive if there exits $\Lambda > 0$ such that for any $u \in H_2^2(M)$

$$\int_M u P_g(u) dv_g \geq \Lambda \|u\|^2_{H_2^2(M)}.$$

Our main result reads as follows.

Theorem 1.3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$ and f a positive function. Suppose that P_g is coercive and at a point x_0 where f attains its maximum the following two conditions hold:*

$$\begin{gathered} \frac{\Delta f(x_0)}{f(x_0)} < \Big(\frac{n(n^2+4n-20)}{3(n+2)(n-4)(n-6)} \frac{1}{(1+\|a\|_r+\|b\|_s)^{n/4}} \\ - \frac{n-2}{3(n-1)} \Big) S_g(x_0) \quad \text{when } n > 6, \\ S_g(x_0) > 0 \quad \text{when } n = 6. \end{gathered} \tag{1.5}$$

Then there is $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, the equation (1.3) has a non trivial weak solution.

For fixed $R \in M$, we define the function ρ on M by

$$\rho(Q) = \begin{cases} d(R,Q) & \text{if } d(R,Q) < \delta(M) \\ \delta(M) & \text{if } d(R,Q) \geq \delta(M) \end{cases} \tag{1.6}$$

where $\delta(M)$ denotes the injectivity radius of M.

For real numbers σ and μ, consider the following equation, in the distribution sense,

$$\Delta^2 u - \nabla^i (\frac{a}{\rho^\sigma} \nabla_i u) + \frac{bu}{\rho^\mu} = \lambda |u|^{q-2} u + f(x)|u|^{N-2} u \tag{1.7}$$

where the functions a and b are smooth on M.

Corollary 1.4. *Let $0 < \sigma < \frac{n}{r} < 2$ and $0 < \mu < \frac{n}{s} < 4$. Suppose that*

$$\frac{\Delta f(x_0)}{f(x_0)} < \frac{1}{3}\Big(\frac{(n-1)n(n^2+4n-20)}{(n^2-4)(n-4)(n-6)} \frac{1}{(1+\|a\|_r+\|b\|_s)^{n/4}} - 1\Big) S_g(x_0)$$

when $n > 6$,

$$S_g(x_0) > 0 \quad \text{when } n = 6.$$

Then there is $\lambda_* > 0$ such that if $\lambda \in (0, \lambda_*)$, the (1.7) possesses a weak non trivial solution $u_{\sigma,\mu} \in M_\lambda$.

In the sharp case $\sigma = 2$ and $\mu = 4$, letting $K(n, 2, \gamma)$ be the best constant in the Hardy-Sobolev inequality given by Theorem 4.1 we obtain the following result.

Theorem 1.5. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$. Let $(u_{\sigma_m,\mu_m})_m$ be a sequence in M_λ such that*

$$J_{\lambda,\sigma,\mu}(u_{\sigma_m,\mu_m}) \leq c_{\sigma,\mu}$$
$$\nabla J_\lambda(u_{\sigma,\mu}) - \mu_{\sigma,\mu} \nabla \Phi_\lambda(u_{\sigma,\mu}) \to 0$$

Suppose that

$$c_{\sigma,\mu} < \frac{2}{n K_0^{n/4} (f(x_0))^{\frac{n-4}{4}}}$$

and

$$1 + a^- \max(K(n, 2, \sigma), A(\varepsilon, \sigma)) + b^- \max(K(n, 2, \mu), A(\varepsilon, \mu)) > 0$$

then the equation

$$\Delta^2 u - \nabla^\mu (\frac{a}{\rho^2} \nabla_\mu u) + \frac{bu}{\rho^4} = f|u|^{N-2} u + \lambda |u|^{q-2} u$$

in the distribution has a weak non trivial solution.

Our paper is organized as follows: in a first section we show that the manifold of constraints is non empty, in the second one we establish a generic existence result to equation 1.3. The third section deals with applications to particular equations which could arise from conformal geometry. In the fourth section and under supplementary assumption we obtain non trivial solution in the critical case. The last section is devoted to tests functions which verify geometric assumptions and by the same way complete the proofs of our claimed theorems in the introduction.

2. The manifold M_λ of constraints is non empty

In this section, we consider on $H_2^2(M)$ the functional

$$J_\lambda(u) = \frac{1}{2} \int_M (|\Delta_g u|^2 - a(x)|\nabla_g u|^2 + b(x) u^2) dv_g - \frac{\lambda}{q} \int_M |u|^q dv_g - \frac{1}{N} \int_M f(x)|u|^N dv_g$$

associated to Equation 1.3. First, we put

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u),\ u \rangle$$

hence

$$\Phi_\lambda(u)=\int_M((\Delta_g u)^2-a(x)|\nabla_g u|^2+b(x)u^2)dv_g-\lambda\int_M|u|^q dv_g-\int_M f(x)|u|^N dv_g.$$

We let

$$M_\lambda=\{u\in H_2^2(M):\ \Phi_\lambda(u)=0 \text{ and } \|u\|\ge\tau>0\}.$$

Proposition 2.1. *The norm*

$$\|u\|=(\int_M|\Delta_g u|^2-a(x)|\nabla_g u|^2+b(x)u^2dv_g)^{1/2}$$

is equivalent to the usual norm on $H_2^2(M)$ if and only if P_g is coercive.

Proof. If P_g is coercive there is $\Lambda>0$ such that for any $u\in H_2^2(M)$,

$$\int_M P_g(u)udv_g\ge\Lambda\|u\|^2_{H_2^2(M)}$$

and since $a\in L^r(M)$ and $b\in L^s(M)$ where $r>\frac{n}{2}$ and $s>\frac{n}{4}$, by Hölder's inequality we obtain

$$\int_M uP_g(u)dv_g\le\|\Delta_g u\|_2^2+\|a\|_{\frac{n}{2}}\|\nabla_g u\|^2_{2^*}+\|b\|_{\frac{n}{4}}\|u\|_N^2$$

where $2^*=2n/(n-2)$.

The Sobolev's inequalities lead to: for any $\eta>0$,

$$\|\nabla_g u\|^2_{2^*}\le\max((1+\eta)K(n,1)^2,A_\eta)\int_M(|\nabla_g^2u|^2+|\nabla_g u|^2)dv_g$$

where $K(n,1)$ denotes the best Sobolev's constant in the embedding $H_1^2(\mathbb{R}^n)\hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n)$, and for any $\epsilon>0$,

$$\|u\|_N^2\le\max((1+\varepsilon)K_0,B_\varepsilon)\|u\|^2_{H_2^2(M)}$$

where in this latter inequality K_0 is the best Sobolev's constant in the embedding $H_1^2(M)\hookrightarrow L^{\frac{2n}{n-2}}(M)$ and B_ϵ the corresponding (see [3]). Now by the well known formula (see [3, page 115])

$$\int_M|\nabla_g^2u|^2dv_g=\int_M(|\Delta_g u|^2-R_{ij}\nabla^i u\nabla^j u)dv_g$$

where R_{ij} denote the components of the Ricci curvature, there is a constant $\beta>0$ such that

$$\int_M|\nabla_g^2u|^2dv_g\le\int_M|\Delta_g u|^2+\beta|\nabla_g u|^2dv_g$$

so we obtain

$$\|\nabla_g u\|^2_{2^*}\le(\beta+1)\max((1+\eta)K(n,1)^2,A_\eta)\int_M(|\Delta_g u|^2+|\nabla_g u|^2+u^2)dv_g$$

and we infer that

$$\begin{aligned}\int_M P_g(u)udv_g\le\|u\|^2_{H_2^2(M)}&+(\beta+1)\|a\|_{\frac{n}{2}}\max((1+\eta)K(n,1)^2,A_\eta)\|u\|^2_{H_2^2(M)}\\&+\|b\|_{\frac{n}{4}}\max((1+\varepsilon)K_0,B_\varepsilon)\|u\|^2_{H_2^2(M)}.\end{aligned}$$

Hence

$$\int_M uP_g(u)dv_g$$

$$\leq \underbrace{\max(1, \|b\|_{\frac{n}{4}} \max((1+\varepsilon)K_0, B_\varepsilon), (\beta+1)\|a\|_{\frac{n}{2}} \max((1+\varepsilon)K(n,1)^2, A_\varepsilon))}_{>0} \times \|u\|^2_{H_2^2(M)}.$$

□

Lemma 2.2. *The set M_λ is non empty provided that $\lambda \in (0, \lambda_0)$ where*

$$\lambda_0 = \frac{(2^{q-2} - 2^{q-N})\Lambda^{\frac{N-q}{N-2}}}{V(M)^{(1-\frac{q}{N})}(\max_{x\in M} f(x))^{\frac{2-q}{N-2}}(\max((1+\varepsilon)K(n,2), A_\varepsilon))^{\frac{N-q}{N-2}}}.$$

Proof. The proof of this lemma is the same as in [8], but we give it here for convenience. Let $t > 0$ and $u \in H_2^2(M) - \{0\}$. Evaluating Φ_λ at tu, we obtain

$$\Phi_\lambda(tu) = t^2\|u\|^2 - \lambda t^q \|u\|_q^q - t^N \int_M f(x)|u|^N dv_g.$$

Put

$$\alpha(t) = \|u\|^2 - t^{N-2}\int_M f(x)|u|^N dv(g),$$
$$\beta(t) = \lambda t^{q-2}\|u\|_q^q;$$

by Sobolev's inequality, we obtain

$$\alpha(t) \geq \|u\|^2 - \max_{x\in M} f(x)(\max((1+\varepsilon)K_0, A_\varepsilon))^{N/2}\|u\|^N_{H_2^2(M)} t^{N-2}.$$

By the coercivity of the operator $P_g = \Delta_g^2 - \operatorname{div}_g(a\nabla_g) + b$ there is a constant $\Lambda > 0$ such that

$$\alpha(t) \geq \|u\|^2 - \Lambda^{-N/2} \max_{x\in M} f(x)(\max((1+\varepsilon)K_0, A_\varepsilon))^{\frac{N}{2}}\|u\|^N t^{N-2}.$$

Letting

$$\alpha_1(t) = \|u\|^2 - \Lambda^{-N/2} \max_{x\in M} f(x)(\max((1+\varepsilon)K_0, A_\varepsilon))^{N/2}\|u\|^N t^{N-2}$$

Hölder and Sobolev inequalities lead to

$$\beta(t) \leq \lambda V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\|u\|^q_{H_2^2(M)} t^{q-2}$$

and the coercivity of P_g assures the existence of a constant $\Lambda > 0$ such that

$$\beta(t) \leq \lambda\Lambda^{-q/2} V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\|u\|^q t^{q-2}.$$

Put

$$\beta_1(t) = \lambda\Lambda^{-q/2} V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\|u\|^q t^{q-2}.$$

Let t_0 such $\alpha_1(t_0) = 0$; i.e.,

$$t_0 = \frac{\Lambda^{\frac{N}{2(N-2)}}}{\|u\|(\max_{x\in M} f(x))^{\frac{1}{N-2}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{\frac{N}{2(N-2)}}}$$

Now since $\alpha_1(t)$ is a decreasing and a concave function and $\beta_1(t)$ is a decreasing and convex function, then

$$\min_{t\in(0,\frac{t_0}{2}]} \alpha_1(t) = \alpha_1(\frac{t_0}{2}) = \|u\|^2(1 - 2^{2-N}) > 0,$$

$$\min_{t\in(0,\frac{t_0}{2}]} \beta_1(t) = \beta_1(\frac{t_0}{2}) > 0,$$

where

$$\beta_1(\frac{t_0}{2}) = \frac{2^{2-q}\lambda V(M)^{(1-\frac{q}{N})}\Lambda^{\frac{q-N}{N-2}}\|u\|^2}{(\max((1+\varepsilon)K_0, A_\varepsilon))^{\frac{q-N}{N-2}}(\max_{x\in M} f(x))^{\frac{q-2}{N-2}}}.$$

Consequently $\Phi_\lambda(tu) = 0$ with $t \in (0, \frac{t_0}{2}]$ has a solution if

$$\min_{t\in(0,\frac{t_0}{2}]} \alpha_1(t) \geq \max_{t\in(0,\frac{t_0}{2}]} \beta_1(t);$$

that is to say

$$0 < \lambda < \frac{(2^{q-2} - 2^{q-N})(\max_{x\in M} f(x))^{\frac{q-2}{N-2}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{\frac{q-N}{N-2}}}{\Lambda^{\frac{N-q}{N-2}}V(M)^{(1-\frac{q}{N})}} = \lambda_0$$

Let $t_1 \in (0, \frac{t_0}{2}]$ such that $\Phi_\lambda(t_1 u) = 0$. If we take $u \in H_2^2(M)$ such that $\|u\| \geq \frac{\rho}{t_1}$ and $v = t_1 u$ we obtain $\Phi_\lambda(v) = 0$ and $\|v\| = t_1\|u\| \geq \rho$; i.e., $v \in M_\lambda$ provided that $\lambda \in (0, \lambda_0)$. □

3. Existence of non trivial solutions in M_λ

The following lemmas whose proofs are similar modulo minor modifications as in [8] give the geometric conditions to the functional J_λ.

Lemma 3.1. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$. For all $u \in M_\lambda$ and all $\lambda \in (0, \min(\lambda_0, \lambda_1))$ there is $A > 0$ such that $J_\lambda(u) \geq A > 0$ where*

$$\lambda_1 = \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{q/2}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K(n,2), A_\varepsilon))^{q/2}\tau^{q-2}}.$$

Lemma 3.2. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$. The following assertions are true:*

(i) *$\langle \nabla\Phi_\lambda(u), u\rangle < 0$ for all $u \in M_\lambda$ and for all $\lambda \in (0, \min(\lambda_0, \lambda_1))$.*
(ii) *The critical points of J_λ are points of M_λ.*

Now, we show that J_λ satisfies the Palais-Smale condition on M_λ provided that $\lambda > 0$ is sufficiently small. The result is given by the following lemma whose proof is different from the one in the case of smooth coefficients.

Lemma 3.3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Let $(u_m)_m$ be a sequence in M_λ such that*

$$J_\lambda(u_m) \leq c$$
$$\nabla J_\lambda(u_m) - \mu_m \nabla\Phi_\lambda(u_m) \to 0.$$

Suppose that

$$c < \frac{2}{nK_0^{n/4}(f(x_0))^{(n-4)/4}}$$

then there is a subsequence $(u_m)_m$ converging strongly in $H_2^2(M)$.

Proof. Let $(u_m)_m \subset M_\lambda$ and

$$J_\lambda(u_m) = \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\int_M |u_m|^q dv_g .$$

As in the proof of Lemma 3.2, we have

$$J_\lambda(u_m) \geq \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\Lambda^{-q/2}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\|u_m\|^q,$$

$$\begin{aligned} J_\lambda(u_m) &\geq \|u_m\|^2\Big(\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-q/2}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\tau^{q-2}\Big) \\ &> 0\,. \end{aligned}$$

Since $0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{q/2}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K(n,2),A_\varepsilon))^{q/2}\tau^{q-2}}$ and $J_\lambda(u_m) \leq c$, we obtain

$$\begin{aligned} c &\geq J_\lambda(u_m) \\ &\geq \Big[\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}\Big]\|u_m\|^2 > 0 \end{aligned}$$

so

$$\|u_m\|^2 \leq \frac{c}{\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-q/2}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\tau^{q-2}} < +\infty.$$

Then $(u_m)_m$ is a bounded in $H_2^2(M)$. By the compactness of the embedding $H_2^2(M) \subset H_p^k(M)$ ($k = 0,1$; $p < N$) we obtain a subsequence still denoted $(u_m)_m$ such that

$$\begin{gathered} u_m \to u \quad \text{weakly in } H_2^2(M), \\ u_m \to u \quad \text{strongly in } L^p(M) \text{ where } p < N, \\ \nabla u_m \to \nabla u \quad \text{strongly in } L^p(M) \text{ where } p < 2^* = \frac{2n}{n-2} \\ u_m \to u \quad \text{a.e. in } M. \end{gathered}$$

On the other hand since $\frac{2s}{s-1} < N = \frac{2n}{n-4}$, we obtain

$$\begin{aligned} \Big|\int_M b(x)|u_m - u|^2 dv_g\Big| &\leq \|b\|_s \|u_m - u\|^2_{\frac{2s}{s-1}} \\ &\leq \|b\|_s((K_0+\epsilon)\|\Delta(u_m - u)\|_2^2 + A_\epsilon\|u_m - u\|_2^2). \end{aligned}$$

Now taking into account

$$K_0 = \frac{16}{n(n^2-4)(n-4)\omega_n^{n/4}} < 1 \tag{3.1}$$

we obtain

$$\int_M b(x)(u_m - u)^2 dv_g \leq \|b\|_s\|\Delta(u_m - u)\|_2^2 + o(1).$$

By the same process as above, we obtain

$$\int_M a(x)|\nabla(u_m - u)|^2 dv_g \leq \|a\|_r\|\Delta(u_m - u)\|_2^2 + o(1).$$

By Brezis-Lieb lemma, we write

$$\int_M (\Delta_g u_m)^2 dv_g = \int_M (\Delta_g u)^2 dv_g + \int_M (\Delta_g(u_m - u))^2 dv_g + o(1)$$

and

$$\int_M f(x)|u_m|^N dv_g = \int_M f(x)|u|^N dv_g + \int_M f(x)|u_m - u|^N dv_g + o(1).$$

Now we claim that $\mu_m \to 0$ as $m \to +\infty$ Testing with u_m we obtain

$$\langle \nabla J_\lambda(u_m) - \mu_m \nabla\Phi_\lambda(u_m), u_m\rangle = o(1);$$

then

$$\langle \nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m), u_m\rangle = \underbrace{\langle \nabla J_\lambda(u_m), u_m\rangle}_{=0} - \mu_m \langle \nabla \Phi_\lambda(u_m), u_m\rangle = o(1);$$

hence

$$\mu_m \langle \nabla \Phi_\lambda(u_m), u_m\rangle = o(1).$$

By Lemma 3.2, we obtain $\limsup_m \langle \nabla \Phi_\lambda(u_m), u_m\rangle < 0$ so $\mu_m \to 0$ as $m \to +\infty$.

Our last claim is that $u_m \to u$ strongly in $H_2^2(M)$, indeed

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2}\int_M (\Delta_g(u_m - u))^2 dv_g - \frac{1}{N}\int_M f(x)|u_m - u|^N dv_g + o(1).$$

Since $u_m - u \to 0$ weakly in $H_2^2(M)$, testing with $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$, we have $\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u\rangle = o(1)$ and

$$\begin{aligned} &\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u\rangle \\ &= \int_M (\Delta_g(u_m - u))^2 dv_g - \int_M f(x)|u_m - u|^N dv_g = o(1); \end{aligned} \tag{3.2}$$

then

$$\int_M (\Delta_g(u_m - u))^2 dv_g = \int_M f(x)|u_m - u|^N dv_g + o(1),$$

and taking account of (3.2) we obtain

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2}\int_M (\Delta_g(u_m - u))^2 dv_g - \frac{1}{N}\int_M (\Delta_g(u_m - u))^2 dv_g + o(1);$$

i. e.,

$$J_\lambda(u_m) - J_\lambda(u) = \frac{2}{N}\int_M (\Delta_g(u_m - u))^2 dv_g + o(1).$$

Independently, by the Sobolev's inequality, we have

$$\|u_m - u\|_N^2 \le (1+\varepsilon)K_0 \int_M (\Delta_g(u_m - u))^2 dv_g + o(1). \tag{3.3}$$

Since

$$\int_M f(x)|u_m - u|^N dv_g \le \max_{x\in M} f(x)\|u_m - u\|_N^N$$

we infer by (3.3) that

$$\int_M f(x)|u_m - u|^N dv_g \le (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_0^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^N + o(1)$$

and using equality (3.2),

$$o(1) \ge \|\Delta_g(u_m - u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_0^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^N$$

and

$$\begin{aligned} &\|\Delta_g(u_m - u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_0^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^N \\ &= \|\Delta_g(u_m - u)\|_2^2 (1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_0^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^{N-2}) \end{aligned}$$

so if

$$\limsup_{m\to+\infty} \|\Delta_g(u_m - u)\|_2^2 < \frac{1}{K_0^{n/4} (\max_{x\in M} f(x))^{\frac{n}{4}-1}} \tag{3.4}$$

then $u_m \to u$ strongly in $H_2^2(M)$. The condition (3.4) is fulfilled since by Lemma 3.1 $J_\lambda(u) > 0$ on M_λ with λ is as in Lemma 3.1 and by hypothesis,

$$c \geq J_\lambda(u_m) > (J_\lambda(u_m) - J_\lambda(u)) = \frac{2}{n}\int_M (\Delta_g(u_m - u))^2 dv_g$$

and

$$c < \frac{2}{nK_0^{n/4}(\max_{x\in M} f(x))^{\frac{n}{4}-1}}.$$

It is obvious that

$$\Phi_\lambda(u) = 0 \quad \text{and } \|u\| \geq \tau$$

i.e. $u \in M_\lambda$. □

Now we show the existence of a sequence in M_λ satisfying the conditions of Palais-Smale.

Lemma 3.4. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$, then there is a couple $(u_m, \mu_m) \in M_\lambda \times R$ such that $\nabla J_\lambda(u_m) - \mu_m \nabla\Phi_\lambda(u_m) \to 0$ strongly in $(H_2^2(M))^*$ and $J_\lambda(u_m)$ is bounded provide that $\lambda \in (0, \lambda_*)$ with $\lambda_* = \{\min(\lambda_0, \lambda_1), 0\}$.*

Proof. Since J_λ is Gateau differentiable and by Lemma 3.1 bounded below on M_λ it follows from Ekeland's principle that there is a couple $(u_m, \mu_m) \in M_\lambda \times R$ such that $\nabla J_\lambda(u_m) - \mu_m \nabla\Phi_\lambda(u_m) \to 0$ strongly in $(H_2^2(M))'$ and $J_\lambda(u_m)$ is bounded i.e. $(u_m, \mu_m)_m$ is a Palais-Smale sequence on M_λ. □

Now we are in position to establish the following generic existence result.

Theorem 3.5. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$ and f a positive function. Suppose that P_g is coercive and*

$$c < \frac{2}{nK_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}. \tag{3.5}$$

Then there is $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, the equation (1.3) has a non trivial weak solution.

Proof. By Lemma 3.3 and 3.4 there is $u \in H_2^2(M)$ such that

$$J_\lambda(u) = \min_{\varphi\in M_\lambda} J_\lambda(\varphi).$$

By Lagrange multiplicative theorem there is a real number μ such that for any $\varphi \in H_2^2(M)$,

$$\langle \nabla J_\lambda(u), \varphi\rangle = \mu\langle\nabla\Phi_\lambda(u), \varphi\rangle \tag{3.6}$$

and letting $\varphi = u$ in the equation (3.6), we obtain

$$\Phi_\lambda(u) = \langle\nabla J_\lambda(u), u\rangle = \mu\langle\nabla\Phi_\lambda(u), u\rangle.$$

By Lemma 3.2 we obtain that $\mu = 0$ and by equation (3.6), we infer that for any $\varphi \in H_2^2(M)$

$$\langle\nabla J_\lambda(u), \varphi\rangle = 0$$

hence u is weak non trivial solution to equation (1.3) and since by Lemma 3.2, u is a critical points of J_λ. We conclude that $u \in M_\lambda$. □

4. Applications

Let $P \in M$, we define a function on M by

$$\rho_P(Q) = \begin{cases} d(P,Q) & \text{if } d(P,Q) < \delta(M) \\ \delta(M) & \text{if } d(P,Q) \geq \delta(M) \end{cases} \tag{4.1}$$

where $\delta(M)$ is the injectivity radius of M. For brevity we denote this function by ρ. The weighted $L^p(M,\rho^\gamma)$ space will be the set of measurable functions u on M such that $\rho^\gamma |u|^p$ are integrable where $p \geq 1$. We endow $L^p(M,\rho^\gamma)$ with the norm

$$\|u\|_{p,\rho} = (\int_M \rho^\gamma |u|^p dv_g)^{1/p}.$$

In this section we need the Hardy-Sobolev inequality and the Releich-Kondrakov embedding whose proofs are given in [7].

Theorem 4.1. *Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$ and p, q , γ are real numbers such that $\frac{\gamma}{p} = \frac{n}{q} - \frac{n}{p} - 2$ and $2 \leq p \leq \frac{2n}{n-4}$. For any $\epsilon > 0$, there is $A(\epsilon,q,\gamma)$ such that for any $u \in H_2^2(M)$,*

$$\|u\|^2_{p,\rho^\gamma} \leq (1+\epsilon)K(n,2,\gamma)^2 \|\Delta_g u\|_2^2 + A(\epsilon,q,\gamma)\|u\|_2^2 \tag{4.2}$$

where $K(n,2,\gamma)$ is the optimal constant.

In the case $\gamma = 0$, $K(n,2,0) = K(n,2) = K_0^{-1/2}$ is the best constant in the Sobolev's embedding of $H_2^2(M)$ in $L^N(M)$ where $N = \frac{2n}{n-4}$.

Theorem 4.2. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$ and p, q, γ are real numbers satisfying $1 \leq q \leq p \leq \frac{nq}{n-2q}$, $\gamma < 0$ and $l = 1, 2$.*

If $\frac{\gamma}{p} = n\ (\frac{1}{q} - \frac{1}{p}) - l$ then the inclusion $H_l^q(M) \subset L^p(M,\rho^\gamma)$ is continuous. If $\frac{\gamma}{p} > n\ (\frac{1}{q} - \frac{1}{p}) - l$ then inclusion $H_l^q(M) \subset L^p(M,\rho^\gamma)$ is compact.

We consider the equation

$$\Delta_g^2 u + \operatorname{div}_g\Big(\frac{a(x)}{\rho^\sigma}\nabla_g u\Big) + \frac{b(x)}{\rho^\mu}u = \lambda|u|^{q-2}u + f(x)|u|^{N-2}u \tag{4.3}$$

where a and b are smooth functions and ρ denotes the distance function defined by (4.1), $\lambda > 0$ in some interval $(0,\lambda_*)$, $1 < q < 2$, σ, μ will be precise later and we associate to (4.3) on $H_2^2(M)$ the functional

$$\begin{aligned} J_\lambda(u) = \frac{1}{2}\int_M ((\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma}|\nabla_g u|^2 + \frac{b(x)}{\rho^\mu}u^2)dv_g \\ - \frac{\lambda}{q}\int_M |u|^q dv_g - \frac{1}{N}\int_M f(x)|u|^N dv_g. \end{aligned}$$

If we put

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u), u\rangle$$

we obtain

$$\Phi_\lambda(u) = \int_M (\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma}|\nabla_g u|^2 + \frac{b(x)}{\rho^\mu}u^2 dv_g - \lambda\int_M |u|^q dv_g - \int_M f(x)|u|^N dv_g.$$

Theorem 4.3. *Let $0 < \sigma < \frac{n}{s} < 2$ and $0 < \mu < \frac{n}{p} < 4$. Suppose that*

$$\sup_{u \in H_2^2(M)} J_{\lambda,\sigma,\mu}(u) < \frac{2}{n\ K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}$$

then there is $\lambda_* > 0$ such that if $\lambda \in (0, \lambda_*)$, equation (4.3) possesses a weak non trivial solution $u_{\sigma,\mu} \in M_\lambda$.

Proof. Let $\tilde{a} = \frac{a(x)}{\rho^\sigma}$ and $\tilde{b} = \frac{b(x)}{\rho^\mu}$, so if $\sigma \in (0, \min(2, \frac{n}{s}))$ and $\mu \in (0, \min(4, \frac{n}{p}))$, obliviously $\tilde{a} \in L^s(M)$, $\tilde{b} \in L^p(M)$, where $s > \frac{n}{2}$ and $p > \frac{n}{4}$. Theorem 4.3 is a consequence of Theorem 3.5. □

5. The critical cases $\sigma = 2$ and $\mu = 4$

In the cases $\sigma = 2$ and $\mu = 4$ the Hardy-Sobolev inequality proved in case of manifolds by the first author in [7] and is formulated in Theorem 4.1 is no longer valid, so we consider the subcritical cases $0 < \sigma < 2$ and $0 < \mu < 4$ and we tend σ to 2and μ to 4. This can be done successfully by adding an appropriate assumption and by using the Lebesgue dominated converging theorem.

By section four, for any $\sigma \in (0, \min(2, \frac{n}{s}))$ and $\mu \in (0, \min(4, \frac{n}{p}))$, there is a solution $u_{\sigma,\mu} \in M_\lambda$ of equation (1.3). Now we are going to show that the sequence $(u_{\sigma,\mu})_{\sigma,\mu}$ is bounded in $H_2^2(M)$. Evaluating $J_{\lambda,\sigma,\mu}$ at $u_{\sigma,\mu}$

$$J_{\lambda,\sigma,\mu}(u_{\sigma,\mu}) = \frac{1}{2}\|u_{\sigma,\mu}\|^2 - \frac{1}{N}\int_M f(x)|u_{\sigma,\mu}|^N dv_g - \frac{1}{q}\lambda \int_M |u_{\sigma,\mu}|^q dv_g$$

and taking account of $u_{\sigma,\mu} \in M_\lambda$, we infer that

$$J_{\lambda,\sigma,\mu}(u_{\sigma,\mu}) = \frac{N-2}{2N}\|u_{\sigma,\mu}\|^2 - \lambda\frac{N-q}{Nq}\int_M |u_{\sigma,\mu}|^q dv_g.$$

For a smooth function a on M, denotes by $a^- = \min(0, \min_{x\in M}(a(x))$. Let $K(n, 2, \sigma)$ the best constant and $A(\varepsilon, \sigma)$ the corresponding constant in the Hardy-Sobolev inequality given in Theorem 4.1.

Theorem 5.1. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$. Let $(u_m)_m = (u_{\sigma_m,\mu_m})_m$ be a sequence in M_λ such that*

$$J_{\lambda,\sigma,\mu}(u_m) \leq c_{\sigma,\mu}$$
$$\nabla J_\lambda(u_m) - \mu_{_{\sigma,\mu}} \nabla\Phi_\lambda(u_m) \to 0.$$

Suppose that

$$c_{\sigma,\mu} < \frac{2}{n\ K(n,2)^{n/4}(\max_{x\in M} f(x))^{(n-4)/4}}$$

and

$$1 + a^- \max(K(n, 2, \sigma), A(\varepsilon, \sigma)) + b^- \max(K(n, 2, \mu), A(\varepsilon, \mu)) > 0.$$

Then the equation

$$\Delta^2 u - \nabla^\mu(\frac{a}{\rho^2}\nabla_\mu u) + \frac{bu}{\rho^4} = f|u|^{N-2}u + \lambda|u|^{q-2}u$$

has a non trivial solution in the sense of distributions.

Proof. Let $(u_m)_m \subset M_{\lambda,\sigma,\mu}$,

$$J_{\lambda,\sigma,\mu}(u_m) = \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\int_M |u_m|^q dv_g$$

As in proof of Theorem 3.5, we obtain

$$\begin{aligned} J_{\lambda,\sigma,\mu}(u_m) \geq \|u_m\|^2 \Big(& \frac{N-2}{2N} \\ & - \lambda\frac{N-q}{Nq}\Lambda_{\sigma,\mu}^{-q/2} V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K(n,2), A_\varepsilon))^{q/2}\tau^{q-2}\Big) > 0 \end{aligned}$$

where

$$0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda_{\sigma,\mu}^{q/2}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K(n,2), A_\varepsilon))^{q/2}\tau^{q-2}}.$$

First we claim that

$$\lim_{(\sigma,\mu)\to(2^-,4^-)} \inf \Lambda_{\sigma,\mu} > 0.$$

Indeed, if $\nu_{1,\sigma,\mu}$ denotes the first nonzero eigenvalue of the operator

$$P_g = \Delta_g^2 - \operatorname{div}(\frac{a}{\rho^\sigma}\nabla_g) + \frac{b}{\rho^\mu},$$

then clearly $\Lambda_{\sigma,\mu} \geq \nu_{1,\sigma,\mu}$. Suppose on the contrary that $\lim_{(\sigma,\mu)\to(2^-,4^-)} \inf \Lambda_{\sigma,\mu} = 0$, then $\liminf_{(\sigma,\mu)\to(2^-,4^-)} \nu_{1,\sigma,\mu} = 0$. Independently, if $u_{\sigma,\mu}$ is the corresponding eigenfunction to $\nu_{1,\sigma,\mu}$ we have

$$\begin{aligned} \nu_{1,\sigma,\mu} &= \|\Delta u_{\sigma,\mu}\|_2^2 + \int_M \frac{a|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma} dv_g + \int_M \frac{b u_{\sigma,\mu}^2}{\rho^\mu} dv_g \\ &\geq \|\Delta u_{\sigma,\mu}\|_2^2 + a^- \int \frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma} dv_g + b^- \int_M \frac{u_{\sigma,\mu}^2}{\rho^\mu} dv_g \end{aligned} \tag{5.1}$$

where $a^- = \min(0, \min_{x\in M} a(x))$ and $b^- = \min(0, \min_{x\in M} b(x))$. The Hardy-Sobolev's inequality given by Theorem 4.1 leads to

$$\int_M \frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma} dv_g \leq C(\|\nabla|\nabla u_{\sigma,\mu}|\|^2 + \|\nabla u_{\sigma,\mu}\|^2),$$

and since

$$\|\nabla|\nabla u_{\sigma,\mu}|\|^2 \leq \|\nabla^2 u_{\sigma,\mu}\|^2 \leq \|\Delta u_{\sigma,\mu}\|^2 + \beta\|\nabla u_{\sigma,\mu}\|^2$$

where $\beta > 0$ is a constant and it is well known that for any $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that

$$\|\nabla u_{\sigma,\mu}\|^2 \leq \varepsilon\|\Delta u_{\sigma,\mu}\|^2 + c\|u_{\sigma,\mu}\|^2.$$

Hence

$$\int_M \frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma} dv_g \leq C(1+\varepsilon)\|\Delta u_{\sigma,\mu}\|^2 + A(\varepsilon)\|u_{\sigma,\mu}\|^2 \tag{5.2}$$

Now if $K(n,2,\sigma)$ denotes the best constant in inequality (5.2) we obtain that for any $\varepsilon > 0$,

$$\int_M \frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma} dv_g \leq (K(n,2,\sigma)^2 + \varepsilon)\|\Delta u_{\sigma,\mu}\|^2 + A(\varepsilon,\sigma)\|u_{\sigma,\mu}\|^2. \tag{5.3}$$

By inequalities (4.2), (5.1) and (5.3), we have

$$\begin{aligned} \nu_{1,\sigma,\mu} \geq (1 &+ a^- \max(K(n,2,\sigma), A(\varepsilon,\sigma)) \\ &+ b^- \max(K(n,2,\mu), A(\varepsilon,\mu)))(\|\Delta u_{\sigma,\mu}\|^2 + \|u_{\sigma,\mu}\|^2) \end{aligned}$$

So if

$$1 + a^- \max(K(n,2,\sigma), A(\varepsilon,\sigma)) + b^- \max(K(n,2,\mu), A(\varepsilon,\mu)) > 0$$

then we obtain $\lim_{\sigma,\mu}(u_{\sigma,\mu}) = 0$ and $\|u_{\sigma,\mu}\| = 1$ a contradiction. The reflexivity of $H_2^2(M)$ and the compactness of the embedding $H_2^2(M) \subset H_p^k(M)$ ($k = 0, 1$; $p < N$), imply that up to a subsequence, we have

$$\begin{gathered}
u_m \to u \quad \text{weakly in } H_2^2(M), \\
u_m \to u \quad \text{strongly in } L^p(M),\ p < N, \\
\nabla u_m \to \nabla u \quad \text{strongly in } L^p(M),\ p < 2^* = \frac{2n}{n-2}, \\
u_m \to u \quad \text{a. e. in } M.
\end{gathered}$$

The Brézis-Lieb lemma allows us to write

$$\int_M (\Delta_g u_m)^2 dv_g = \int_M (\Delta_g u)^2 dv_g + \int_M (\Delta_g (u_m - u))^2 dv_g + o(1)$$

and

$$\int_M f(x)|u_m|^N dv_g = \int_M f(x)|u|^N dv_g + \int_M f(x)|u_m - u|^N dv_g + o(1).$$

Now by the boundedness of the sequence $(u_m)_m$, we have that $u_m \to u$ weakly in $H_2^2(M)$, $\nabla u_m \to \nabla u$ weakly in $L^2(M, \rho^{-2})$ and $u_m \to u$ weakly in $L^2(M, \rho^{-4})$; i.e., for any $\varphi \in L^2(M)$,

$$\int_M \frac{a(x)}{\rho^2} \nabla u_m \nabla \varphi dv_g = \int_M \frac{a(x)}{\rho^2} \nabla u \nabla \varphi dv_g + o(1)$$

and

$$\int_M \frac{b(x)}{\rho^4} u_m \varphi dv_g = \int_M \frac{b(x)}{\rho^4} u \varphi dv_g + o(1).$$

For every $\phi \in H_2^2(M)$ we have

$$\begin{aligned}
&\int_M \Big(\Delta_g^2 u_m + \operatorname{div}_g \big(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m \big) + \frac{b(x)}{\rho^{\delta_m}} u_m \Big) \phi dv_g \\
&= \int_M (\lambda |u_m|^{q-2} u_m + f(x)|u_m|^{N-2} u_m) \phi dv_g.
\end{aligned} \tag{5.4}$$

By the weak convergence in $H_2^2(M)$, we have immediately that

$$\int_M \phi \Delta_g^2 u_m dv_g = \int_M \phi \Delta_g^2 u dv_g + o(1)$$

and

$$\begin{aligned}
&\int_M \big(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m - \frac{a(x)}{\rho^2} \nabla_g u \big) \phi dv_g \\
&= \int_M \big(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m + \frac{a(x)}{\rho^2} (\nabla_g u_m - \nabla_g u_m) - \frac{a(x)}{\rho^2} \nabla_g u \big) \phi dv_g
\end{aligned}$$

Then

$$\begin{aligned}
&\Big|\int_M \Big(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m - \frac{a(x)}{\rho^2}\nabla_g u\Big)\phi dv_g\Big| \\
&\le \Big|\int_M \Big(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m - \frac{a(x)}{\rho^2}\nabla_g u_m\Big)\phi dv_g\Big| + \Big|\int_M (\frac{a(x)}{\rho^2}\nabla_g u_m - \frac{a(x)}{\rho^2}\nabla_g u)\phi dv_g\Big| \\
&\le \int_M |a(x)\phi\nabla_g u_m|\Big|\frac{1}{\rho^{\sigma_m}} - \frac{1}{\rho^2}\Big| dv_g + \Big|\int_M \frac{a(x)}{\rho^2}\nabla_g(u_m - u)\phi dv_g\Big|.
\end{aligned} \tag{5.5}$$

The weak convergence in $L^2(M,\rho^{-2})$ and the Lebesgue's dominated convergence theorem imply that the second right hand side of (5.5) goes to 0. For the third term of the left hand side of (5.3), we write

$$\int_M \Big(\frac{b(x)}{\rho^{\delta_m}}u_m - \frac{b(x)}{\rho^4}u\Big)\phi dv_g = \int_M \Big(\frac{b(x)}{\rho^{\delta_m}}u_m - \frac{b(x)}{\rho^4}u_m + \frac{b(x)}{\rho^4}u_m - \frac{b(x)}{\rho^4}u\Big)\phi dv_g$$

and

$$\begin{aligned}
&\Big|\int_M (\frac{b(x)}{\rho^{\delta_m}}u_m - \frac{b(x)}{\rho^4}u)\phi dv_g\Big| \\
&\le \int_M |b(x)\phi u_m|\Big|\frac{1}{\rho^{\delta_m}} - \frac{1}{\rho^4}\Big| dv_g + \Big|\int_M \frac{b(x)}{\rho^4}(u_m - u)\phi dv_g\Big|.
\end{aligned} \tag{5.6}$$

Here also the weak convergence in $L^2(M,\rho^{-4})$ and the Lebesgue's dominated convergence allows us to affirm that the left hand side of (5.6) converges to 0.

It remains to show that $\mu_m \to 0$ as $m \to +\infty$ and $u_m \to u$ strongly in $H_2^2(M)$ but this is the same as in the proof of Theorem 3.5 which implies also $u \in M_\lambda$. □

6. Test Functions

In this section, we give the proof of the main theorem to do so, we consider a normal geodesic coordinate system centered at x_0. Denote by $S_{x_0}(\rho)$ the geodesic sphere centered at x_0 and of radius ρ ($\rho < d$ which is the injectivity radius). Let $d\Omega$ be the volume element of the $n-1$-dimensional Euclidean unit sphere S^{n-1} and put

$$G(\rho) = \frac{1}{\omega_{n-1}}\int_{S(\rho)} \sqrt{|g(x)|}d\Omega$$

where ω_{n-1} is the volume of S^{n-1} and $|g(x)|$ the determinant of the Riemannian metric g. The Taylor's expansion of $G(\rho)$ in a neighborhood of x_0 is given by

$$G(\rho) = 1 - \frac{S_g(x_0)}{6n}\rho^2 + o(\rho^2)$$

where $S_g(x_0)$ denotes the scalar curvature of M at x_0. Let $B(x_0,\delta)$ be the geodesic ball centered at x_0 and of radius δ such that $0 < 2\delta < d$ and denote by η a smooth function on M such that

$$\eta(x) = \begin{cases} 1 & \text{on } B(x_0,\delta) \\ 0 & \text{on } M - B(x_0,2\delta). \end{cases}$$

Consider the radial function

$$u_\epsilon(x) = (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)})^{\frac{n-4}{8}}\frac{\eta(\rho)}{((\rho\theta)^2+\epsilon^2)^{\frac{n-4}{2}}}$$

with

$$\theta = (1 + \|a\|_r + \|b\|_s)^{1/n}$$

where $\rho = d(x_0, x)$ is the distance from x_0 to x and $f(x_0) = \max_{x\in M} f(x)$. For further computations we need the following integrals: for any real positive numbers p, g such that $p - q > 1$ we put

$$I_p^q = \int_0^{+\infty} \frac{t^q}{(1+t)^p} dt\,.$$

The following relations are immediate

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q, \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q.$$

6.1. Application to compact Riemannian manifolds of dimension $n > 6$.

Theorem 6.1. *Let (M, g) be a compact Riemannian manifold of dimension $n > 6$. Suppose that at a point x_0 where f attains its maximum the following condition*

$$\frac{\Delta f(x_0)}{f(x_0)} < \frac{1}{3}\Big(\frac{(n-1)n(n^2+4n-20)}{(n^2-4)(n-4)(n-6)} \frac{1}{(1+\|a\|_r + \|b\|_s)^{n/4}} - 1\Big) S_g(x_0)$$

holds. Then (1.2) *has a non trivial solution with energy*

$$J_\lambda(u) < \frac{1}{K_0^{n/4} (\max_{x\in M} f(x))^{\frac{n}{4}-1}}.$$

Proof. The proof of Theorem 6.1 reduces to show that the condition (3.5) of Theorem 3.5 is satisfied and since by Lemma 2.2 there is a $t_0 > 0$ such that $t_0 u_\epsilon \in M_\lambda$ for sufficiently small λ, so it suffices to show that

$$\sup_{t>0} J_\lambda(t u_\epsilon) < \frac{1}{K_0^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n}{4}-1}}.$$

To compute the term $\int_M f(x)|u_\epsilon(x)|^N dv_g$, we need the following Taylor's expansion of f at the point x_0

$$f(x) = f(x_0) + \frac{\partial^2 f(x_0)}{2\partial y^i \partial y^j} y^i y^j + o(\rho^2)$$

and also that of the Riemannian measure

$$dv_g = 1 - \frac{1}{6} R_{ij}(x_0) y^i y^j + o(\rho^2)$$

where $R_{ij}(x_0)$ denotes the Ricci tensor at x_0. The expression of $\int_M f(x)|u_\epsilon(x)|^N dv_g$ is well known (see for example [11]) and is given in case $n > 6$ by

$$\int_M f(x)|u_\epsilon(x)|^N dv_g = \frac{\theta^{-n}}{K_0^{n/4} (f(x_0))^{\frac{n-4}{4}}} \Big(1 - (\frac{\Delta f(x_0)}{2(n-2) f(x_0)} + \frac{S_g(x_0)}{6(n-2)}) \epsilon^2 + o(\epsilon^2)\Big)$$

where K_0 is given by (3.1) and $\omega_n = 2^{n-1} I_n^{\frac{n}{2}-1} \omega_{n-1}$ and ω_n is the volume of S^n, the standard unit sphere of R^{n+1} endowed with its round metric.

Now the restriction of $|\frac{\partial u_\epsilon}{\partial \rho}|$ to the geodesic ball $B(x_0, \delta)$ is computed as follows

$$|\frac{\partial u_\epsilon}{\partial \rho}|_{B(x_0,\delta)} = |\nabla u_\epsilon| = \theta^{-2}(n-4)(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)})^{\frac{n-4}{8}} \frac{\rho}{((\frac{\rho}{\theta})^2 + \epsilon^2)^{\frac{n-2}{2}}}$$

and Since $a \in L^r(M)$ with $r > n/2$ we have

$$\int_{B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g \le \theta^{-4}(n-4)^2\Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}} \|a\|_r \omega_{n-1}^{1-\frac{1}{r}} \times \Big(\int_0^\delta \frac{\rho^{\frac{2r}{r-1}+n-1}}{((\frac{\rho}{\theta})^2+\epsilon^2)^{\frac{(n-2)r}{r-1}}}\Big(\int_{S(\rho)}\sqrt{|g(x)|}d\Omega\Big)d\rho\Big)^{\frac{r-1}{r}}$$

Since

$$\int_{S(\rho)}\sqrt{|g(x)|}d\Omega = \omega_{n-1}\Big(1-\frac{S_g(x_0)}{6n}\rho^2+o(\rho^2)\Big)$$

we obtain

$$\int_{B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g \le \theta^{-4}(n-4)^2(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)})^{\frac{n-4}{4}} \|a\|_r \omega_{n-1}^{1-\frac{1}{r}} \times \Big(\int_0^\delta \frac{\rho^{\frac{2r}{r-1}+n-1}}{((\rho\theta)^2+\epsilon^2)^{\frac{(n-2)r}{r-1}}}d\rho\Big(1-\frac{S_g(x_0)}{6n}\rho^2+o(\rho^2)\Big)\Big)^{\frac{r-1}{r}}$$

and by the following change of variable

$$t=(\frac{\rho\theta}{\epsilon})^2 \quad \text{i.e. } \rho=\frac{\epsilon}{\theta}\sqrt{t}$$

we obtain

$$\begin{aligned}
&\int_{B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g \\
&\le \theta^{-n\frac{r}{r-1}}(n-4)^2\Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}} \|a\|_r \omega_{n-1}^{1-\frac{1}{r}} \epsilon^{-(n-4)+2-\frac{n}{r}} \\
&\times \Big(\int_0^{(\frac{\delta\theta}{\epsilon})^2} \frac{t^{\frac{n-2}{2}+\frac{r}{r-1}}}{(t+1)^{\frac{(n-2)r}{r-1}}}dt - \frac{S_g(x_0)}{6n}\theta^{-2}\epsilon^2\int_0^{(\frac{\delta\theta}{\epsilon})^2} \frac{t^{\frac{n}{2}+\frac{r}{r-1}}}{(t+1)^{\frac{(n-2)r}{r-1}}}dt + o(\epsilon^2)\Big)^{\frac{r-1}{r}}.
\end{aligned}$$

Letting $\epsilon \to 0$ we obtain

$$\begin{aligned}
&\int_{B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g \\
&\le 2^{-1+\frac{1}{r}}\theta^{-n(1-\frac{1}{r})}(n-4)^2(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)})^{\frac{n-4}{4}} \|a\|_r \omega_{n-1}^{1-\frac{1}{r}} \epsilon^{-(n-4)+2-\frac{n}{r}} \\
&\quad \times (I_{\frac{(n-2)r}{r-1}}^{\frac{n-2}{2}+\frac{r}{r-1}} - \theta^{-2}\frac{S_g(x_0)}{6n} I_{\frac{(n-2)r}{r-1}}^{\frac{n}{2}+\frac{r}{r-1}}\epsilon^2 + o(\epsilon^2))^{\frac{r-1}{r}}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g \\
&\le 2^{-1+\frac{1}{r}}\theta^{-n\frac{r}{r-1}}(n-4)^2\Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}} \|a\|_r \omega_{n-1}^{1-\frac{1}{r}} \epsilon^{\epsilon^{-(n-4)+2-\frac{n}{r}}} \\
&\quad \times I_{\frac{(n-2)r}{r-1}}^{1+\frac{n-2}{2}\cdot\frac{r-1}{r}}\Big[1-\frac{r-1}{r}\theta^2\frac{S_g(x_0)}{6n} I_{\frac{(n-2)r}{r-1}}^{\frac{n}{2}+\frac{r}{r-1}} I_{\frac{(n-2)r}{r-1}}^{-\frac{n-2}{2}-\frac{r}{r-1}}\epsilon^2+o(\epsilon^2)\Big].
\end{aligned}$$

It remains to compute the integral $\int_{B(x_0,2\delta)-B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g$.

First we remark that

$$\Big|\int_{(\frac{\delta\theta}{\epsilon})^2}^{(\frac{2\delta\theta}{\epsilon})^2} h(t)\frac{t^q}{(t+1)^p}dt\Big| \le C(\frac{1}{\epsilon})^{2(q-p+1)} = C\epsilon^{2(p-q-1)}$$

and since $p-q=n-4\ge 3$, we obtain

$$\int_{(\frac{\delta\theta}{\epsilon})^2}^{(\frac{2\delta\theta}{\epsilon})^2} h(t)\frac{i^q}{(t+1)^p}dt = o(\epsilon^2)$$

and then

$$\int_{B(x_0,2\delta)-B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g = o(\epsilon^2). \tag{6.1}$$

Finally we obtain

$$\begin{aligned}
&\int_M a(x)|\nabla u_\epsilon|^2 dv_g \\
&\le 2^{-1+\frac{1}{r}}\theta^{-n\frac{r}{r-1}}(n-4)^2\Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}}\|a\|_r\omega_{n-1}^{1-\frac{1}{r}}\epsilon^{-(n-4)+2-\frac{n}{r}} \\
&\quad\times\Big(I_{\frac{(n-2)r}{r-1}}^{1+\frac{n-2}{2}\cdot\frac{r-1}{r}} + o(\epsilon^2)\Big).
\end{aligned}$$

Letting

$$A = K_0^{n/4}\frac{(n-4)^{\frac{n}{4}+1}\times(\omega_{n-1})^{\frac{r-1}{r}}}{2^{\frac{r-1}{r}}}(n(n^2-4))^{\frac{n-4}{4}}\Big(I_{\frac{(n-2)r}{r-1}}^{\frac{n-2}{2}+\frac{r}{r-1}}\Big)^{\frac{r-1}{r}} \tag{6.2}$$

we obtain

$$\int_M a(x)|\nabla u_\epsilon|^2 dv_g \le \epsilon^{2-\frac{n}{r}}\theta^{-n\frac{r}{r-1}}\frac{A}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\|a\|_r(1+o(\epsilon^2)).$$

Now we compute

$$\int_M b(x)u_\epsilon^2 dv_g = \int_{B(x_0,\delta)} b(x)u_\epsilon^2 dv_g + \int_{B(x_0,2\delta)-B(x_0,\delta)} b(x)u_\epsilon^2 dv_g$$

and since $b\in L^s(M)$ with $s>\frac{n}{4}$, we have

$$\int_M b(x)u_\epsilon^2 dv_g \le \|b\|_s\|u_\epsilon\|_{\frac{2s}{s-1}}^2.$$

Independently,

$$\begin{aligned}
\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_0,\delta)} &= \Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}} \\
&\quad\times\Big(\int_0^\delta \frac{\rho^{n-1}}{((\rho\theta)^2+\epsilon^2)^{\frac{(n-4)s}{(s-1)}}}\Big(\int_{S(r)}\sqrt{|g(x)|}d\Omega\Big)dr\Big)^{\frac{s-1}{s}}
\end{aligned}$$

and

$$\int_{S(r)}\sqrt{|g(x)|}d\Omega = \omega_{n-1}\Big(1-\frac{S_g(x_0)}{6n}\rho^2+o(\rho^2)\Big).$$

Consequently,

$$\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_0,\delta)} = \Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}}$$

$$\omega_{n-1}^{\frac{s-1}{s}} \times \Big(\int_0^{\delta} \frac{\rho^{n-1}}{((\rho\theta)^2+\epsilon^2)^{\frac{(n-4)s}{(s-1)}}}\Big(1-\frac{S_g(x_0)}{6n}\rho^2+o(\rho^2)\Big)d\rho\Big)^{\frac{s-1}{s}}.$$

And putting $t=(\rho\theta/\epsilon)^2$, we obtain

$$\begin{aligned}
\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_0,\delta)} &= \Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}}(\omega_{n-1})^{\frac{s-1}{s}}\epsilon^{-n+4+4-\frac{n}{s}} \\
&\quad\times\Big(\frac{\epsilon^n\theta^{-n}}{2}\int_0^{(\frac{\delta\theta}{\epsilon})^2}\frac{t^{\frac{n}{2}-1}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt \\
&\quad-\frac{\theta^{-n-2}S_g(x_0)}{12n}\epsilon^{n+2}\int_0^{(\frac{\delta\theta}{\epsilon})^2}\frac{t^{\frac{n}{2}}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt+o(\epsilon^{n+2})\Big)^{\frac{s-1}{s}}.
\end{aligned}$$

Letting $\epsilon\to 0$, we obtain

$$\begin{aligned}
\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_0,\delta)} &= \Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}}(\omega_{n-1})^{\frac{s-1}{s}}\epsilon^{-n+4+4-\frac{n}{s}} \\
&\quad\times\theta^{-n\frac{s}{s-1}}(\frac{\epsilon^n}{2})^{\frac{s-1}{s}}\Big(\int_0^{+\infty}\frac{t^{\frac{n}{2}}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt \\
&\quad-\frac{S_g(x_0)}{12n}\epsilon^2\theta^{-2}\int_0^{+\infty}\frac{t^{\frac{n}{2}+1}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt+o(\epsilon^2)\Big)^{\frac{s-1}{s}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_0,\delta)} \\
&=\Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}}(\omega_{n-1})^{\frac{s-1}{s}}\epsilon^{-n+4+4-\frac{n}{s}}\theta^{-n\frac{s}{s-1}}(\frac{\epsilon^n}{2})^{\frac{s-1}{s}} \\
&\quad\times\Big(\int_0^{+\infty}\frac{t^{\frac{n}{2}}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt-\theta^{-2}\frac{S_g(x_0)}{12n}\epsilon^2\int_0^{+\infty}\frac{t^{\frac{n}{2}+1}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt+o(\epsilon^2)\Big)^{\frac{s-1}{s}},
\end{aligned}$$

or

$$\begin{aligned}
\|u_\epsilon\|^2_{\frac{2s}{s-1}} &= \Big(\frac{(n-4)n(n^2-4)}{f(x_0)}\Big)^{\frac{n-4}{4}}\Big(\frac{\omega_{n-1}}{2}\Big)^{\frac{s-1}{s}}\epsilon^{4-\frac{n}{s}}\theta^{-n\frac{s}{s-1}} \\
&\quad\times\Big[(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}})^{\frac{s-1}{s}}-\frac{\theta^{-2}(s-1)S_g(x_0)}{12n\ s}(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}})^{-\frac{1}{s}}I^{\frac{n}{2}+1}_{\frac{(n-4)s}{(s-1)}}\epsilon^2+o(\epsilon^2)\Big]
\end{aligned}$$

Finally, by the same method as in equality (6.1), we obtain

$$\begin{aligned}
&\int_M b(x)u_\epsilon^2dv_g \\
&\le\|b\|_s(\frac{(n-4)n(n^2-4)}{f(x_0)})^{\frac{n-4}{4}}(\frac{\omega_{n-1}}{2})^{\frac{s-1}{s}}\epsilon^{4-\frac{n}{s}}\theta^{-n\frac{s}{s-1}}\Big(\Big(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}}\Big)^{\frac{s-1}{s}}+o(\epsilon^2)\Big).
\end{aligned}$$

Putting

$$B=K_0^{n/4}((n-4)n(n^2-4))^{\frac{n-4}{4}}(\frac{\omega_{n-1}}{2})^{\frac{s-1}{s}}\Big(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}}\Big)^{\frac{s-1}{s}} \tag{6.3}$$

we obtain

$$\int_M b(x)u_\epsilon^2dv_g\le\epsilon^{4-\frac{n}{s}}\theta^{-n\frac{s}{s-1}}\frac{\|b\|_sB}{K_0^{\frac{n}{4}}(f(x_0))^{\frac{n-4}{4}}}(1+o(\epsilon^2)).$$

The computation of $\int_M (\Delta u_\epsilon)^2 dv_g$ is well known see for example ([11]) and is given by

$$\int_M (\Delta u_\epsilon)^2 dv_g = \frac{\theta^{-n}}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\Big(1 - \frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_0)\epsilon^2 + o(\epsilon^2)\Big).$$

Summarizing, we obtain

$$\begin{aligned}
&\int_M (\Delta u_\epsilon)^2 - a(x)|\nabla u_\epsilon|^2 + b(x)u_\epsilon^2 dv_g \\
&\leq \frac{\theta^{-n}}{K_0^{n/4} f(x_0)^{\frac{n-4}{4}}}\Big(1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}} A\|a\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}} B\|b\|_s \\
&\quad - \frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_0)\epsilon^2 + o(\epsilon^2)\Big).
\end{aligned}$$

Now, we have

$$\begin{aligned}
J_\lambda(tu_\epsilon) &\leq J_0(tu_\epsilon) = \frac{t^2}{2}\|u_\epsilon\|^2 - \frac{t^N}{N}\int_M f(x)|u_\epsilon(x)|^N dv_g \\
&\leq \frac{\theta^{-n}}{K_0^{n/4} f(x_0)^{\frac{n-4}{4}}}\Big\{\frac{1}{2}t^2(1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}} A\|a\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}} B\|b\|_s) - \frac{t^N}{N} \\
&\quad + \Big[\Big(\frac{\Delta f(x_0)}{2(n-2)f(x_0)} + \frac{S_g(x_0)}{6(n-1)}\Big)\frac{t^N}{N} - \frac{1}{2}t^2\frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_0)\Big]\epsilon^2\Big\} \\
&\quad + o(\epsilon^2)
\end{aligned}$$

and letting ϵ be small enough so that

$$1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}} A\|a\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}} B\|b\|_s \leq (1 + \|a\|_r + \|b\|_s)^{\frac{4}{n}}$$

and since the function $\varphi(t) = \alpha\frac{t^2}{2} - \frac{t^N}{N}$, with $\alpha > 0$ and $t > 0$, attains its maximum at $t_0 = \alpha^{\frac{1}{N-2}}$ and

$$\varphi(t_0) = \frac{2}{n}\alpha^{n/4}.$$

Consequently,

$$\begin{aligned}
J_\lambda(tu_\epsilon) &\leq \frac{2\theta^{-n}}{nK_0^{n/4} f(x_0)^{\frac{n-4}{4}}}\Big\{1 + \|a\|_r + \|b\|_s + \Big[\Big(\frac{\Delta f(x_0)}{2(n-2)f(x_0)} + \frac{S_g(x_0)}{6(n-1)}\Big)\frac{t_0^N}{N} \\
&\quad - \frac{1}{2}t_0^2\frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_0)\Big]\epsilon^2\Big\} + o(\epsilon^2).
\end{aligned}$$

Taking into account the value of θ and putting

$$R(t) = \Big(\frac{\Delta f(x_0)}{2(n-2)f(x_0)} + \frac{S_g(x_0)}{6(n-1)}\Big)\frac{t^N}{N} - \frac{1}{2}\frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_0)t^2$$

we obtain

$$\sup_{t\geq 0} J_\lambda(tu_\epsilon) < \frac{2}{nK_0^{n/4}(\max_{x\in M} f(x))^{\frac{n}{4}-1}}$$

provided that $R(t_0) < 0$; i.e.,

$$\frac{\Delta f(x_0)}{f(x_0)} < \Big(\frac{n(n^2+4n-20)}{3(n+2)(n-4)(n-6)}\frac{1}{(1 + \|a\|_r + \|b\|_s)^{n/4}} - \frac{n-2}{3(n-1)}\Big) S_g(x_0).$$

Which completes the proof. $\square$

6.1.1. *Application to compact Riemannian manifolds of dimension $n = 6$.*

Theorem 6.2. *In case $n = 6$, we suppose that at a point x_0 where f attains its maximum $S_g(x_0) > 0$. Then the equation (1.2) has a non trivial solution.*

Proof. The same calculations as in case $n > 6$ gives us

$$\int_M f(x)|u_\epsilon(x)|^N dv_g = \frac{\theta^{-n}}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\Big(1-\Big(\frac{\Delta f(x_0)}{2(n-2)f(x_0)}+\frac{S_g(x_0)}{6(n-2)}\Big)\epsilon^2+o(\epsilon^2)\Big).$$

Also, we have

$$\int_M a(x)|\nabla u_\epsilon|^2 dv_g \le \frac{\|a\|_r A}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\epsilon^{2-\frac{n}{r}}\theta^{-\frac{r}{r-1}}(1+o(\epsilon^2))$$

and

$$\int_M b(x)u_\epsilon^2 dv_g \le \frac{\|b\|_s B}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\epsilon^{4-\frac{n}{s}}\theta^{-\frac{s}{s-1}} + (1+o(\epsilon^2)).$$

where A and B are given by (6.2) and (6.3) respectively for $n = 6$. The computations of the term $\int_M (\Delta u_\epsilon)^2 dv_g$ are well known (see for example [11])

$$\begin{aligned}
&\int_M (\Delta u_\epsilon)^2 dv(g) \\
&= \theta^{-n}(n-4)^2\Big(\frac{(n-4)n(n^2-4)}{f(x_0)}\Big)^{\frac{n-4}{4}}\frac{\omega_{n-1}}{2} \\
&\quad\times\Big(\frac{n(n+2)(n-2)}{(n-4)}I_n^{\frac{n}{2}-1} - \frac{2}{n}\theta^{-2}S_g(x_0)\epsilon^2\log(\frac{1}{\epsilon^2}) + O(\epsilon^2)\Big).
\end{aligned}$$

$$\int_M (\Delta u_\epsilon)^2 dv_g = \frac{\theta^{-n}}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\Big(1 - \frac{2(n-4)}{n^2(n^2-4)I_n^{\frac{n}{2}-1}}S_g(x_0)\epsilon^2\log(\frac{1}{\epsilon^2}) + O(\epsilon^2)\Big).$$

Now summarizing and letting ϵ so that

$$1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}}A\|b\|_s + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}}B\|a\|_r \le (1+\|a\|_r+\|b\|_s)^{\frac{4}{n}}$$

we obtain

$$\begin{aligned}
J_\lambda(u_\epsilon) &\le \frac{1}{2}\|u_\epsilon\|^2 - \frac{1}{N}\int_M f(x)|u_\epsilon(x)|^N dv_g \\
&\le \frac{\theta^{-n}}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\Big[\frac{t^2}{2}(1+\|a\|_r+\|b\|_s)^{1-\frac{4}{n}} - \frac{t^N}{N} \\
&\quad - \frac{n-4}{n^2(n^2-4)I_n^{\frac{n}{2}-1}}\theta^{-2}S_g(x_0)t^2\epsilon^2\log(\frac{1}{\epsilon^2})\Big] + O(\epsilon^2).
\end{aligned}$$

The same arguments as in the case $n > 6$ allow us to infer that

$$\max_{t\ge 0} J_\lambda(tu_\epsilon) < \frac{2}{n\ K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}$$

if $S_g(x_0) > 0$. Which completes the proof. □

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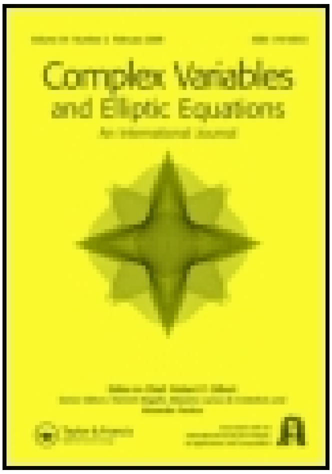

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Multiple solutions to singular fourth order elliptic equations on compact manifolds

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Multiple solutions to singular fourth order elliptic equations on compact manifolds

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Using the method of Nehari manifolds, we prove the existence of at least two distinct weak solutions to elliptic equation of four order with singularities and with critical Sobolev growth.

Keywords: singular fourth order elliptic equation; Hardy inequality; Sobolev's exponent growth; Nehari manifold

AMS Subject Classification: 58J05

1. Introduction

Fourth order elliptic equations have been intensively investigated the last decades particularly after the discovery of an important conformally invariant operator by Paneitz on 4 – dimensional Riemannian manifolds [1] and whose definition was extended to higher dimension by Branson [2].This operator is closely related to the problem of prescribed Q-curvature. Many works have been devoted to this subject (see [3–21]). Let (M, g) be a compact smooth Riemannian manifold of dimension $n \geq 5$ with a metric g. We denote by $H_2^2(M)$ the standard Sobolev space which is the completion of the space $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{2,2} = \sum_{k=0}^{k=2} \left\|\nabla^k \varphi\right\|_2$$

where $\|.\|_2$ denotes the $L^2(M)$ -norm. $H_2^2(M)$ will be endowed with the equivalent suitable norm:

$$\|u\|_{H_2^2(M)} = \left(\int_M \left((\Delta_g u)^2 + |\nabla_g u|^2 + u^2\right) dv_g\right)^{\frac{1}{2}}.$$

Recently, Madani [22], has considered the Yamabe problem with singularities which he solved under some geometric conditions. The first author in [8] considered singular fourth order elliptic equations with singularities of the form

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$$\Delta^2 u - \nabla^i (a(x)\nabla_i u) + b(x)u = f\,|u|^{N-2}\,u \tag{1}$$

where the functions a and b are in $L^s(M)$, $s > \frac{n}{2}$ and in $L^p(M)$, $p > \frac{n}{4}$ respectively, $N = \frac{2n}{n-4}$ is the Sobolev critical exponent in the embedding $H_2^2(R^n) \hookrightarrow L^N(R^n)$. He established the following result:

THEOREM 1.1 *Let (M, g) be a compact n-dimensional Riemannian manifold, $n \geq 6$, $a \in L^s(M)$, $b \in L^p(M)$, with $s > \frac{n}{2}$, $p > \frac{n}{4}$, $f \in C^\infty(M)$ a positive function and $P \in M$ such that $f(P) = \max_{x\in M} f(x)$.*

For $n \geq 10$, or $n = 9$ and $\frac{9}{4} < p < 11$ or $n = 8$ and $2 < p < 5$ or $n = 7$ and $\frac{7}{2} < s < 9$, $\frac{7}{4} < p < 3$, suppose that

$$\frac{n^2+4n-20}{6(n-6)(n^2-4)}S_g(P) - \frac{n-4}{2n(n-2)}\frac{\Delta f(P)}{f(P)} > 0.$$

For $n = 6$ and $\frac{3}{2} < p < 2$, $3 < s < 4$, suppose that

$$S_g(P) > 0.$$

Then the Equation (1) has a non trivial weak solution u in $H_2^2(M)$. Moreover if $a \in H_1^s(M)$, then
$u \in C^{0,\beta}$, for some $\beta \in \left(0, 1 - \frac{n}{4p}\right)$.

For fixed $R \in M$, we define the function ρ on M by

$$\rho(Q) = \begin{cases} d(R,Q) & \text{if } d(R,Q) < \delta(M) \\ \delta(M) & \text{if } d(R,Q) \geq \delta(M) \end{cases} \tag{2}$$

where $\delta(M)$ denotes the injectivity radius of M.

In this paper, we are concerned with the following problem: for real numbers σ and μ, consider the equation in the distribution sense

$$\Delta^2 u - \nabla^i(a\rho^{-\mu}\nabla_i u) + \rho^{-\alpha} b u = \lambda\,|u|^{q-2}\,u + f(x)\,|u|^{N-2}\,u \tag{3}$$

where the functions a and b are smooth on M, $1 < q < 2$ and λ a real parameter. Denote by P_g the operator defined on $H_2^2(M)$ by $u \to P_g(u) = \Delta^2 u - \nabla^i(a\rho^{-\mu}\nabla_i u) + \rho^{-\alpha} bu$. We look for multiple solutions to Equation (3). Our main results state as follows:

THEOREM 1.2 *Let (M, g) be a compact n-dimensional Riemannian manifold, $n \geq 6$, a, b, f are smooth functions with f a positive function and $x_o \in M$ such that $f(x_o) = \max_{x\in M} f(x)$. Let $0 < \sigma < 2$ and $0 < \mu < 4$. Suppose that the operator P_g is coercive and*

$$\begin{cases} \frac{\Delta f(x_o)}{2(n-2)f(x_o)} + \frac{S_g(x_o)}{6(n-1)} < 0 \text{ and } S_g(x_o) > 0 & \text{in case } n > 6 \\ S_g(x_o) > 0 & \text{in case } n = 6. \end{cases}$$

Then there is $\lambda_* > 0$ such that if $\lambda \in (0, \lambda_*)$, the Equation (3) possesses at least two distinct non trivial solutions in the distribution sense.

The proof of Theorem 1.2 relies on the following Hardy–Sobolev type inequality (for a proof see [6]).

LEMMA 1.3 *Let (M, g) be a compact n- dimensional Riemannian manifold, p, q and γ real numbers satisfying*

$$1 \leq q \leq p \leq \frac{nq}{n-2q}, \quad n > 2q, \quad \frac{\gamma}{p} = -2 + n\left(\frac{1}{q} - \frac{1}{p}\right) > -\frac{n}{p}.$$

For any $\varepsilon > 0$, there is a constant $A(\varepsilon, q, \gamma)$ such that

$$\forall f \in H_2^q(M), \quad \|f\|_{p,\rho^\gamma}^q \leq (1+\varepsilon)\, K^q(n, q, \gamma) \left\|\nabla^2 f\right\|_q^q + A(\varepsilon, q, \gamma)\, \|f\|_q^q .$$

In the particular case $\gamma = 0$, $K(n, q, 0) = K(n, q)$ is the best constant in Sobolev inequality of the embedding $H_2^2(R^n) \hookrightarrow L^q(R^n)$.

For brevity along all this work we put $K_o = K(n, 2)$.

Let σ and μ be as in Theorem 1.2, the Hardy–Sobolev inequality given by Lemma 1.3 leads to the following inequality

$$\int_M \frac{|\nabla u|^2}{\rho^\sigma} dv_g \leq C(\|\nabla |\nabla u|\|^2 + \|\nabla u\|^2)$$

where $C > 0$ is a constant.

Now since

$$\|\nabla |\nabla u|\|^2 \leq \left\|\nabla^2 u\right\|^2 \leq \|\Delta u\|^2 + \beta \|\nabla u\|^2$$

where $\beta > 0$ is a constant and taking account of the following well known inequality: for any $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that

$$\|\nabla u\|^2 \leq \varepsilon \|\Delta u\|^2 + c \|u\|^2$$

we infer that:

$$\int_M \frac{|\nabla u|^2}{\rho^\sigma} dv_g \leq C(1+\varepsilon) \|\Delta u\|^2 + A(\varepsilon, \sigma) \|u\|^2 . \tag{4}$$

Let $K(n, 1, \sigma)^2$ be the best constant in inequality (4) and $K(n, 2, \mu)^2$ be the best one in

$$\int_M \frac{u^2}{\rho^\mu} dv_g \leq C(1+\varepsilon) \|\Delta u\|^2 + A(\varepsilon, \mu) \|u\|^2 .$$

For any $0 < \sigma < 2$ and $0 < \mu < 4$, denote by $u_{\sigma,\mu}$ the solution of Equation (3) given in Theorem 1.2. In the sharp case $\sigma = 2$ and $\mu = 4$, we obtain the following result:

THEOREM 1.4 *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$. Suppose that the operator P_g is coercive and let $\left(u_{\sigma,\mu}\right)_{(\sigma,\mu)\in]0,2[\times]0,4[}$ be a sequence in M_λ such that:*

$$\begin{cases} J_{\lambda,\sigma,\mu}(u_{\sigma,\mu}) \to a_{\sigma,\mu} \\ \nabla J_\lambda(u_{\sigma,\mu}) \to 0 \end{cases} .$$

Assume moreover that

$$\left|a_{\sigma,\mu}\right| < \frac{2}{n\, K_o^{\frac{n}{4}} \left(f(x_\circ)\right)^{\frac{n-4}{4}}}$$

and

$$1 + a^- \max K(n,1,2)^2 + b^- \max K(n,2,4)^2 > 0$$

then the equation

$$\Delta^2 u - \nabla^\mu \left(\frac{a}{\rho^2}\nabla_\mu u\right) + \frac{bu}{\rho^4} = f\,|u|^{N-2}\,u + \lambda\,|u|^{q-2}\,u$$

has at least two distinct non trivial solutions in distribution sense. Where for a smooth function w on M, $w^- = \min\left(0, \min_{x\in M} w(x)\right)$.

We consider the energy functional J_λ defined by: for each $u \in H_2^2(M)$

$$J_\lambda(u) = \frac{1}{2}\int_M \left((\Delta u)^2 + a(x)\rho^{-\sigma}\,|\nabla u|^2 + b(x)\rho^{-\mu}u^2\right) dv_g - \frac{\lambda}{q}\int_M |u|^q\, dv_g - \frac{1}{N}\int_M f(x)\,|u|^N\, dv_g.$$

Put

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u),\; u\rangle\,,$$

then

$$\Phi_\lambda(u) = \int_M \left((\Delta u)^2 + a(x)\rho^{-\sigma}\,|\nabla u|^2 + b(x)\rho^{-\mu}u^2\right) dv_g - \lambda\int_M |u|^q\, dv_g - \int_M f(x)\,|u|^N\, dv_g$$

and

$$\langle \nabla\Phi_\lambda(u), u\rangle = 2\int_M \left((\Delta u)^2 + a(x)\rho^{-\sigma}\,|\nabla u|^2 + b(x)\rho^{-\mu}u^2\right) dv_g - \lambda q\int_M |u|^q\, dv_g - N\int_M f(x)\,|u|^N\, dv_g.$$

It is well-known that the solutions of the Equation (3) are critical points of the energy functional J_λ. On the other hand the Nehari minimization problem writes as follows:

$$\alpha_\lambda = \inf_{u\in N_\lambda} J_\lambda(u)$$

where

$$N_\lambda = \left\{u \in H_2^2(M)\setminus\{0\} :\; \Phi_\lambda(u) = 0\right\}.$$

Note that N_λ contains every solution of the Equation (3) and splits in three parts

$$N_\lambda^+ = \{u \in N_\lambda :\; \langle \nabla\Phi_\lambda(u),\; u\rangle > 0\}$$
$$N_\lambda^- = \{u \in N_\lambda :\; \langle \nabla\Phi_\lambda(u),\; u\rangle < 0\}$$
$$N_\lambda^0 = \{u \in N_\lambda :\; \langle \nabla\Phi_\lambda(u),\; u\rangle = 0\}.$$

Since the operator P_g is coercive we obtain as in [9] that

$$\|u\| = \left(\int_M \left((\Delta u)^2 + a(x)\rho^{-\sigma} |\nabla u|^2 + b(x)\rho^{-\mu} u^2\right) dv_g)\right)^{\frac{1}{2}}$$

is an equivalent norm to the usual one on $H_2^2(M)$.

2. Some preparatory lemmas

Before stating the proofs of our main results as in [24], we give some nice properties of N_λ^+, N_λ^- and N_λ^0.
Put

$$\lambda_\circ = \frac{(N-2)\, q\ \Lambda^{\frac{q}{2}}}{2\,(N-q)\, V(M)^{1-\frac{q}{N}}\,(\max(K_\circ, A_\varepsilon))^{\frac{q}{2}}} \tag{5}$$

where Λ is the constant of the coercivity of the operator P_g, $V(M)$ is the volume of the manifold M and $K_\circ$, A_ε are the constants appearing in the Sobolev inequality of the embedding $H_2^2(R^n) \hookrightarrow L^p(R^n)$. The following lemma shows that the minimizers of J_λ on N_λ are critical points for J_λ.

Lemma 2.1 *Let $\lambda \in (0, \lambda_\circ)$, if v is a local minimizer for $J_\lambda$on N_λ and $v \notin N_\lambda^0$, then* $\nabla J_\lambda(v) = 0$.

Proof If v is a local minimizer for $J_\lambda$on N_λ, then by Lagrange multipliers theorem, there is a real number θ such that for any $\varphi \in H_2^2(M)$

$$\langle \nabla J_\lambda(v), \varphi\rangle = \theta\,\langle \nabla\Phi_\lambda(v), \varphi\rangle.$$

If $\theta = 0$, then the lemma is proved. If it is not the case we pick $\varphi = v$ and we use the assumption that $v \in N_\lambda$ to infer

$$\langle \nabla J_\lambda(v), v\rangle = \theta\,\langle \nabla\Phi_\lambda(v), v\rangle = 0$$

which contradicts that $v \notin N_\lambda^0$. □

Now we give some technical lemmas:

Lemma 2.2 *There is $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$ the set N_λ^0 is empty.*

Proof Suppose that for every $\lambda > 0$ there is $\lambda' \in (0, \lambda)$ such that $N_{\lambda'}^0 \neq \emptyset$ and let $u \in N_{\lambda'}^0$ i.e.

$$\langle \nabla\Phi_{\lambda'}(u), u\rangle = 2\,\|u\|^2 - \lambda' q\,\|u\|_q^q - N\int_M f(x)\,|u|^N\,dv_g = 0$$

and by the fact that

$$\Phi_{\lambda'}(u) = \|u\|^2 - \lambda'\,\|u\|_q^q - \int_M f(x)\,|u|^N\,dv_g = 0$$

we get

$$\|u\|^2 = \frac{N-q}{2-q} \int_M f(x)\, |u|^N \, dv_g \tag{6}$$

and also

$$\lambda' \, \|u\|_q^q = \frac{N-q}{2-q} \int_M f(x)\, |u|^N \, dv_g. \tag{7}$$

Independently by the Sobolev inequality and the coerciveness of the operator P_g we obtain

$$\int_M f(x)\, |u|^N \, dv_g \leq \max_{x\in M} f(x) \Lambda^{-\frac{N}{2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \, \|u\|^N \tag{8}$$

where Λ denotes the constant of the coercivity. From (6) and (8) we deduce that

$$\|u\| \geq \left[\frac{(N-q)\, \Lambda^{-\frac{N}{2}} \, ((\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \max_{x\in M} f(x)}{(2-q)} \right]^{\frac{1}{2-N}}. \tag{9}$$

Now consider the functional $I_{\lambda'} : N_\lambda \to \mathbb{R}$ given by

$$I_{\lambda'}(u) = \left[\left(\frac{N-q}{2-q} \right)^{\frac{q}{2}} \frac{2-q}{N-q} \right]^{\frac{2}{2-q}} \left(\frac{\|u\|^q}{\lambda' \, \|u\|_q^q} \right)^{\frac{2}{q-2}} - \int_M f(x)\, |u|^N \, dv_g.$$

If $u \in N^0_{\lambda'}$, then (6) and (7) give

$$I_{\lambda'}(u) = \left[\left(\frac{N-q}{2-q} \right)^{\frac{q}{2}} \frac{2-q}{N-q} \right]^{\frac{2}{2-q}} \left[\frac{\left(\frac{N-q}{2-q} \int_M f(x)\, |u|^N \, dv_g \right)^{\frac{q}{2}}}{\frac{N-2}{2-q} \int_M f(x)\, |u|^N \, dv_g} \right]^{\frac{2}{q-2}}$$

$$- \int_M f(x)\, |u|^N \, dv_g = 0. \tag{10}$$

Putting

$$\theta = \left[\left(\frac{N-q}{2-q} \right)^{\frac{q}{2}} \frac{2-q}{N-q} \right]^{\frac{2}{2-q}} = \frac{2-q}{N-q}$$

and taking account of the coerciveness of the operator P_g, the Sobolev inequality and equality (6) we get:

$$I_{\lambda'}(u) = \theta \frac{\Lambda^{\frac{q}{N}}}{\lambda'^{\frac{q}{2}} V(M)^{1-\frac{q}{N}} \, (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2N}}}$$

$$I_{\lambda'}(u) \geq \theta \left(\frac{\|u\|^q}{\lambda' \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \, \|u\|^q} \right)^{\frac{2}{q-2}}$$

$$- \int_M f(x)\, |u|^N \, dv_g.$$

That is to say

$$I_{\lambda'}(u) \geq \theta \left(\frac{\Lambda^{\frac{q}{2}}}{\lambda' V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}} \right)^{\frac{2}{q-2}} - \left(\left(\frac{N-q}{2-q} \right) \Lambda^{-\frac{N}{2}} ((\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \max_{x \in M} f(x) \right)^{\frac{2}{2-N}}.$$

Hence, if λ is sufficiently small, so as $\lambda' > 0$ and $I_{\lambda'}(u) > 0$ for all $u \in N^0_{\lambda'}$. This contradicts (10). So there is $\lambda_1 > 0$, such that for any $\lambda \in (0, \lambda_1)$, the set $N^0_\lambda = \emptyset$. □

From Lemma 2.2, N_λ splits as $N_\lambda = N^+_\lambda \cup N^-_\lambda$ where $0 < \lambda < \lambda_1$. We define

$$\alpha_\lambda = \inf_{u \in N_\lambda} J_\lambda(u), \ \alpha^+_\lambda = \inf_{u \in N^+_\lambda} J_\lambda(u) \ \text{ and } \ \alpha^-_\lambda = \inf_{u \in N^-_\lambda} J_\lambda(u)$$

Lemma 2.3 *For each $\lambda \in (0, \lambda_\circ)$, the functional J_λ is bounded from below on N^+_λ.*

Proof If $u \in N^+_\lambda$,then by the Sobolev's inequality, we deduce that:

$$J_\lambda(u) = \frac{N-2}{2N} \|u\|^2 - \frac{\lambda (N-q)}{qN} \|u\|^q_q$$
$$J_\lambda(u) \geq \frac{N-2}{2N} \|u\|^2 - \lambda \frac{N-q}{Nq} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q_{H^2_2(M)}$$

and taking account of the coerciveness of the operator P_g, we infer that

$$J_\lambda(u) \geq \frac{N-2}{2N} \|u\|^2 - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q$$

where Λ is the constant of coercivity of the operator P_g.

If $\|u\| \geq 1$, then

$$J_\lambda(u) \geq \left[\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \right] \|u\|^q .$$

So, if

$$0 < \lambda < \frac{(N-2)\, q\, \Lambda^{\frac{q}{2}}}{2\,(N-q)\, (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} V(M)^{1-\frac{q}{N}}} = \lambda_\circ$$

we get

$$J_\lambda(u) > 0.$$

If $u \in N^+_\lambda$ with $\|u\| < 1$, we obtain that:

$$J_\lambda(u) > -\lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}.$$

Thus J_λ is bounded from below on N_λ. □

As a consequence of Lemma 2.1 we obtain:

LEMMA 2.4 *If* $\lambda \in (0, \lambda_\circ)$, *then*

$$\alpha_\lambda^+ = \inf_{u \in N_\lambda^+} J_\lambda(u) < 0.$$

Proof If $u \in N_\lambda^+$, then

$$J_\lambda(u) = \frac{N-2}{2N} \|u\|^2 - \frac{\lambda(N-q)}{Nq} \|u\|_q^q$$

and since

$$\langle \nabla \Phi_\lambda(u), u \rangle = 2 \|u\|^2 - \lambda q \|u\|_q^q - N \int_M f(x) |u|^N dv_g > 0$$

we get

$$J_\lambda(u) \le \frac{\lambda(N-q)}{N} \left(\frac{1}{2} - \frac{1}{q} \right) \|u\|_q^q < 0$$

i.e.

$$\inf_{u \in N_\lambda^+} J_\lambda(u) < 0.$$

□

LEMMA 2.5 *For every* $\lambda \in (0, \min(\lambda_0, \lambda_1))$,

$$\alpha_\lambda^- = \inf_{u \in N_\lambda^-} J_\lambda(u) > 0.$$

Proof If $u \in N_\lambda^-$, then

$$J_\lambda(u) = \frac{N-2}{2N} \|u\|^2 - \frac{\lambda(N-q)}{Nq} \|u\|_q^q$$

and since

$$\langle \nabla \Phi_\lambda(u), u \rangle = 2 \|u\|^2 - \lambda q \|u\|_q^q - N \int_M f(x) |u|^N dv_g < 0 \tag{11}$$

we infer that

$$\|u\|^2 > \frac{\lambda (N-q)}{(N-2)} \|u\|_q^q . \tag{12}$$

By the Sobolev inequality and from the coerciveness of the operator P_g, there exists a constant $\Lambda > 0$, such that

$$J_\lambda(u) \ge \frac{N-2}{2N} \|u\|^2 - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q .$$

So if $u \in N_\lambda^-$ and $\|u\| \ge 1$,

$$J_\lambda(u) \ge \left[\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \right] \|u\|^q \tag{13}$$

hence if

$$0<\lambda<\frac{(N-2)\,q\;\Lambda^{\frac{q}{2}}}{2\,(N-q)\;V(M)^{1-\frac{q}{N}}\,(\max((1+\varepsilon)K_\circ,A_\varepsilon))^{\frac{q}{2}}}=\lambda_\circ$$

then

$$J_\lambda(u)>0.$$

In case $u\in N_\lambda^-$ and $\|u\|<1$, by the Sobolev inequality, the inequality (11) and the coerciveness of the operator P_g, we obtain

$$0<\xi\le\|u\|<1$$

where

$$\xi=\left[\frac{(2-q)\,\Lambda^{\frac{N}{2}}\,(\max((1+\varepsilon)K_\circ,A_\varepsilon))^{-\frac{N}{2}}}{(N-q)\max_{x\in M}f(x)}\right]^{\frac{1}{N-2}}$$

and Λ is the constant of coerciveness of the operator P_g.

The inequality (13) becomes

$$J_\lambda(u)\ge\frac{N-2}{2N}\xi^2-\lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ,A_\varepsilon))^{\frac{q}{2}}$$

Hence, if we put

$$\lambda_2=\frac{\frac{(N-2)}{2(N-q)}\xi^2\Lambda^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}}(\max(K_\circ,A_\varepsilon))^{\frac{q}{2}}}\tag{14}$$

and take $\lambda\in(0,\min(\lambda_0,\lambda_1,\lambda_2))$ we obtain

$$J_\lambda(u)\ge C>0$$

where C is a constant depending on $N=\frac{2n}{n-4}$, Λ, $V(M)$, K_o and A_ε. So

$$\inf_{u\in N_\lambda^-}J_\lambda(u)>0.$$

□

For each $u\in H_2^2(M)-\{0\}$, define

$$E(t)=t^{2-q}\,\|u\|^2-t^{N-q}\int_M f\,|u|^N\,dv_g$$

so $E(0)=0$ and $E(t)$ goes to $-\infty$ as $t\to+\infty$. Also for $t>0$, we have

$$E'(t)=(2-q)\,t^{1-q}\,\|u\|^2-(N-q)t^{N-q-1}\int_M f\,|u|^N\,dv_g$$

and $E'(t)=0$ at

$$t_o=\left(\frac{2-q}{N-q}\right)^{\frac{1}{N-2}}\left(\frac{\|u\|^2}{\int_M f\,|u|^N\,dv_g}\right)^{\frac{1}{N-2}}.$$

Hence $E(t)$ achieves its maximum at t_o and it is increasing on $[0,t_o)$ and decreasing on $[t_o,+\infty)$.

Evaluating the function E at t_o, we get

$$E(t_o) = \left(\frac{2-q}{N-q}\right)^{\frac{2-q}{N-2}} \left(\frac{\|u\|^2}{\int_M f\,|u|^N\,dv_g}\right)^{\frac{2-q}{N-q}} \|u\|^2$$

$$- \left(\frac{2-q}{N-q}\right)^{\frac{N-q}{N-2}} \left(\frac{\|u\|^2}{\int_M f\,|u|^N\,dv_g}\right)^{\frac{N-q}{N-2}} \int_M f\,|u|^N\,dv_g$$

$$= \frac{N-2}{N-q}\left(\frac{2-q}{N-q}\right)^{\frac{2-q}{N-2}} \frac{\|u\|^{\frac{2(N-q)}{N-2}}}{\left(\int_M f\,|u|^N\,dv_g\right)^{\frac{2-q}{N-2}}}.$$

By the Sobolev inequality we get for any $\epsilon > 0$,

$$\int_M f\,|u|^N\,dv_g \le \|f\|_\infty \left(\left(K_o^2+\epsilon\right)\|\Delta u\|_2^2 + A(\epsilon)\,\|u\|_2^2\right)^{\frac{N}{2}}$$

$$\le \|f\|_\infty \max\left(K_o^2+\epsilon, A(\epsilon)\right)^{\frac{N}{2}} \|u\|_{H_2^2}^N$$

$$\le \Lambda^{-\frac{N}{2}} \|f\|_\infty \max\left(K_o^2+\epsilon, A(\epsilon)\right)^{\frac{N}{2}} \|u\|^N$$

$$= C^{\frac{N}{2}} \|f\|_\infty \|u\|^N$$

where Λ is the constant of the coercivity, K_o the best constant in the Sobolev inequality and $A(\epsilon)$ the correspondent constant, $\|f\|_\infty = \sup_{x\in M}|f(x)|$ and $C = \Lambda^{-1}\max\left(K_o^2+\epsilon,\right.$ $\left.A(\epsilon)\right)$.

Consequently

$$E(t_o) \ge \frac{N-2}{N-q}\left(\frac{2-q}{N-q}\right)^{\frac{2-q}{N-q}} C^{\frac{N(q-2)}{2(N-2)}} \|f\|_\infty \|u\|^q. \tag{15}$$

Independently and in the same way as above we get

$$\|u\|_q^q \le \Lambda^{-\frac{q}{2}} vol(M)^{1-\frac{q}{N}} C^{\frac{q}{2}} \|u\|^q. \tag{16}$$

Hence

$$E(0) = 0 < \lambda\,\|u\|_q^q \le E(t_o)$$

provided that

$$\lambda \le \frac{\frac{N-2}{N-q}\left(\frac{2-q}{N-q}\right)^{\frac{2-q}{N-q}} \|f\|_\infty}{V(M)^{1-\frac{q}{2}} C^{\frac{N-q}{N-2}}}.$$

Consequently by the nature of the function $E(t)$ we infer the existence of t^-, t^+ with $0 < t^+ < t^o < t^-$ such that

$$\lambda\,\|u\|_q^q = E(t^+) = E(t^-). \tag{17}$$

and

$$E'(t^+) > 0 > E'(t^-).$$

Now we evaluate Φ_λ at t^-u and at t^+u to get

$$\begin{aligned}\Phi_\lambda(t^-u) &= \left\langle \nabla J_\lambda\left(t^-u\right), t^-u\right\rangle \\ &= \left(t^-\right)^2 \|u\|^2 - \left(t^-\right)^N \int_M f\,|u|^N\,dv_g - \lambda\left(t^-\right)^q \|u\|_q^q \\ &= \left(t^-\right)^q \left(\left(t^-\right)^{2-q}\|u\|^2 - \left(t^-\right)^{N-q}\int_M f\,|u|^N\,dv_g - \lambda\|u\|_q^q\right)\end{aligned}$$

and by (17) we deduce that

$$\Phi_\lambda(t^-u) = 0$$

and also we get

$$\Phi_\lambda(t^+u) = 0.$$

Moreover, we have

$$\left\langle \nabla\Phi_\lambda(t^-u), t^-u\right\rangle = 2\left(t^-\right)^2\|u\|^2 - N\left(t^-\right)^N\int_M f\,|u|^N\,dv_g - q\left(t^-\right)^q\lambda\|u\|_q^q$$

and taking account of (17) we infer that

$$\left\langle \nabla\Phi_\lambda(t^-u), t^-u\right\rangle = (2-q)\left(t^-\right)^2\|u\|^2 - (N-q)\left(t^-\right)^N\int_M f\,|u|^N\,dv_g$$

and again by (17) we obtain

$$\begin{aligned}&\left\langle \nabla\Phi_\lambda(t^-u), t^-u\right\rangle \\ &= \left(t^-\right)^{1+q}\left((2-q)\left(t^-\right)^{1-q}\|u\|^2 - (N-q)\left(t^-\right)^{N-q-1}\int_M f\,|u|^N\,dv_g\right) \\ &= \left(t^-\right)^{1+q} E'\left(t^-\right) < 0\end{aligned}$$

that means that $t^-u \in N_\lambda^-$. By similar procedure we get also $t^+u \in N_\lambda^+$.

3. Existence of a local minimizer for J_λ on N_λ^+ and N_λ^-

In this section we focus on the existence of a local minimum of J_λ on N_λ^+ and N_λ^-: to do so we will be in need of the following Hardy–Sobolev inequality and Releich–Kondrakov embedding respectively whose proofs are given in [6]. The weighted space $L^p(M, \rho^\gamma)$ will be the set of measurable functions u on M such that $\rho^\gamma |u|^p$ are integrable where $p \geq 1$ and γ are real numbers. We endow $L^p(M, \rho^\gamma)$ with the norm:

$$\|u\|_{p,\rho} = \left(\int_M \rho^\gamma |u|^p\,dv_g\right)^{\frac{1}{p}}.$$

THEOREM 3.1 *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$ and p, q , γ real numbers such that $\frac{\gamma}{p} = n\left(\frac{1}{q} - \frac{1}{p}\right) - 2 > -\frac{n}{p}$ and $2 \leq p \leq \frac{2n}{n-4}$.*

For any $\epsilon > 0$, there is $A(\epsilon, q, \gamma)$ such that for any $u \in H_2^2(M)$

$$\|u\|_{p,\rho^\gamma}^2 \leq (1+\epsilon)K(n,2,\gamma)^2\|\Delta u\|_2^2 + A(\epsilon,q,\gamma)\|u\|_2^2$$

where $K(n, 2, \gamma)$ is the optimal constant.

In case $\gamma = 0$, $K(n,2,0)^2 = K(n,2)^2 = K_o$ is the best constant in the Sobolev's embedding of $H_2^2(M)$ in $L^N(M)$ where $N = \frac{2n}{n-4}$.

THEOREM 3.2 *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$ and p, q, γ real numbers satisfying $1 \leq q \leq p \leq \frac{nq}{n-2q}$, $\gamma < 0$ and $l = 1{,}2$.*

If $\frac{\gamma}{p} = n\,(\frac{1}{q} - \frac{1}{p}) - l > -\frac{n}{p}$ then the inclusion $H_l^q(M) \subset L^p(M, \rho^\gamma)$ is continuous. If $\frac{\gamma}{p} > n\,(\frac{1}{q} - \frac{1}{p}) - l$ then inclusion $H_l^q(M) \subset L^p(M, \rho^\gamma)$ is compact.

The following variant of the Ekeland's variational principle will be useful:

LEMMA 3.3 *If V is a Banach space and $J \in C^1(V,R)$ is bounded from below, then there exists a minimizing sequence (u_n) for J in V such that $J(u_n) \to \inf_V J$ and $J'(u_n) \to 0$ as $n \to \infty$.*

Now we state the following lemma:

LEMMA 3.4 *For any $\lambda \in (0, \lambda_\circ)$*

(i) *There exists a minimizing sequence $(u_m)_m \subset N_\lambda$ such that $J_\lambda(u_m) = \alpha_\lambda + o(1)$ and $\nabla J_\lambda(u_m) = o(1)$*
(ii) *There exists a minimizing sequence $(u_m)_m \subset N_\lambda^+$(respectively $(u_m)_m \subset N_\lambda^-$) such that $J_\lambda(u_m) = \alpha_\lambda^+ + o(1)$ and $\nabla J_\lambda(u_m) = o(1)$ (respectively $J_\lambda(u_m) = \alpha_\lambda^- + o(1)$ and $\nabla J_\lambda(u_m) = o(1)$).*

Proof By Lemma 2.3 and the Enkland's variational principle (see 3.3) J_λ admits a Palais–Smale sequence at level α_λ in N_λ(the same is also true for (ii)). □

Now, we establish the existence of a local minimum for J_λ on N_λ^+

THEOREM 3.5 *Let $\lambda \in (0, \lambda_\circ)$, and suppose that a sequence $(u_m)_m \subset N_\lambda^+$ fulfils the following assumptions:*

$$\begin{cases} J_\lambda(u_m) = a_\lambda^+ + o(1) \\ \nabla J_\lambda(u_m) = o(1) \text{ in } H_2^2(M)' \end{cases}$$

with

$$\left|a_\lambda^+\right| < \frac{2}{n\, K_\circ^{\frac{n}{4}} (\max_{x \in M} f(x))^{\frac{n}{4}-1}}. \tag{C1}$$

Then the functional J_λ has a minimizer u^+ in N_λ^+ which satisfies

(i) $J_\lambda(u^+) = \alpha_\lambda^+ < 0$,
(ii) *u^+ is a non trivial weak solution of Equation (3).*

Proof Let $(u_m)_m \subset N_\lambda^+$ be a Palais–Smale sequence for J_λ on N_λ^+ i.e.

$$J_\lambda(u_m) = a_\lambda^+ + o(1) \text{ and } \nabla J_\lambda(u_m) = o(1) \text{ in } H_2^2(M)'.$$

So

$$\begin{aligned} a_\lambda^+ + o(1) &= J_\lambda(u_m) - \frac{1}{q}\langle \nabla J_\lambda(u_m), u_m \rangle \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_m\|^2 + \left(\frac{1}{q} - \frac{1}{N}\right)\int_M f\,|u_m|^N\, dv_g \\ &\leq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_m\|^2 + \left(\frac{1}{q} - \frac{1}{N}\right)\frac{2-q}{N-q}\|u_m\|^2 \\ &= \frac{(q-2)(N-2)}{2qN}\|u_m\|_2^2 \end{aligned}$$

Hence

$$\|u_m\|^2 \leq \frac{2qN}{(q-2)(N-2)} a_\lambda^+ + o(1)$$

so the sequence $(u_m)_m$ is bounded sequence in $H_2^2(M)$ and by the well known Sobolev's embedding, we get a subsequence such that

$$u_m \to u^+ \text{ weakly in } H_2^2(M).$$

$$u_m \to u^+ \text{ strongly in } L^p(M) \quad \text{for } 1 < p < N = \frac{2n}{n-4}.$$

$$\nabla u_m \to \nabla u^+ \text{ strongly in } L^q(M) \quad \text{for } 1 < q < 2^* = \frac{2n}{n-2}.$$

$$u_m \to u^+ \quad \text{a.e. in } M.$$

And also by Theorem 3.2 we have:

$$u_m \to u^+ \text{ strongly in } L^2(M, \rho^{-\mu}) \quad \text{for } 0 < \mu < 4$$

and

$$\nabla u_m \to \nabla u^+ \text{ strongly in } L^2(M, \rho^{-\sigma}) \quad \text{for } 0 < \sigma < 2.$$

First we prove that u^+ is a weak solution to Equation (3). In fact, from the above convergences and the following claims: since

$$\left\| |u_m|^{N-2} u_m \right\|_{\frac{N}{N-1}} = \|u_m\|_N^{N-1} \leq C\, \|u_m\|_{H_2^2}^{N-1} < +\infty$$

where C is a constant, and for every $\varphi \in H_2^2$, $f\varphi \in L^N(M)$ so we get that

$$\int_M f\,|u_m|^{N-2} u_m \varphi dv_g \to \int f\left|u^+\right|^{N-2} u^+ \varphi dv_g. \tag{18}$$

Also we have

$$\left\| |u_m|^{q-2} u_m \right\|_{\frac{N}{q-1}} = \left\| |u_m|^{q-2} u_m \right\|_N^{q-1} < +\infty$$

and since $u^+ \in H_2^2(M) \subset L^N(M) \subset L^{\frac{N}{N-q+1}}(M)$, we deduce that

$$\int_M |u_m|^{q-2} u_m \varphi dv_g \to \int_M \left|u^+\right|^{q-2} u^+ \varphi dv_g.$$

On other hand we have:

$$\langle \nabla J_\lambda(u_m), \varphi\rangle = \int_M \left(\Delta^2 u_m\varphi + div\left(\frac{a}{\rho^\sigma}\nabla u_m\right)\varphi + \frac{b}{\rho^\mu}u_m\varphi\right)dv_g$$
$$- \int_M f\,|u_m|^{N-2}\,u_m\varphi dv_g - \lambda\int_M |u_m|^{q-2}\,u_m\varphi dv_g \to \left\langle \nabla J_\lambda(u^+), \varphi\right\rangle$$

or

$$\left\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u^+), \varphi\right\rangle = 0.$$

By letting $m \to \infty$, we obtain

$$\left\langle \nabla J_\lambda(u^+), \varphi\right\rangle = 0.$$

Obviously $u^+ \in N_\lambda$. We claim that $u^+ \in N_\lambda^+$, since if it is not the case $u^+ \in N_\lambda^-$, thus $\left\langle \nabla J_\lambda(u^+), u^+\right\rangle = 0$ and $\left\langle \nabla \Phi_\lambda(u^+), u^+\right\rangle < 0$, which implies that $J_\lambda(u^+) > 0$, a contradiction with the claim in Lemma 2.4.

Then,

$$J_\lambda(u^+) = \alpha_\lambda^+ < 0.$$

u^+ is a weak solution of Equation (3).

Now, we have to show that u^+ is non trivial solution.

By Brézis–Lieb Lemma (see [23]), we have:

$$\left\|\Delta_g u_m\right\|_2^2 - \left\|\Delta_g u^+\right\|_2^2 = \left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2 + o(1) \tag{19}$$

and

$$\int_M f(x)\left(|u_m|^N - \left|u^+\right|^N\right)dv_g = \int_M f(x)\left|u_m - u^+\right|^N dv_g + o(1). \tag{20}$$

Now from the relations (19), (20) and the strong convergences of $\nabla u_m \to \nabla u^+$ and $u_m \to u^+$ in $L^2\left(M, \rho^{-\sigma}\right)$ and $L^2\left(M, \rho^{-\mu}\right)$ respectively we obtain that:

$$\begin{aligned} J_\lambda(u_m) &= J_\lambda(u_m) - J_\lambda(u^+) + J_\lambda(u^+) \\ &= \frac{1}{2}\left\|\Delta\left(u_m - u^+\right)\right\|_2^2 - \frac{1}{N}\int_M f\left|u_m - u^+\right|^N dv_g + J_\lambda(u^+) + o(1). \end{aligned} \tag{21}$$

Since $u_m - u^+ \to 0$ weakly in $H_2^2(M)$, we test by $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$ and get:

$$\begin{aligned} &\left\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u^+), u_m - u^+\right\rangle \\ &\quad = \left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2 - \int_M f(x)\left|u_m - u^+\right|^N dv_g = o(1). \end{aligned}$$

and by the fact that u^+ is a solution to Equation (3) we deduce that:

$$\begin{aligned} J_\lambda(u_m) = &\left(\frac{1}{2} - \frac{1}{N}\right)\int_M f(x)\left|u_m - u^+\right|^N dv_g + \left(\frac{1}{2} - \frac{1}{N}\right)\int_M f(x)\left|u^+\right|^N dv_g \\ &+ \lambda\left(\frac{1}{2} - \frac{1}{q}\right)\|u_+\|_q^q + o\left(1\right) \end{aligned}$$

and using again the Lieb–Brézis lemma and taking in mind $1 < q < 2$ we obtain:

$$J_\lambda(u_m) \le \left(\frac{1}{2} - \frac{1}{N}\right) \int_M f(x)\,|u_m|^N\, dv_g + o(1)$$
$$= \frac{2}{n} \int_M f(x)\,|u_m|^N\, dv_g + o(1).$$

Hence

$$\left|a_\lambda^+\right| \le \frac{2}{n} \limsup_m \int_M f(x)\,|u_m|^N\, dv_g \tag{22}$$

Independently, by Brézis–Lieb lemma, we have

$$\int_M f(x)\left|u_m - u^+\right|^N dv_g + o(1)$$
$$= \int_M f(x)\left(|u_m|^N - \left|u^+\right|^N\right) dv_g + o(1).$$

Writing

$$|u_m|^N - |u_m|^{N-2}\left(u_m - u^+\right)^2 = 2\,|u_m|^{N-2}\, u_m u^+ - |u_m|^{N-2}\left(u^+\right)^2$$

and taking account of (18) and the following convergence

$$\int_M f(x)\,|u_m|^{N-2}\left(u^+\right)^2 dv_g \to \int_M f(x)\left|u^+\right|^N dv_g$$

which is obtained by the same way as (18), we infer that:

$$\int_M f(x)\left(|u_m|^N - \left|u^+\right|^N\right) dv_g$$
$$= \int_M |u_m|^{N-2}\left(u_m - u^+\right)^2 dv_g + o(1)$$
$$\le \left(\int_M f(x)\,|u_m|^N\, dv_g\right)^{1-\frac{2}{N}} \left(\int_M f(x)\left|u_m - u^+\right|^N dv_g\right)^{\frac{2}{N}}.$$

Using again equality (22) and the Sobolev inequality, we deduce:

$$\left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2$$
$$\le \left(\int_M f(x)\,|u_m|^N\, dv_g\right)^{1-\frac{2}{N}} \left(\int_M f(x)\left|u_m - u^+\right|^N dv_g\right)^{\frac{2}{N}} + o(1)$$
$$\le \left(\int_M f(x)\,|u_m|^N\, dv_g\right)^{1-\frac{2}{N}} \left(\max_{x\in M} f(x)\right)^{\frac{2}{N}} \left\|u_m - u^+\right\|_N^2 + o(1)$$
$$\le \left(\int_M f(x)\,|u_m|^N\, dv_g\right)^{1-\frac{2}{N}} \left(\max_{x\in M} f(x)\right)^{\frac{2}{N}} \left((K_o + \varepsilon)\left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2\right) + o(1).$$

Consequently,

$$\left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2 \left(1 - (K_o + \varepsilon)\left(\max_{x\in M} f(x)\right)^{\frac{2}{N}} \left(\int_M f(x)\,|u_m|^N\, dv_g\right)^{1-\frac{2}{N}}\right) \le o(1)$$

so if

$$\limsup_{m} \int_M f(x) |u_m|^N dv_g < \frac{1}{K_o^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n}{4}-1}} \tag{23}$$

then

$$\lim_n \left\| \Delta_g \left(u_m - u^+ \right) \right\|_2 = 0$$

and $u_m \to u^+$ strongly in H_2^2.

Since by assumption

$$\left| a_\lambda^+ \right| < \frac{2}{n K_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}}$$

we get the condition (23). □

THEOREM 3.6 *Let $\lambda \in (0, \lambda_\circ)$ and suppose that a sequence $(u_m)_m \subset N_\lambda^-$ fulfils*

$$\begin{cases} J_\lambda(u_m) \to a_\lambda^- + o(1) \\ \nabla J_\lambda(u_m) \to 0 \text{ in } H_2^2(M)' \end{cases}$$

with

$$a_\lambda^- < \frac{2}{n\, K_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n}{4}-1}}. \tag{24}$$

Then the functional J_λ has a minimizer u^- in N_λ^- and it satisfies:

(i) $J_\lambda(u^-) = \alpha_\lambda^- > 0$,
(ii) *u^- is a non trivial solution of equation (1).*

Proof The proof is similar to that of Theorem 3.5, so we omit it. □

Remark 1 The non trivial solutions u^+ and u^- of Equation (1) given by Theorems 3.5 and 3.6 are distinct since $u^+ \in N_\lambda^+$, $u^- \in N_\lambda^-$ and $N_\lambda^+ \cap N_\lambda^- = \emptyset$.

4. The sharp case $\sigma = 2$ and $\mu = 4$

By Section 3, for any $\sigma \in (0, 2)$ and $\mu \in (2, 4)$, there is a weak solution $u_{\sigma,\mu}^+ \in N_\lambda^+$ (resp. $u_{\sigma,\mu}^- \in N_\lambda^-$) of Equation (3). Now we are going to show that the sequences $\left(u_{\sigma,\mu}^+\right)_{\sigma,\mu}$ and $\left(u_{\sigma,\mu}^-\right)_{\sigma,\mu}$ are bounded in $H_2^2(M)$. First of all we have

$$J_{\lambda,\sigma,\mu}(u_{\sigma,\mu}) = \frac{1}{2} \left\| u_{\sigma,\mu} \right\|^2 - \frac{1}{N} \int_M f(x) \left| u_{\sigma,\mu} \right|^N dv_g - \frac{1}{q} \lambda \int_M \left| u_{\sigma,\mu} \right|^q dv_g$$

and since $u_{\sigma,\mu} \in N_\lambda$, we infer that

$$J_{\lambda,\sigma,\mu}(u_{\sigma,\mu}) = \frac{N-2}{2N} \left\| u_{\sigma,\mu} \right\|^2 - \lambda \frac{N-q}{Nq} \int_M \left| u_{\sigma,\mu} \right|^q dv_g.$$

For a smooth function φ on M, denotes by $\varphi^- = \min(0, \min_{x\in M}\varphi(x))$. Let $K(n,2,\sigma)$ be the best constant in the Hardy–Sobolev inequality and $A(\varepsilon,\sigma)$ the correspondent constant.

THEOREM 4.1 *Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$. For every $(\sigma,\mu) \in (0,2)\times(0,4)$ let $(u_{m,\sigma,\mu})_{m\in N}$ be a sequence in $N^+_{\lambda,\sigma,\mu}$ such that*

$$\begin{cases} J_{\lambda,\sigma,\mu}(u_{m,\sigma,\mu}) = a^+_{\lambda,\sigma,\mu} + o(1) & (resp. a^-_{\lambda,\sigma,\mu} + o(1)) \\ J_{\lambda,\sigma,\mu}(u_{m,\sigma,\mu}) \to 0 \text{ in } H^2_2(M)' & \text{as } m \to \infty \end{cases}.$$

Suppose that

$$\left|a^+_{\lambda,\sigma,\mu}\right| < \frac{2}{n\,K(n,2)^{\frac{n}{4}}\left(\max_{x\in M} f(x)\right)^{\frac{n-4}{4}}} \times \left(\text{resp.} a^-_{\lambda,\sigma,\mu} < \frac{2}{n\,K(n,2)^{\frac{n}{4}}\left(\max_{x\in M} f(x)\right)^{\frac{n-4}{4}}}\right)$$

and

$$\frac{1}{2} + a^- K(n,1,2)^2 + b^- K(n,2,4)^2 > 0$$

then the equation

$$\Delta^2 u - \nabla^\mu\left(\frac{a}{\rho^2}\nabla_\mu u\right) + \frac{bu}{\rho^4} = f\,|u|^{N-2}u + \lambda\,|u|^{q-2}u \tag{25}$$

has a non trivial weak solution $u^+ \in N^+_{\lambda,\sigma,\mu}$.

Proof In a first step we have to show that the sequence $(\Lambda_{\sigma,\mu})_{\sigma,\mu}$ of constants of coercivity does not converge to 0 as $(\sigma,\mu) \to (2^-,4^-)$. Since the operator P_g is coercive, the first $\lambda_{\sigma,\mu}$ eigenvalue of P_g is positive and we could consider it as the constant of coercivity. To achieve this task, let $\delta,\delta' \in (0,\delta_M)$ with $\delta < \delta'$ where δ_M is the injectivity radius of M. If $u_{\sigma,\mu}$ denotes the eigenfunction corresponding to $\lambda_{\sigma,\mu}$ with $\|u_{\sigma,\mu}\|_2 = 1$ and suppose that $\lambda_{\sigma,\mu} \to 0$ as $(\sigma,\mu) \to (2^-,4^-)$. It is obvious that

$$\begin{aligned} \lambda_{\sigma,\mu} &= \frac{1}{2}\int_M (\Delta u_{\sigma,\mu})^2 dv_g + \frac{1}{2}\int_M a\frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma} dv_g + \frac{1}{2}\int_M b\frac{(u_{\sigma,\mu})^2}{\rho^\mu} dv_g \\ &= \frac{1}{2}\int_{B(P,\delta)} (\Delta u_{\sigma,\mu})^2 dv_g + \frac{1}{2}\int_{B(P,\delta)} a\frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma} dv_g + \frac{1}{2}\int_{B(P,\delta)} b\frac{(u_{\sigma,\mu})^2}{\rho^\mu} dv_g \\ &\quad + \frac{1}{2}\int_{M-B(P,\delta)} (\Delta u_{\sigma,\mu})^2 dv_g + \frac{1}{2}\int_{M-B(P,\delta)} a\frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma} dv_g \\ &\quad + \frac{1}{2}\int_{M-B(P,\delta)} b\frac{(u_{\sigma,\mu})^2}{\rho^\mu} dv_g \end{aligned} \tag{26}$$

where $B(P,\delta)$ is the geodesic ball of center P and of radius δ. Let $\eta \in C^\infty(M)$ such that

$$\eta_\delta(x) = \begin{cases} 1 & \text{if } x \in B(P,\delta) \\ 0 & \text{if } x \notin B(P,\delta') \end{cases}$$

On the other hand if $\exp_P^{-1} : B(P,\delta') \to B(0,\delta')$ is the normal chart at the point P and for any $Q \in B(P,\delta')$, $\rho(Q) = d(P,Q) = |x|$, let $\widetilde{u}_{\sigma,\mu} = u_{\sigma,\mu} o \exp_P^{-1}$, the support of $\widetilde{u}_{\sigma,\mu}$ is in $\left\{x \in R^n : |x| < \delta'\right\}$and since it is well known that there exists $\varepsilon > 0$ such that:

$$(1-\varepsilon)^{n-1}\, dx \le dv_g \le (1+\varepsilon)^{n-1} dx.$$

Consequently and if we put:

$$v_{\sigma,\mu} = \eta_\delta u_{\sigma,\mu}$$

we get:

$$\int_{B(P,\delta)} \frac{\left(u_{\sigma,\mu}\right)^2}{\rho^\mu}\, dv_g \le I_1 = \int_{B(P,\delta')} \frac{\left(v_{\sigma,\mu}\right)^2}{\rho^\mu}\, dv_g = \int_{R^n} \frac{\left|\widetilde{v}_{\sigma,\mu}\right|^2}{|x|^\mu} \sqrt{|g|}dx$$
$$\le (1+\varepsilon)^{n-1} \int_{R^n} \frac{\left|\widetilde{v}_{\sigma,\mu}\right|^2}{|x|^\mu}\, dx$$

and by applying twice the Hardy inequality we get:

$$\begin{aligned}
I_1 &\le (1+\varepsilon)^{n-1} C \int_{R^n} \frac{\left|\widetilde{\nabla}\widetilde{v}_{\sigma,\mu}\right|^2}{|x|^{\mu-2}}\, dx \\
&\le (1+\varepsilon)^{n-1} C \int_{R^n} \left|\widetilde{\nabla}\left|\widetilde{\nabla}\widetilde{v}_{\sigma,\mu}\right|\right|^2 dx \\
&\le (1+\varepsilon)^{n-1} C \int_{R^n} \left|\widetilde{\nabla}^2\widetilde{v}_{\sigma,\mu}\right|^2 dx = (1+\varepsilon)^{n-1} C \int_{R^n} (\widetilde{\Delta}\widetilde{v}_{\sigma,\mu})^2 dx \\
&\le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{n-1} C \int_{R^n} (\widetilde{\Delta}\widetilde{v}_{\sigma,\mu})^2 \sqrt{|g|}dx = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{n-1} C \int_{B(P,\delta')} (\Delta v_{\sigma,\mu})^2 dv_g
\end{aligned} \tag{27}$$

where C is a universal constant.

Hence

$$\left(\int_{B(P,\delta)} \frac{\left(u_{\sigma,\mu}\right)^2}{\rho^\mu}\, dv_g\right)^{\frac{1}{2}} \le K_{\delta'}(n,2,\mu) \int_{B(P,\delta')} (\Delta v_{\sigma,\mu})^2 dv_g. \tag{28}$$

where $K_{\delta'}(n,2,\mu)$ is the best constant in (27) with $K_{\delta'}(n,2,\mu) \to K(n,2,\mu)$ as $\delta' \to 0$.

The same computations lead to

$$\int_{B(P,\frac{\delta}{2})} \frac{\left|\nabla v_{\sigma,\mu}\right|^2}{\rho^\sigma}\, dv_g \le K_\delta(n,1,\sigma) \int_{B(0,\delta)} \left(\Delta v_{\sigma,\mu}\right)^2 dv_g \tag{29}$$

where $K_{\delta'}(n,1,\sigma) \to K(n,1,\sigma)$ as $\delta' \to 0$.

Now plugging (28) and (29) in (26) we obtain:

$$\begin{aligned}
2\lambda_{\sigma,\mu} \ge& \left(\frac{1}{2} + a^- K_{\delta'}(n,1,\sigma) + b^- K_{\delta'}(n,2,\mu)\right) \int_{B(P,\delta)} \left(\Delta u_{\sigma,\mu}\right)^2 dv_g \\
&+ \frac{1}{2} \int_{B(P,\delta)} \left(\Delta u_{\sigma,\mu}\right)^2 dv_g + \left(a^- K_{\delta'}(n,1,\sigma) + b^- K_{\delta'}(n,2,\mu)\right) \\
&\times \int_{B(P,\delta')\setminus B(P,\delta)} \left(\Delta v_{\sigma,\mu}\right)^2 dv_g.
\end{aligned} \tag{30}$$

On the other hand direct computations give:

$$\begin{aligned}&\int_{B(P,\delta')\setminus B(P,\delta)} \left(\Delta v_{\sigma,\mu}\right)^2 dv_g \\ &= \int_{B(P,\delta')\setminus B(P,\delta)} (\Delta\eta_\delta)^2 u_{\sigma,\mu}^2 dv_g + 4\int_{B(P,\delta')\setminus B(P,\delta)} \left\langle \nabla\eta_\delta, \nabla u_{\sigma,\mu}\right\rangle^2 dv_g \\ &\quad + \int_{B(P,\delta')\setminus B(P,\delta)} \eta_\delta^2 \left(\Delta u_{\sigma,\mu}\right)^2 dv_g - 4\int_{B(P,\delta')\setminus B(P,\delta)} \left\langle \nabla\eta_\delta, \nabla u_{\sigma,\mu}\right\rangle u_{\sigma,\mu} (\Delta\eta_\delta)\, dv_g \\ &\quad + 2\int_{B(P,\delta')\setminus B(P,\delta)} \eta_\delta u_{\sigma,\mu} \Delta\eta_\delta \Delta u_{\sigma,\mu} dv_g - 4\int_{B(P,\delta')\setminus B(P,\delta)} \eta_\delta \left\langle \nabla\eta_\delta, \nabla u_{\sigma,\mu}\right\rangle \Delta u_{\sigma,\mu} dv_g.\end{aligned}$$

Now noting that $|\nabla\eta_{\delta'}| \le C\delta'$, $\left|\Delta\eta'_\delta\right| \le C\delta'^2$, thanks to the Hölder inequality we infer that:

$$\begin{aligned}&\int_{B(P,\delta')\setminus B(P,\delta)} \left(\Delta v_{\sigma,\mu}\right)^2 dv_g \\ &\le \int_{B(P,\delta')\setminus B(P,\delta)} \left(\Delta u_{\sigma,\mu}\right)^2 dv_g + C\delta'^4 \int_{B(P,\delta')\setminus B(P,\delta)} u_{\sigma,\mu}^2 dv_g \\ &\quad + 4C\delta'^2 \int_{B(P,\delta')\setminus B(P,\delta)} \left|\nabla u_{\sigma,\mu}\right|^2 dv_g \\ &\quad + 4C\delta'^3 \left(\int_{B(P,\delta')\setminus B(P,\delta)} \left|\nabla u_{\sigma,\mu}\right|^2 dv_g\right)^{\frac{1}{2}} \left(\int_{B(P,\delta')\setminus B(P,\delta)} \left|u_{\sigma,\mu}\right|^2 dv_g\right)^{\frac{1}{2}} \\ &\quad + 2C\delta'^2 \left(\int_{B(P,\delta')\setminus B(P,\delta)} \left|u_{\sigma,\mu}\right|^2 dv_g\right)^{\frac{1}{2}} \left(\int_{B(P,\delta')\setminus B(P,\delta)} \left(\Delta u_{\sigma,\mu}\right)^2 dv_g\right)^{\frac{1}{2}} \\ &\quad + 4C\delta' \left(\int_{B(P,\delta')\setminus B(P,\delta)} \left|\nabla u_{\sigma,\mu}\right|^2 dv_g\right)^{\frac{1}{2}} \left(\int_{B(P,\delta')\setminus B(P,\delta)} \left(\Delta u_{\sigma,\mu}\right)^2 dv_g\right)^{\frac{1}{2}}\end{aligned}$$

where $C > 0$ is a universal constant.

Hence

$$\int_{B(P,\delta')\setminus B(P,\delta)} \left(\Delta v_{\sigma,\mu}\right)^2 dv_g \le \int_{B(P,\delta')\setminus B(P,\delta)} \left(\Delta u_{\sigma,\mu}\right)^2 dv_g + O(\delta'). \tag{31}$$

If we choose δ' close to δ such that

$$\begin{aligned}&\frac{1}{2}\int_{B(P,\delta)} \left(\Delta u_{\sigma,\mu}\right)^2 dv_g + \left(a^- K_{\delta'}(n,1,\sigma) + b^- K_{\delta'}(n,2,\mu)\right) \\ &\quad \times \int_{B(P,\delta')\setminus B(P,\delta)} \left(\Delta u_{\sigma,\mu}\right)^2 dv_g \ge 0\end{aligned}$$

we deduce by (30) that

$$2\lambda_{\sigma,\mu} \ge \left(\frac{1}{2} + a^- K_{\delta'}(n,1,\sigma) + b^- K_{\delta'}(n,2,\mu)\right) \int_{B(P,\delta')} \left(\Delta v_{\sigma,\mu}\right)^2 dv_g \ge 0$$

so if $\frac{1}{2} + a^- K_{\delta'}(n,1,\sigma) + b^- K_{\delta'}(n,2,\mu) > 0$, we get

$$0 = \int_{B(P,\delta')} \left(\Delta v_{\sigma,\mu}\right)^2 dv_g = \int_{B(P,\delta')} \left|\nabla v_{\sigma,\mu}\right|^2 dv_g.$$

Hence $v_{\sigma,\mu} = \eta_{\delta'} u_{\sigma,\mu} = 0$ almost everywhere on M, then $u_{\sigma,\mu} = 0$ a.e. in the ball $B(P,\delta)$ and since P is arbitrary chosen in M we conclude that $u_{\sigma,\mu} = 0$ a.e. in M which contradicts the fact $\|u_{\sigma,\mu}\|_2 = 1$. Hence $\liminf_{(\sigma,\mu)\to(2^-,4^-)} \lambda_{\sigma,\mu} > 0$.

In a second step, we will show that the equation (25) has a non trivial weak solution. For simplicity we write $(u_m)_m$ instead of $(u_{m,\sigma,\mu})_m$. Let $(u_m)_m \subset N^+_{\lambda,\sigma,\mu}$, by the same arguments as in the proof of Theorem 3.5 we obtain that:

$$\|u_m\|^2_{\sigma,\mu} \le \frac{2qN}{(q-2)(N-2)} a^+_{\lambda,\sigma,\lambda} + o(1)$$

where

$$\|u_m\|^2_{\sigma,\mu} = \|\Delta u_m\|_2^2 + \int_M \left(\frac{a(x)}{\rho^\sigma} |\nabla_g u_m|^2 + \frac{b(x)}{\rho^\mu} u_m^2 \right) dv_g$$

and

$$\left| a^+_{\lambda,\sigma,\lambda} \right| < \frac{2}{n\, K(n,2)^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}}.$$

By Theorem 3.6, the Equation (3) admits a non trivial weak solution $u^+_{\sigma,\mu}$ in $H_2^2(M)$ i.e. for every $\varphi \in H_2^2(M)$:

$$\int_M \left(\Delta u^+_{\sigma,\mu} \Delta\varphi + \frac{a}{\rho^\sigma} \nabla u^+_{\sigma,\mu} \nabla\varphi + \frac{b}{\rho^\mu} u^+_{\sigma,\mu}\varphi \right) dv_g$$
$$= \lambda \int_M \left| u^+_{\sigma,\mu} \right|^q dv_g + \int_M f \left| u^+_{\sigma,\mu} \right|^N dv_g$$

where

$$0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)} \Lambda^{\frac{q}{2}}_{\sigma,\mu}}{V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K(n,2), A_\varepsilon))^{\frac{q}{2}} \tau^{q-2}} = \lambda_{o,\sigma,\mu}.$$

We have to show that the sequence $\left(u^+_{\sigma,\mu}\right)_{\alpha,\mu}$ converges up to a subsequence to a non trivial weak solution $u^+ \in H_2^2(M)$ to equation (25). Since $\left(u^+_{\sigma,\mu}\right)_{\alpha,\mu}$ is a bounded sequence in $H_2^2(M)$, by reflexivity of this latter space and the compactness of the embedding $H_2^2(M) \subset H_p^k(M)$ ($k = 0,1$; $p < N$), we obtain up to a subsequence we have: as $(\sigma,\mu) \to (2^-, 4^-)$

$$u^+_{\sigma,\mu} \to u^+ \text{ weakly in } H_2^2(M)$$
$$u^+_{\sigma,\mu} \to u^+ \text{ strongly in } L^p(M), \quad p < N$$
$$\nabla u^+_{\sigma,\mu} \to \nabla u^+ \text{ strongly in } L^p(M), \quad p < 2^* = \frac{2n}{n-2}$$

and

$$u^+_{\sigma,\mu} \to u^+ \text{ a.e. in } M.$$

Moreover by Theorem 3.2, we have: $\nabla u^+_{\sigma,\mu} \to \nabla u^+$ weakly in $L^2(M,\rho^{-2})$ and $u^+_{\sigma,\mu} \to u^+$ weakly in $L^2(M,\rho^{-4})$ i.e. for any $\varphi \in H_2^2(M)$, we have:

$$\int_M \frac{a(x)}{\rho^2} \nabla u^+_{\sigma,\mu} \nabla\varphi dv_g = \int_M \frac{a(x)}{\rho^2} \nabla u^+ \nabla\varphi dv_g + o(1)$$

and

$$\int_M \frac{b(x)}{\rho^4} u^+_{\sigma,\mu} \varphi dv_g = \int_M \frac{b(x)}{\rho^4} u^+ \varphi dv_g + o(1).$$

As a consequence of these latter equalities, we obtain:

$$\begin{aligned}&\int_M \left(\frac{a(x)}{\rho^\sigma} \nabla u^+_{\sigma,\mu} - \frac{a(x)}{\rho^2} \nabla u^+ \right) \nabla \varphi dv_g \\ &= \int_M \left(\frac{a(x)}{\rho^\sigma} \nabla u^+_{\sigma,\mu} - \frac{a(x)}{\rho^2} \left(\nabla u^+_{\sigma,\mu} - \nabla u^+_{\sigma,\mu} \right) - \frac{a(x)}{\rho^2} \nabla u^+ \right) \nabla \varphi dv_g.\end{aligned}$$

Hence

$$\begin{aligned}&\left| \int_M \left(\frac{a(x)}{\rho^\sigma} \nabla u^+_{\sigma,\mu} - \frac{a(x)}{\rho^2} \nabla u^+ \right) \nabla \phi dv_g \right| \\ &\quad \le \left| \int_M \left(\frac{a(x)}{\rho^\sigma} \nabla u^+_{\sigma,\mu} - \frac{a(x)}{\rho^2} \nabla_g u^+_{\sigma,\mu} \right) \nabla \phi dv_g \right| \\ &\qquad + \left| \int_M \left(\frac{a(x)}{\rho^2} \nabla u^+_{\sigma,\mu} - \frac{a(x)}{\rho^2} \nabla u^+ \right) \nabla \phi dv_g \right| \\ &\quad \le \int_M \left| \frac{1}{\rho^\sigma} - \frac{1}{\rho^2} \right| \left| a(x) \nabla_g u^+_{\sigma,\mu} \right| dv_g + \left| \int_M \frac{a(x)}{\rho^2} \nabla_g \left(u^+_{\sigma,\mu} - u^+ \right) \right| dv_g.\end{aligned}$$

By the weak convergence of $\left(u^+_{\sigma,\mu}\right)_{\sigma,\mu}$ in $L^2(M, \rho^{-2})$ and the dominated Lebesgue's convergence theorem, we obtain that:

$$\int_M \left(\frac{a(x)}{\rho^\sigma} \nabla u^+_{\sigma,\mu} - \frac{a(x)}{\rho^2} \nabla u^+ \right) dv_g = o(1).$$

Also, we have:

$$\begin{aligned}&\int_M \left(\frac{b(x)}{\rho^\mu} u^+_{\sigma,\mu} - \frac{b(x)}{\rho^4} u^+ \right) dv_g \\ &= \int_M \left(\frac{b(x)}{\rho^\mu} u^+_{\sigma,\mu} - \frac{b(x)}{\rho^4} u^+_{\sigma,\mu} + \frac{b(x)}{\rho^4} u^+_{\sigma,\mu} - \frac{b(x)}{\rho^4} u^+ \right) dv_g\end{aligned}$$

so

$$\begin{aligned}&\left| \int_M \left(\frac{b(x)}{\rho^\mu} u^+_{\sigma,\mu} - \frac{b(x)}{\rho^4} u^+ \right) dv_g \right| \\ &\le \int_M \left| b(x) u^+_{\sigma,\mu} \right| \left| \frac{1}{\rho^\mu} - \frac{1}{\rho^4} \right| dv_g + \left| \int_M \frac{b(x)}{\rho^4} \left(u^+_{\sigma,\mu} - u^+ \right) dv_g \right|\end{aligned}$$

and by the same arguments, we obtain that:

$$\int_M \left(\frac{b(x)}{\rho^\mu} u^+_{\sigma,\mu} - \frac{b(x)}{\rho^4} u^+ \right) \phi dv_g = o(1).$$

Resuming we get that u^+ is a weak solution to Equation (25).

Now, we have to show that u^+ is non trivial and $u^+ \in N^+_\lambda$, but this is the same as in Theorem 3.5 and . To achieve the proof of Theorem 4.1 we must show that equation (25) has a second solution $u^- \in N^-_\lambda$. □

5. Test functions

To give the proof of the main result, we consider a normal geodesic coordinate system centred at x_o. Let $S_{x_o}(\rho)$ the geodesic sphere centred at x_o and of radius ρ strictly less that the injectivity radius d. Let dv_h be the volume element of the $n-1$ -dimensional Euclidean unit sphere S^{n-1} endowed with its canonical metric and put

$$G(\rho) = \frac{1}{\omega_{n-1}} \int\limits_{S(\rho)} \sqrt{|g(x)|}dv_h$$

where ω_{n-1} is the volume of S^{n-1} and $|g(x)|$ the determinant of the Riemannian metric g. The Taylor's expansion of $G(\rho)$ in a neighborhood of x_o expresses as

$$G(\rho) = 1 - \frac{S_g(x_\circ)}{6n}\rho^2 + o(\rho^2)$$

where $S_g(x_\circ)$ is the scalar curvature of M at $x_\circ$.
If $B(x_\circ, \delta)$ is the geodesic ball centred at $x_\circ$ and of radius δ such that $0 < 2\delta < d$, we consider the following cutoff smooth function η on M

$$\eta(x) = \begin{cases} 1 & \text{on } B(x_o, \delta) \\ 0 & \text{on } M - B(x_o, 2\delta) \end{cases} .$$

Define the following radial function

$$u_\epsilon(x) = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{8}} \frac{\eta(\rho)}{((\rho\theta)^2 + \epsilon^2)^{\frac{n-4}{2}}}$$

where $\theta > 1$ and $\rho = d(x_o, x)$ is the distance from x_o to x and $f(x_\circ) = \max_{x\in M} f(x)$. We need also the following integrals: for any real positive numbers p, q such that $p - q > 1$ we put

$$I_p^q = \int_0^{+\infty} \frac{t^q}{(1+t)^p} dt$$

which fulfill the following relations

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \quad \text{and} \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q .$$

5.1. *Proof of Theorem 1.2 in case* dim$(M) > 6$

Proof The proof of Theorem 1.2 reduces to show that the condition (C1) of Theorem 3.5 which is the same condition (24) of Theorem 3.6 is satisfied and since at the end of Section 1, we have shown that for a given $u \in H_2^2(M)$ there exist two real numbers $t^- > 0$ and $t^+ > 0$ such that $t^- u \in N_\lambda^-$ and $t^+ u \in N_\lambda^+$ for sufficiently small λ, so it suffices to show that

$$\sup_{t>0} J_\lambda(tu_\epsilon) < \frac{2}{nK_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n}{4}-1}} .$$

The expression of $\int_M f(x)\,|u_\epsilon(x)|^N\,dv_g$ is well known (see e.g. [10]) and is given in case $n > 6$ by

$$\int_M f(x)\,|u_\epsilon(x)|^N\,dv_g = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1-\left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)}+\frac{S_g(x_\circ)}{6(n-2)}\right)\epsilon^2+o(\epsilon^2)\right).$$

The following estimation is computed in [9] and is given by

$$\int_M \frac{a(x)}{\rho^\sigma}|\nabla u_\epsilon|^2\,dv_g \le 2^{-1+\frac{1}{r}}\theta^{-n\frac{r}{r-1}}(n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}}\left\|\frac{a}{\rho^\sigma}\right\|_r\omega_{n-1}^{1-\frac{1}{r}}\epsilon^{-(n-4)+2-\frac{n}{r}} \times\left(I_{\frac{(n-2)r}{r-1}}^{1+\frac{n-2}{2}\cdot\frac{r-1}{r}}+o(\epsilon^2)\right).$$

Letting

$$A = K_\circ^{\frac{n}{4}}\frac{(n-4)^{\frac{n}{4}+1}\times(\omega_{n-1})^{\frac{r-1}{r}}}{2^{\frac{r-1}{r}}}(n(n^2-4))^{\frac{n-4}{4}}\left(I_{\frac{(n-2)r}{r-1}}^{\frac{n-2}{2}+\frac{r}{r-1}}\right)^{\frac{r-1}{r}} \tag{32}$$

we obtain

$$\int_M a(x)\,|\nabla u_\epsilon|^2\,dv_g \le \epsilon^{2-\frac{n}{r}}\theta^{-n\frac{r}{r-1}}\frac{A}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left\|\frac{a}{\rho^\sigma}\right\|_r\left(1+o(\epsilon^2)\right).$$

Also the estimation of the third term of J_λ is computed in [9] as

$$\int_M \frac{b(x)}{\rho^\mu}u_\epsilon^2 dv_g \le \|b\|_s\left(\frac{(n-4)n(n^2-4)}{f(x_\circ)}\right)^{\frac{n-4}{4}}\left(\frac{\omega_{n-1}}{2}\right)^{\frac{s-1}{s}}\epsilon^{4-\frac{n}{s}}\theta^{-n\frac{s}{s-1}} \times\left(\left(I_{\frac{(n-4)s}{(s-1)}}^{\frac{n}{2}}\right)^{\frac{s-1}{s}}+o(\epsilon^2)\right).$$

Putting

$$B = K_\circ^{\frac{n}{4}}((n-4)n(n^2-4))^{\frac{n-4}{4}}\left(\frac{\omega_{n-1}}{2}\right)^{\frac{s-1}{s}}\left(I_{\frac{(n-4)s}{(s-1)}}^{\frac{n}{2}}\right)^{\frac{s-1}{s}} \tag{33}$$

we get

$$\int_M b(x)u_\epsilon^2 dv_g \le \epsilon^{4-\frac{n}{s}}\theta^{-n\frac{s}{s-1}}\frac{\left\|\frac{b}{\rho^\mu}\right\|_s B}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1+o(\epsilon^2)\right).$$

The computation of $\int_M (\Delta u_\epsilon)^2\,dv_g$ is well known see for example [10] and is given by

$$\int_M (\Delta u_\epsilon)^2\,dv_g = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1-\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2+o(\epsilon^2)\right).$$

Resuming we get

$$\int_M \left((\Delta u_\epsilon)^2 + a(x)\,|\nabla u_\epsilon|^2 + b(x)u_\epsilon^2\right) dv_g$$
$$\leq \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}\, f(x_\circ)^{\frac{n-4}{4}}} \left(1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{r}{r-1}} A \left\|\frac{a}{\rho^\sigma}\right\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{s}{s-1}} B \left\|\frac{b}{\rho^\mu}\right\|_s - \frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_\circ)\epsilon^2 + o(\epsilon^2)\right).$$

Now, we have

$$J_\lambda(tu_\epsilon) \leq J_o(tu_\epsilon) = \frac{t^2}{2}\|u_\epsilon\|^2 - \frac{t^N}{N}\int_M f(x)\,|u_\epsilon(x)|^N dv_g$$
$$\leq \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}\, f(x_\circ)^{\frac{n-4}{4}}} \left\{\frac{1}{2}t^2\left(1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{r}{r-1}} A \left\|\frac{a}{\rho^\sigma}\right\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{s}{s-1}} B \left\|\frac{b}{\rho^\mu}\right\|_s\right) - \frac{t^N}{N} + \left[\left(\frac{\Delta f(x_o)}{2(n-2)f(x_o)} + \frac{S_g(x_o)}{6(n-1)}\right)\frac{t^N}{N} - \frac{1}{2}t^2\frac{n^2+4n-20}{6\left(n^2-4\right)(n-6)}S_g(x_o)\right]\epsilon^2\right\} + o\left(\epsilon^2\right)$$

and letting ϵ small enough so that

$$1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{r}{r-1}} A \left\|\frac{a}{\rho^\sigma}\right\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{s}{s-1}} B \left\|\frac{b}{\rho^\mu}\right\|_s \leq \theta^{\frac{4}{n}}$$

and since the function $\varphi(t) = \alpha\frac{t^2}{2} - \frac{t^N}{N}$, with $\alpha > 0$ and $t > 0$, attains its maximum at $t_o = \alpha^{\frac{1}{N-2}}$ and

$$\varphi(t_o) = \frac{2}{n}\alpha^{\frac{n}{4}}.$$

Consequently, we get

$$J_\lambda(tu_\epsilon) \leq \frac{2}{nK_\circ^{\frac{n}{4}}\, f(x_\circ)^{\frac{n-4}{4}}}\left\{1 + \left[\left(\frac{\Delta f(x_o)}{2(n-2)f(x_o)} + \frac{S_g(x_o)}{6(n-1)}\right)\frac{t_o^N}{N} - \frac{1}{2}\frac{n^2+4n-20}{6\left(n^2-4\right)(n-6)}S_g(x_o)\,t_o^2\right]\epsilon^2\right\} + o\left(\epsilon^2\right).$$

So if $\frac{\Delta f(x_o)}{2(n-2)f(x_o)} + \frac{S_g(x_o)}{6(n-1)} < 0$ and $S_g(x_o) > 0$ we obtain:

$$\sup_{t\geq 0} J_\lambda(tu_\epsilon) < \frac{2}{nK_\circ^{\frac{n}{4}}\left(\max_{x\in M} f(x)\right)^{\frac{n}{4}-1}}$$

which completes the proof. □

5.2. *Proof of Theorem 1.2 in case* dim$(M) = 6$

Proof In case $n = 6$ the only term whose expression differs from the case $n > 6$ is the first term of J_λ and is given (see e.g. [10]) by

$$\int_M (\Delta u_\epsilon)^2 dv_g = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1 - \frac{2(n-4)}{n^2(n^2-4)I_n^{\frac{n}{2}-1}} S_g(x_\circ)\epsilon^2\log\left(\frac{1}{\epsilon^2}\right) + O(\epsilon^2)\right).$$

Letting ϵ small enough so that

$$1+\epsilon^{2-\frac{n}{r}}\theta^{-\frac{r}{r-1}}A\left\|\frac{a}{\rho^{\sigma}}\right\|_{r}+\epsilon^{4-\frac{n}{s}}\theta^{-\frac{s}{s-1}}B\left\|\frac{b}{\rho^{\mu}}\right\|_{s}\leq\theta^{\frac{4}{n}}$$

where A and B are given by (32) and (33), we get

$$\begin{aligned}J_{\lambda}\left(u_{\epsilon}\right)&\leq\frac{1}{2}\left\|u_{\epsilon}\right\|^{2}-\frac{1}{N}\int_{M}f(x)\left|u_{\epsilon}(x)\right|^{N}dv_{g}\\&\leq\frac{\theta^{-n}}{K_{\circ}^{\frac{n}{4}}\left(f\left(x_{\circ}\right)\right)^{\frac{n-4}{4}}}\left[\frac{1}{2}\theta^{\frac{4}{n}}t^{2}-\frac{t^{N}}{N}\right.\\&\qquad\left.-\frac{n-4}{n^{2}\left(n^{2}-4\right)I_{n}^{\frac{n}{2}-1}}S_{g}(x_{\circ})t^{2}\epsilon^{2}\log\left(\frac{1}{\epsilon^{2}}\right)\right]+O(\epsilon^{2}).\end{aligned}$$

So if $S_{g}(x_{\circ})>0$, we infer that:

$$\max_{t\geq0}J_{\lambda}\left(tu_{\epsilon}\right)<\frac{2}{n\,K_{\circ}^{\frac{n}{4}}\left(f\left(x_{\circ}\right)\right)^{\frac{n-4}{4}}}$$

provided that

$$S_{g}(x_{\circ})>0.$$

Which achieves the proof. □

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On singular elliptic equations involving critical Sobolev exponent

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On singular elliptic equations involving critical Sobolev exponent

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Abstract. Given a $n-$dimensional compact Riemannian manifold (M, g) with $n \geq 5$, we consider the following semi-linear elliptic equation :

$$P_g(u) := \Delta_g^2 u + div_g \left(a(x)\nabla_g u\right) + b(x)u = f(x)\left|u\right|^{N-2} u + \lambda h(x)\left|u\right|^{q-2} u$$

where the functions a, b and h are in suitable Lebesgue spaces, $2 < q < N$ and $\lambda > 0$ a real parameter, f is a smooth positive function and the operator P_g is coercive. Under some additional conditions, we obtain results concering the existence of strong solutions of the above equation in $H_2^2(M)$.

1. Introduction

In 1983 Paneitz discovered a particular conformally fourth-order operator defined on $4-$dimensional smooth Riemannian manifolds [1]. In 1987, Branson generalized the definition to higher dimensions in [2] as follows. Let (M, g) be smooth, compact $n-$dimensional Riemannian manifold with $n \geq 5$, and $u \in C^4(M)$. The Paneitz-Branson operator P_g^n is then defined via [2]:

$$P_g^n(u) = \Delta_g^2 u - div_g \left(a_n(x)du\right) + Q_g^n u$$

where

$$a_n(x) = \frac{(n-2)^2+4}{2(n-2)(n-1)} S_g.g - \frac{4}{n-2} Ric_g,$$
$$Q_g^n = \frac{1}{(n-1)(n-4)} \Delta_g S_g + \frac{n^3-4n^2+16n-16}{4(n-1)^2(n-2)^2(n-4)} S_g^2 - \frac{4}{(n-4)(n-2)^2} \left|Ric_g\right|^2,$$

being Δ_g, S_g and Ric_g the Laplace-Beltrami operator, the scalar and the Ricci curvatures of g, respectively. The Paneitz-Branson operator enjoys interesting conformal properties that are very similar to those of the conformal Laplacian operator. Remark that if $\tilde{g} = \varphi^{\frac{4}{n-4}} g$, with φ a positive function of class $C^4(M)$, is a conformal metric to g, then for all $u \in C^4(M)$, $P_g^n(u\varphi) = \varphi^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(u)$. In particular, if $u \equiv 1$ then $P_g^n(\varphi) = Q_{\tilde{g}}^n \varphi^{\frac{n+4}{n-4}}$.

Many interesting results on Paneitz-Branson operator and related topics have been recently obtained by several authors, we refer the reader to Refs. [3]- [10]. Here we recall a few of these results that are pertinent to our investigation, see the list P1)-P3) below.

Let (M,g) be an $n-$dimensional compact, smooth and oriented Riemannian manifold with $n \geq 5$, $H_2^2(M)$ be the standard Sobolev space consisting of function in $L^2(M)$ whose derivatives up to the second order are in $L^2(M)$, and let $N = \frac{2n}{n-4}$ be the associated Sobolev critical exponent. Now, we define the best constant $K_\circ$ of the embedding $H_2^2(\mathbb{R}^n) \subset L^N(\mathbb{R}^n)$ given by

$$\frac{1}{K_o} = \frac{n(n^2-4)(n-4)\omega_n^{\frac{4}{n}}}{16}$$

where ω_n is the volume of the unit Euclidean $n-$sphere (S^n, h).

P1) In 2002, F. Robert and P. Esposito in [10] considered the following equation

$$\Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u = f(x)\left|u\right|^{N-2} u + h(x)\left|u\right|^{q-2} u$$

where: i) $a \in \Lambda_{(2,0)}^{+\infty}(M)$ is a smooth symmetric $(2,0)$-tensor field, ii) b, h, f are smooth functions in M, with f positive, and iii) $2 < q < N$. They established the following remarkable result:

Theorem 1 *Let (M,g) be an $n-$dimensional compact Einsteinian manifold with $n \geq 8$. Assume that P_g^n is coercive and let $f \in C^\infty(M)$, $f > 0$ such that there exists $x_\circ \in M$ with $f(x_\circ) = \max_{x\in M} f(x)$, $\Delta f(x_\circ) = 0$ and*

$$\frac{4(n^2-4n-4)}{3(n+2)}\left|Weyl_g(x_\circ)\right|^2 + (n-6)(n-8)\frac{\Delta_g^2 f(x_\circ)}{f(x_\circ)} + 2(n-6)(n-8)\frac{\langle \nabla_g f(x_\circ), Ric_g(x_\circ)\rangle}{f(x_\circ)} > 0.$$

Then, there exists $\tilde{g}$ conformal to g such that $Q_{\tilde{g}}^n(x) = f(x)$.

P2) In 2010, M. Benalili in [5] considered the equation:

$$\Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u = f(x)\left|u\right|^{N-2} u \tag{1}$$

where f is a positive $C^\infty-$function on M, $a \in L^r(M)$ and $b \in L^s(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$. He established the following result:

Theorem 2 *Let (M,g) is an $n-$dimensional compact manifold with $n \geq 8$ and for $2 < p < 5$, $\frac{9}{4} < s < 11$ or $n = 7$, $\frac{7}{2} < p < 9$, $\frac{7}{4} < s < 9$ assume that there exists $x_\circ \in M$ such that $f(x_\circ) = \max_{x\in M} f(x)$ and*

$$\frac{n^2+4n-20}{6(n-6)(n^2-4)}S_g(x_\circ) + \frac{(n-4)}{2n(n-2)}\frac{\Delta_g f(x_\circ)}{f(x_\circ)} > 0.$$

For $n = 6$, $\frac{3}{2} < p < 2$, $3 < s < 4$, assume that $S_g(x_\circ) > 0$. Then, the equation (1) has a weak solution in $H_2^2(M)$.

P3) Recently, M. Benalili and the author proved in [7], the following result:

Theorem 3 *Let (M,g) be a compact manifold of dimension $n \geq 6$, $a \in L^r(M)$, $b \in L^s(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$, $0 < q < 2$ and f a positive $C^\infty-$function on M. We suppose that P_g is coercive and the existence of a point $x_\circ \in M$ such $f(x_\circ) = \max_{x\in M} f(x)$ and*

$$\begin{cases} \frac{\Delta_g f(x_\circ)}{f(x_\circ)} < \frac{1}{3}\left(\frac{(n-1)n\left(n^2+4n-20\right)}{(n-6)(n-4)(n^2-4)}\left(1+\|a\|_r+\|b\|_s\right)^{-\frac{4}{n}} - \right) S_g(x_\circ) & \text{if } n > 6 \\ S_g(x_\circ) > 0 & \text{if } n = 6 \end{cases}$$

Then, there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, the equation

$$\Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u = f(x)\,|u|^{N-2}\,u + \lambda\,|u|^{q-2}\,u$$

has a weak solution in $H_2^2(M)$.

In this paper, we look for solutions to the following semi-linear elliptic equation:

$$P_g(u) := \Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u = f(x)\,|u|^{N-2}\,u + \lambda h(x)\,|u|^{q-2}\,u \tag{2}$$

where $a \in L^r(M)$, $b \in L^s(M)$ and $h \in L^d(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$ and $d > \frac{N}{N-q} := \alpha$, $2 < q < N$ and $\lambda > 0$ a real parameter. In doing so, we assume the following conditions:

(h^1) The operator P_g is coercive, that is: $\exists \Lambda > 0 : \langle P_g(u); u\rangle \geq \Lambda\,\|u\|^2_{H_2^2(M)}, \forall u \in H_2^2(M)$;
(h^2) The function h doesn't vanish almost everywhere on M.

(h^3) The parameter λ fulfills $0 < \lambda < \lambda_1$ with

$$\lambda_1 := \frac{q\,(N-2)}{2\,(N-q)}\Lambda^{\frac{q}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{-\frac{q}{2}}\,\|h\|_\alpha^{-1}.$$

Our main results state as follows:

Theorem 4 *Let (M, g) be an $n-$dimensional compact, smooth and oriented Riemannian manifold with $n > 6$ and f a smooth positive function on M. Let $a \in L^r(M)$, $b \in L^s(M)$ and $h \in L^d(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$, $d > \frac{N}{N-q}$ and $2 < q < N$. We assume that the conditions (h^1), (h^2) and (h^3) are satisfied and that there exists $x_\circ \in M$ such that $f(x_o) = \max_{x\in M} f(x)$ and*

$$\left(\frac{n(n-2\sqrt{6}+2)(n+2\sqrt{6}+2)-(n-6)(n-4)^3(n+2)}{3(n+2)(n-4)^2(n-6)\left(1+\|a\|_r+\|b\|_s\right)^{\frac{4}{n}}}S_g(x_\circ) - \frac{(n-4)\Delta f(x_\circ)}{2f(x_\circ)}\right) > 0.$$

Then, the equation (2) possesses a nontrivial solution in $H_2^2(M)$.

Theorem 5 *Let (M, g) be a compact, smooth and oriented Riemannian manifold of dimension $n = 6$ under the same conditions of theorem 4 with*

$$S_g(x_\circ) > 0$$

Then, the equation (2) possesses a nontrivial solution in $H_2^2(M)$.

2. Generic existence result

Throughout this section, we consider the energy functional J_λ, for each $u \in H_2^2(M)$,

$$J_\lambda(u) = \frac{1}{2}\int_M\left((\Delta_g u)^2 - a(x)\,|\nabla_g u|^2 + b(x)u^2\right)dv(g) - \frac{\lambda}{q}\int_M h(x)\,|u|^q\,dv(g) - \frac{1}{N}\int_M f(x)\,|u|^N\,dv(g)$$

First, we have the following lemma, whose proof is easy and can be found in [7].

Lemma 6 $\|u\| = (\int_M\left((\Delta_g u)^2 - a(x)\,|\nabla_g u|^2 + b(x)u^2\right)dv(g))^{\frac{1}{2}}$ *is an equivalent norm of the usual one of $H_2^2(M)$ if only if the operator P_g is coercive.*

The main tool to prove our result is the Mountain-Pass lemma of Ambrossetti-Rabinowitz given by the following lemma:

Lemma 7 *Let $J \in C^1(E, \mathbb{R})$ where $(E, \|.\|)$ is a Banach space. We assume that:*
(i) $J(0) = 0$.
(ii) $\exists r, R > 0$ such that $J(u) \geq R > 0$ for all $u \in E$ such that $\|u\| = r$.
(iii) $\exists v \in E$ such that $\limsup_{t \longrightarrow +\infty} J(tv) < 0$.

If

$$c = \min_{\eta \in \Gamma} \max_{t \in [0,1]} \left(J\left(\eta(t)\right)\right) \qquad \text{where} \quad \Gamma = \left\{ \eta \in C^1\left([0;1]\,; E\right) : \eta\left(0\right) = 0, \eta\left(1\right) = v \right\}$$

then there exists a sequence $(u_n)_n$ in E such that:

$$J\left(u_n\right) \longrightarrow c \qquad \text{and} \qquad \nabla J\left(u_n\right) \longrightarrow 0 \qquad \text{in } E^*$$

where E^* is the dual space of E. Moreover, we have that: $c \leq \sup_{t \geq 0} J(tv)$.

It is easily seen that J_λ is a C^1 functional and its Fréchet derivative is given by:

$$\begin{aligned}\left\langle \nabla J_\lambda\left(u\right), v\right\rangle = \int_M \left(\Delta_g u.\Delta_g v - a(x) g(\nabla_g u, \nabla_g v) + b(x) uv\right) dv(g) \quad + \\ -\lambda \int_M h(x) \left|u\right|^{q-2} uv dv(g) - \int_M f(x) \left|u\right|^{N-2} uv dv(g).\end{aligned}$$

Moreover, the functional J_λ verifies the Mountain-Pass conditions, namely:

Lemma 8 *Suppose that the conditions of (h^1), (h^2) and (h^3) of section 1 are satisfied. Then J_λ fulfills the following properties*
1-There exist constants $r, R > 0$ such that $J_\lambda(u) \geq R > 0$, $\|u\| = r$.
2-There exists $v \in H_2^2(M)$, with $\|v\| > r$, such that $J_\lambda(v) < 0$.

Lemma 9 *Let (M, g) be a $n-$dimensional compact, smooth and oriented Riemannian manifold with $n \geq 5$ and suppose that conditions (h^1)- (h^2) are satisfied. Then each Palais-Smale sequence at level c_λ is bounded in $H_2^2(M)$.*

Proof. The proof follows from the coerciveness of the operator P_g, the Sobolev's inequality and the condition (h^2). ■

Theorem 10 *Let (M, g) is an $n-$dimensional compact, smooth and oriented Riemannian manifold with $n \geq 5$. Let $(u_m)_m$ be a Palais-Smale sequence at level c_λ. Assume that conditions (h^1)-(h^2) and (h^3) are satisfied and that*

$$c_\lambda < \frac{1}{(1+\varepsilon)^{\frac{n}{n-4}} K_\circ^{\frac{n}{n-4}} \max_{x \in M} f(x)}.$$

Then, there is a subsequence of $(u_m)_m$ converging strongly in $H_2^2(M)$.

Proof. We follows closely the method used in [7]. ■

3. The sharp case

Let $P \in M$, we define the distance function ρ on M by

$$\rho_P(Q) = \begin{cases} d(P,Q) & \text{if} \quad d(P,Q) < i_g(M) \\ \delta(M) & \text{if} \quad d(P,Q) \geq i_g(M) \end{cases}$$

and $i_g(M)$ is the injectivity radius of M. Furthermore, we define the space $L^p(M, \rho^\gamma)$ as follows.

Definition 11 *Let (M, g) be a compact $5 \leq n$-dimensional Riemannian manifold. We consider the space $L^p(M, \rho^\gamma)$ where $1 \leq p \leq +\infty$ of measurable functions u on M such that $\rho^\gamma |u|^p$ is integrable, i.e.*

$$\|u\|^p_{p,\rho^\gamma} := \int_M \rho^\gamma |u|^p \, dv(g) < +\infty$$

Now, we use the following Hardy-Sobolev inequalities proven in [5] (the Hardy-Sobolev inequalities for the singular Yamabe equation was proven in [9]).

Theorem 12 *[5] Let (M, g) be a compact $5 \leq n$-dimensional Riemannian manifold and p, q and γ three real numbers satisfying $\frac{\gamma}{p} = \frac{n}{q} - \frac{n}{p} - 2$ and $2 \leq p \leq \frac{2n}{n-4}$.*
For any $\epsilon > 0$, there is a constant $A(\epsilon, q, \gamma)$ such that

$$\forall u \in H^2_2(M) : \|u\|^2_{p,\rho^\gamma} \leq (1+\epsilon) K(n,2,\gamma)^2 \|\Delta_g u\|^2_2 + A(\epsilon, q, \gamma) \|u\|^2_2$$

In particular: $K(n, 2, 0)^2 = K_\circ$ is the optimal constant of Sobolev inequality.

Theorem 13 *[5] Let (M, g) be a compact $5 \leq n$dimensional Riemannian manifold and p, q and γ three real numbers satisfying: $1 \leq q \leq p \leq \frac{nq}{n-2q}$ and $\gamma < 0$.*
•.If $\frac{\gamma}{p} = n(\frac{1}{q} - \frac{1}{p}) - 2$, then the imbedding $H^q_2(M) \subset L^p(M, \rho^\gamma)$ is continuous.
•.If $\frac{\gamma}{p} > n(\frac{1}{q} - \frac{1}{p}) - 2$, then the imbedding $H^q_2(M) \subset L^p(M, \rho^\gamma)$ is compact.

We consider the following equation:

$$\Delta^2_g u + div_g \left(\frac{a(x)}{\rho^\sigma} \nabla_g u \right) + \frac{b(x)}{\rho^\mu} u = f(x) |u|^{N-2} u + \lambda \frac{h(x)}{\rho^\beta} |u|^{q-2} u \tag{3}$$

where a, b and h are three smooth functions and the distance function defined before in section 1, $2 < q < N$ and $\lambda > 0$ a real parameter. The energy functional J_λ: $H^2_2(M) \longrightarrow \mathbb{R}$ associated to equation (3) is defined as:

$$\begin{aligned} J_\lambda(u) = \quad & \frac{1}{2} \int_M \left((\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\mu} u^2 \right) dv(g) + \\ & -\frac{\lambda}{q} \int_M \frac{h(x)}{\rho^\beta} |u|^q \, dv(g) - \frac{1}{N} \int_M f(x) |u|^N \, dv(g), \end{aligned}$$

where $u \in H^2_2(M)$ and it is well-known that the critical points of J_λ are the weak solutions of (3).

Theorem 14 *Let $0 < \sigma < \frac{n}{r} < 2$, $0 < \mu < \frac{n}{s} < 4$ and $0 < \beta < \frac{N}{d} < N - q$. We suppose that the conditions (h^1), (h^2) and (h^3) are satisfied and*

$$\sup_{u \in H^2_2(M)} J^{\sigma,\mu,\beta}_\lambda(u) < \frac{2}{n K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}$$

Then, the equation (3) has a non trivial solution $u_{\sigma,\mu,\beta} \in H^2_2(M)$.

Proof. The result follows in that if we put $\widetilde{a} = \frac{a(x)}{\rho^\sigma}$, $\tilde{b}(x) = \frac{b(x)}{\rho^\mu}$ and $\tilde{h}(x) = \frac{h(x)}{\rho^\beta}$, then $\widetilde{a} \in L^r(M)$, $\tilde{b} \in L^s(M)$ and $\tilde{h} \in L^d(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$ and $d > \frac{N}{N-q}$. ■

4. Critical cases

Strategies developed in [7] and [8] enable us to derive another result, that refers to the critical cases when $\sigma = 2$, $\mu = 4$, and $\beta = \frac{n(q-2)}{2} - 2q$.

Theorem 15 *Let (M, g) be an $n-$dimensional compact, smooth and oriented Riemannian manifold with $n \geq 5$ and suppose that the conditions (h^1), (h^2) and (h^3) are satisfied. In addition, let $(u_m)_m := (u_{\sigma_m,\mu_m,\beta_m})_m$ be a sequence in $H_2^2(M)$ such that:*

$$\begin{cases} J_\lambda^{\sigma,\mu,\beta}(u_m) \to c_\lambda^{\sigma,\mu,\beta} & \text{for all} \quad n \in \mathbb{N} \\ \nabla J_\lambda^{\sigma,\mu,\beta}(u_m) \to 0 & \text{weakly in} \quad H_2^2(M) \end{cases} \quad \text{with } c_\lambda^{\sigma,\mu,\beta} < \frac{2}{nK_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \tag{4}$$

and

$$1 + a^- \max(K(n, 2, \sigma); \quad A(\epsilon, \sigma)) + b^- \max(K(n, 2, \mu); \quad A(\epsilon, \mu)) > 0. \tag{5}$$

Then, the equation

$$\Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^\sigma}\nabla_g u\right) + \frac{b(x)}{\rho^\mu}u = f(x)\,|u|^{N-2}\,u + \lambda\frac{h(x)}{\rho^\beta}\,|u|^{q-2}\,u$$

has a nontrivial solution $u_{\sigma,\mu,\beta} \in H_2^2(M)$.

Proof. We follow closely the method used in [7] and [8]. First by using the condition (5) we obtain, as in [7], that the sequence $(\Lambda_{\alpha,\mu})_{\alpha,\mu}$ of constants of coerciveness of the operator $u \to \Delta_g^2 u + div_g\left(\frac{a(x)}{\rho^\sigma}\nabla_g u\right) + \frac{b(x)}{\rho^\mu}u$ is bounded below by a constant $\Lambda > 0$ as $(\alpha, \mu) \to (2^-, 4^-)$. Let $(u_m)_m \subset H_2^2(M)$, such that :

$$J_\lambda^{\sigma,\mu,\beta}(u_m) = c_\lambda^{\sigma,\mu,\beta} + o(1) \qquad \text{and} \qquad \nabla J_\lambda^{\sigma,\mu,\beta}(u_m) = o(1) \qquad \text{in } \left(H_2^2(M)\right)^*$$

Then we have:

$$J_\lambda^{\sigma,\mu,\beta}(u_m) - \frac{1}{N}\left\langle J_\lambda^{\sigma,\mu,\beta}(u_m), u_m\right\rangle = \left(\frac{1}{2} - \frac{1}{N}\right)\|u_m\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{N}\right)\int_M h(x)\,|u_m|^q\,dv(g)$$

By Hölder and Sobolev inequalities, we get that

$$J_\lambda^{\sigma,\mu,\beta}(u_m) - \frac{1}{N}\left\langle \nabla J_\lambda^{\sigma,\mu,\beta}(u_m), u_m\right\rangle = c_\lambda^{\sigma,\mu,\beta} + o(1)$$

and

$$c_\lambda^{\sigma,\mu,\beta} + o(1) \geq \left(\frac{1}{2} - \frac{1}{N}\right)\|u_m\|^2 - \left(\frac{1}{q} - \frac{1}{N}\right)(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\,\|h\|_\alpha\,\|u_m\|_{H_2^2(M)}^q$$

In addition the hypothesis (h^1) and (h^2) are satisfied and if we have $\|u_n\| \geq 1$, then we obtain

$$\|u_m\| \leq \left[\left(\frac{N-2}{2} - \lambda\frac{N-q}{q}\Lambda^{-\frac{q}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\,\|h\|_\alpha\right)^{-1} Nc_\lambda^{\sigma,\mu,\beta}\right]^{\frac{1}{q}} + o(1)$$

Then $(u_m)_m$ is bounded in $H_2^2(M)$. The rest of the proof is the same as in Theorem 10. ■

Concluding remark. To prove main Theorems given in the Introduction, let $\delta \in \left(0, \frac{i_g(M)}{2}\right)$ and $\eta \in C^\infty(M)$ such that:

$$\eta(x) = \begin{cases} 1 & \text{if} \quad x \in B(x_\circ, \delta) \\ 0 & \text{if} \quad x \in M - B(x_\circ, 2\delta) \end{cases}$$

For $\epsilon > 0$, we define the radial function u_ϵ by:

$$u_\epsilon(x) := \frac{\eta(x)}{\left(\epsilon^2 + (\xi\rho)^2\right)^{\frac{n-4}{2}}} \qquad \text{with} \quad \xi = (1 + \|a\|_r + \|b\|_s)^{\frac{1}{n}} . \tag{6}$$

We next point out that, by resorting to the strategy outlined in [7, 8], the function given by (6) can be proved to verify condition (4) of the generic theorem. This step completes our discussion on the solutions of Equation (2).

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Résumé

L'objet de cette thèse est l'étude, sur les variétés riemanniennes compactes l'existence et la multiplicité de solutions pour une équation aux dérivées partielles elliptiques de quatrième ordre. Les techniques utilisées basées sur la théorie des points critique de la fonctionnelle d'énergie.

Mots-clés

EDP - EDP avec singularité - Problème elliptiques non linéaire du quatrième ordre - Problème critique non linéaire - Quatrième ordre - Variété riemannienne compacte –Variété de Nehari- Exposent critique de Sobolev - Exposent critique de Hardy-Sobolev - Opérateur de Paneitz-Branson - Q-courbure.

الملخص

الغرض من هذا البحث هو دراسة عن ريمانتبات المدمجة متعددة وجود حلول لمعادلة تفاضلية جزئية من الشكل البيضوي ذات الدرجة الرابعة.ا لتقنيات المستخدمة مبنية على نظرية نقطة حرجة . الحلول التي يتم الحصول عليها عبارة عن نقاط حرجة للدالة المرفقة لها عبارة.

الكلمات المفتاحية

معادلة تفاضلية جزئية من الشكل البيضوي ذات الدرجة الرابعة - ريمانتبات المدمجة متعددة– نقاط حرجة

Abstract

The subject of this thesis consists in the study, on compact riemannian manifold with existence and multiplicity of solution to a class of fourth order nonlinear elliptic equations. The technique used relies on critical points of functional restricted to suitable manifold.

Keywords

EDP – Singular EDP - Non linear Elliptic fourth order – Compact riemannian manifold – Nehari manifold – Sobolev Exponent - Hardy-Sobolev exponent - Paneitz-Branson operator - Q-curvature.