Dédicaces

Thanks to very powerful Allah I could completed this work which I dedicate particularly,

■ with the memory of my dear father who left us for a long time,

■ with my dear mother,

■ with all my family,

■ with all my friends.

Acknowledgements

I first of all make a point of thanking very cordially Pr, Mohammed Bouchekif for having to give opportunity to me of launching me in this mathematical great adventure. I could not say to which point I am grateful to him, so much for his human generosity than for his high teaching and mathematical competence. I enormously profited from his enthusiasm, his availability and the great freedom that it granted to me. It also knew to give again me confidence during time of doubts and it is partly thanks to him that I am now decided to continue the adventure and to undertake an academic career.

My recognition also goes to the president of the jury, Pr, Yebdri Mustapha, like with the professors, members of my Ph.D. examining board, Pr, Bougima Sidi Mohammed, Pr, Messirdi Bekkai, Pr, Lakmech Abdelkader and Dr, Abdellaoui Boumedienne for their relevant reading and remarks. My thanks go also members and teams of the Laboratory dynamic systems and with the department of mathematics.

I would still like to thank all my colleagues.

My greater gratitude also goes to my family, and particularly to my mother, for her support, its great wisdom and its glance always so serene on the world, which bring to me much.

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Introduction

In this thesis, we consider a class of scalar elliptic equations or elliptic systems involving singular weights and critical exponents of general form

$$-\operatorname{div}\left(|y|^{-2a}\nabla u\right) - \mu |y|^{-2(a+1)} u = \quad h(y)|y|^{-2_* b}|u|^{2_*-2}u + \lambda g(x) \ \text{ in } \mathbb{R}^N, \ y \neq 0,$$

with each point x in $\mathbb{R}^N$ is written as a pair $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ where k and N are integers such that $N \geq 3$ and k belongs to $\{1,...,N\}$, a,b real numbers, $2_*(a,b)$ critical exponents, $-\infty < \mu < \bar{\mu}_{a,k} := ((k-2(a+1))/2)^2$ and g and h are functions which will defined later. We study problems with various assumptions on a, b and λ.

The choice of such classes rises owing to the fact that the research of solutions in forms of standing waves for an equation of evolution leads sometimes to the cylindrical case [4]. Accordingly, a solitary wave preserves intrinsic properties of particles such as the energy. Owing to their particle-like behaviors, solitary waves can be regarded as a model for extended particles and they arise in many problems of mathematical physics. Thus, this resolution contributes to the development of the modeling of many physical systems in fields as varied as: the theory of the traditional fields and quantum, the plasma physics, see for example [1, 2, 18, 21, 23].

Before summarizing the results consigned in this thesis, briefly, let register this work from the historical point of view to us. Our principal concern, in this work, is to study existence and multiplicity results.

The scalar case. Rigorous mathematical research on the resolution of the elliptic equations containing singular potentials started since the Eighties. Thus, one finds within this framework work of the authors Catrina and Wang [10], Badiale and Tarantello [4] in this field. The originality of these works resides in the techniques used which make possible the proof of the existence and the multiplicity of solutions. Concerning the spherical case, one will retain precursory works of [4, 8, 22], in which the authors used methods of minimization under constraint. As noticed, in 2006, Wang and Zhou [25] studied the same semilinear equation

$$-\Delta u-\mu\left|x\right|^{-2}u=\ \left|u\right|^{2^{*}-2}u+\lambda g\left(x\right)\ \text{ in }\mathbb{R}^{N},\ x\neq 0,$$

with, $0<\mu\leq\bar{\mu}_{0,N}$, $2^{*}=2N/\left(N-2\right)$ and λ is a parameter real. They proved that there exists at least two distinct solutions under some conditions on g by applying Ekeland's variational principle and Mountain Pass theorem without Palais Smale conditions. In 2009, Bouchekif and Matallah [6], extended the work of [25] by studying the quasilinear equation

$$-\operatorname{div}\left(\left|x\right|^{-2a}\nabla u\right)-\mu\left|x\right|^{-2(a+1)}u=\ h\left(x\right)\left|x\right|^{-2_{*}b}\left|u\right|^{2_{*}-2}u+\lambda g\left(x\right)\ \text{ in }\mathbb{R}^{N},\ x\neq 0,$$

where $0<\mu\leq\bar{\mu}_{a,N}$, $-\infty<a<\left(N-2\right)/2,\ a\leq b<a+1,\ 2_{*}=2N/\left(N-2+2\left(b-a\right)\right)$ and λ a positive parameter. They established the existence of two nontrivial solutions by using Ekeland's variational principle and mountain pass theorem under sufficient conditions on functions g and h. .

For the cylindrical case, Badiale et al.[3] and Musina: [17] contributed in the study. As noticed, in 2007, Badiale et al.[3] studied an elliptic equation with decaying cylindrical potential: $|y|^{-\alpha}$ with y in $\mathbb{R}^{k}$ and α positive real. In 2008, Musina: [17] studied the following equation by proving the existence of at least one nonnegative solution

$$-\operatorname{div}\left(\left|y\right|^{-2a}\nabla u\right)=\ \mu\left|y\right|^{-2(a+1)}u+\left|y\right|^{-b}u^{p-1}\ \text{ in }\mathbb{R}^{N},\ y\neq 0,$$

with $-a/2$ instead of a where $a\in\mathbb{R}$, $b=N-p\left(N-2\right)/2$ with $p\in\left(2,2^{*}\right]$ and μ is a parameter real. In 2009, Gazzini and Musina [16] studied the existence and the nonexistence of the same problem by introducing the Hardy-Sobolev-Maz' ya inequality.

The major challenge then consisted in adapting the variational methods which proved reliable for equations where one encountered problems such as: problems of compactness, existence of extermale functions. Thus, the main arguments which one will adopt to free problems will be the constraint defined by the Nehari manifold [19] and the Hardy-Sobolev-Maz' ya inequality

[16].

The singular systems case. Such problems are introduced as models for several physical phenomena related to equilibrium of continuous media which somewhere be perfect insulators, see for example [11, 13]. For more information and connection on problems of this type, the readers may consult in [7, 14] and the references therein.

We quote various results which were obtained.

The first chapter of the thesis recalls the basic definitions which will be frequently used later.

In chapter two, we establish the existence of multiple solutions for nonhomogeneous singular elliptic equations with cylindrical weight by using Ekeland's variational principle and mountain pass theorem without Palais-Smale conditions. This work is an extension of Bouchekif's [6] which interested by the spherical case.

Chapter three is devoted to the study of quasilinear elliptic equations involving decaying cylindrical potentials and critical exponents, which is a generalization of the work of Badiale et al.[3] . We prove the existence of at least two solutions by using the Nehari manifold and Hardy-Sobolev-Maz'ya inequality.

The last chapter, is centered on the study of nonhomogeneous singular elliptic systems involving a singular weakly coupled potential and the Caffarelli-Kohn-Nirenberg critical exponent. Taking as a starting point the work of [7], we give existence results by using the critical point theory under the Nehari manifold as constraint.

List publications:

1) On nonhomogeneous elliptic equations with decaying cylindrical potential and critical exponent, Electron. J. Diff. Eqns., 2011 (54) (2011) $1-10$.

2) On nonhomogeneous singular elliptic equations with cylindrical weight, Submitted, preprint Université de Tlemcen, (2011).

3) On nonhomogeneous singular elliptic systems with critical Caffarelli-Kohn-Nirenberg exponent, Submitted, preprint Université de Tlemcen, (2011).

Chapter 1

Preliminaries

In this chapter, we start by recalling some definitions which will be frequently used throughout the rest of this work.

1.1 Critically point

Let be X a Banache space, J a functional of class C^1 defined in X with values in $\mathbb{R}$. Let us note $J'(u) : X \longrightarrow X'$(dual of X) its derivative within the meaning of Fréchet.

Definition 1.1 *Let $u \in X$, $c \in \mathbb{R}$. The point u is known as critical point of the functional calculus J if $J'(u) = 0$.*
The value c is known as breaking value of J if there exists a critical point u in X such that: $J(u) = c$.

1.2 Mountain Pass Theorem

Let X Banach space, and $J \in C^1(X, \mathbb{R})$ verifying the Palais -Smale condition. Suppose that $J(0) = 0$ and that:
i) $\exists R > 0$ and $\exists r > 0$ such that if $\|u\| = R$, then $J(u) \geq r$;
ii) $\exists (u_0) \in X$ such that $\|u_0\| > R$ and $J(u_0) \leq 0$;

let $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} (J(\gamma(t)))$ where

$$\Gamma = \{\gamma \in C([0,1];X) \text{ such that } \gamma(0) = 0 \text{ et } \gamma(1) = u_0\},$$

then c is critical value of J such that $c \geq r$.

1.3 Theorem (Ekeland's variational principle)

Let $\phi : X \longrightarrow (-\infty, +\infty]$ be a semi-continuous clean function in a lower position limited in a lower position. For each $\varepsilon > 0$ and each $u \in X$ such that $\phi(u) \leq \inf_{x\in X} \phi(u) + \varepsilon$, it exists $v \in X$ such that

(1) $\phi(v) \leq \phi(u)$,

(2) $d(u,v) \leq 1$,

(3) $\phi(v) < \phi(x) + \varepsilon d(x,v)$ for all $x \in X$ such that $x \neq v$.

1.4 Brezis-Lieb Theorem

Let $0 < p < \infty$. Suppose $u_n \longrightarrow u$ a.e. and $\|u_n\|_{L^p(\Omega)} \leq C < \infty$. Then

$$\lim_{n\longrightarrow\infty} \left(\|u_n\|^p_{L^p(\Omega)} - \|u_n - u\|^p_{L^p(\Omega)} \right) = \|u\|^p_{L^p(\Omega)}.$$

1.5 Theorem (Generalized Hardy inequality)

If $1 \leq k \leq N$, we will write a generic point $x \in \mathbb{R}^N$ as $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Let $1 < p < \infty$ and $\alpha + k > 0$. Then, for each $u \in \mathcal{D}\left(\mathbb{R}^N\right)$ the following inequality holds

$$\int_{\mathbb{R}^N} |y|^\alpha |u(x)|^p \, dx \leq \frac{p^p}{(\alpha+k)^p} \int_{\mathbb{R}^N} |\nabla u(x)|^p |y|^{\alpha+p} \, dx.$$

Moreover, the constant $\frac{p^p}{(\alpha+k)^p}$ is optimal.

1.6 The Nehari manifold

Let $J \in C^1(X, \mathbb{R})$ be the Euler functional associated with an elliptic problem on Banach space X. If J is bounded below and has a minimizer on X, then this minimizer is a critical point of J. Hence, it is a solution of the corresponding elliptic problem. However, in many problems J is not bounded below on the whole space X, but is bounded below on an appropriate subset of X and minimizer on this set (if it exist) many give rise to solutions of the corresponding elliptic problem. A good candidate for an appropriate subset of X is the Nahari manifold defined by

$$M = \left\{ u \in X : \ \left\langle J^{'}(u), u \right\rangle = 0 \right\}.$$

1.7 The weighted Sobolev space $X\left(\mathbb{R}^N, |y|^{-\alpha} dx\right)$

Let k, N be such that $N > k \geq 2$ and let $\alpha, \mu > 0$. We define the weighted Sobolev space

$$X := X\left(\mathbb{R}^N, |y|^{-\alpha} dx\right) := \left\{ u \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} |y|^{-\alpha} u^2 dx < +\infty \right\},$$

which is a Hilbert space with respect to the norm defined by

$$\|u\|^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu |y|^{-\alpha} u^2 \right) dx, \text{ for all } u \in X.$$

Whose inner product we denote by

$$\langle u, v \rangle := \int_{\mathbb{R}^N} \left(\nabla u \nabla v - \mu |y|^{-\alpha} uv \right) dx, \text{ for all } u, v \in X.$$

Clearly $X \hookrightarrow \mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$, whence, by well known embedding of $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$, one derives $X \hookrightarrow L^{2_*}\left(\mathbb{R}^N\right)$ and $X \hookrightarrow L^p_{loc}\left(\mathbb{R}^N\right)$ for $1 \leq p \leq 2_*$. In particular the latter embedding is compact if $p < 2_*$ and thus it assures that weak convergence in X implies (up to a subsequence) almost everywhere convergence in $\mathbb{R}^N$.

Set $\mathcal{D} := \mathcal{D}^{1,2}_0\left(\mathbb{R}^N\right) \cap L\left(\mathbb{R}^N, |y|^{-2} dx\right)$, endowed with the scalar product

$$\langle u, v \rangle := \int_{\mathbb{R}^N} \left(\nabla u \nabla v + |y|^{-2} uv \right) dx.$$

For convenience we point out some remarks on the space $\mathcal{D}$.

Lemma 1.1 [17]

(i) $C_c^\infty\left(\left(\mathbb{R}^k \backslash \{0\}\right) \times \mathbb{R}^{N-k}\right)$ *is dense in* $\mathcal{D}$

(ii) If $k > 2$ *then* $\mathcal{D} := \mathcal{D}_0^{1,2}\left(\mathbb{R}^N\right)$

(iii) If $k = 1$ *then* $\mathcal{D} = \mathcal{D}_0^{1,2}\left((0,+\infty) \times \mathbb{R}^{N-1}\right) + \mathcal{D}_0^{1,2}\left((+\infty,0) \times \mathbb{R}^{N-1}\right)$

(iv) If $k \neq 2$ *an equivalent scalar product on* $\mathcal{D}$ *is* $\langle u,v\rangle := \int_{\mathbb{R}^N} \nabla u \nabla v dx.$

1.8 Hardy-Sobolev-Maz'ya inequality

In [16], It states that for any $a \in \mathbb{R}$ and for every real exponent $p \in (2, 2_*]$ there exists a constant $C_{a,p} > 0$ such that

$$C_{a,p}\left(\int_{\mathbb{R}^N} |y|^{-b} |u|^p \, dx\right)^{2/p} \leq \int_{\mathbb{R}^N} \left(|y|^a |\nabla u|^2 - \bar{\mu}_{a,k} |y|^{a-2} |u|^2\right) dx,$$

for any $u \in C_c^\infty\left(\mathbb{R}^N\right)$ if $a > 2-k$ and for any $u \in C_c^\infty\left(\mathbb{R}_0^N\right)$ if $a \leq 2-k$.
where $b = N - p\left(\frac{N-2+a}{2}\right)$ and $\bar{\mu}_{a,k} := \left(\frac{k-2+a}{2}\right)^2$.
The Maz'ya inequality states that

$$C_{a,p}\left(\int_{\mathbb{R}^N} |y|^{-b} |u|^p \, dx\right)^{2/p} \leq \int_{\mathbb{R}^N} |y|^a |\nabla u|^2 \, dx, \ \forall u \in C_c^\infty\left(\mathbb{R}^N\right).$$

Chapter 2

On nonhomogeneous singular elliptic equations with cylindrical weight

In this chapter, we establish the existence of multiple solutions for nonhomogeneous singular elliptic equations with cylindrical weight, by using Ekeland's variational principle and mountain pass theorem without Palais-Smale conditions.

2.1 Introduction

This chapter deals with the existence and multiplicity of solutions to the following problem

$$(\mathcal{P}_{\lambda,\mu})\begin{cases} -\Delta u-\mu\left|y\right|^{-2}u= & h\left(y\right)\left|y\right|^{-b}\left|u\right|^{p-2}u+\lambda g\left(x\right)\ \text{ in } \mathbb{R}^N,\ y\neq 0\\ u\in\mathcal{D}_0^{1,2}, \end{cases}$$

where each point x in $\mathbb{R}^N$ is written as a pair $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$ with k and N are integers such that $N\geq 3$ and k belongs to $\{2,...,N\}$, $b=N-p\left(N-2\right)/2$ with $p\in\left(2,2^*\right]$ and $2^*=2N/\left(N-2\right)$ is the critical Sobolev exponent, λ and μ are positive parameters, $g\in\mathcal{H}_\mu^{'}\cap C\left(\mathbb{R}^N\right)$ not identically equal to 0 and h is a bounded positive function on $\mathbb{R}^k$. $\mathcal{H}_\mu^{'}$ is the dual of $\mathcal{H}_\mu$.

By $\mathcal{D}_0^{1,2}=\mathcal{D}_0^{1,2}\left(\left(\mathbb{R}^k\backslash\{0\}\right)\times\mathbb{R}^{N-k}\right)$ and $\mathcal{H}_\mu=\mathcal{H}_\mu\left(\left(\mathbb{R}^k\backslash\{0\}\right)\times\mathbb{R}^{N-k}\right)$, we denote the closure of $C_0^\infty\left(\left(\mathbb{R}^k\setminus\{0\}\right)\times\mathbb{R}^{N-k}\right)$ with respect to the norms

$$\|u\|_0=\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{1/2}\text{ and }\|u\|_\mu=\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2-\mu\,|y|^{-2}\,|u|^2\right)dx\right)^{1/2}$$

for $\mu<\bar{\mu}_k$ and $k\neq 2$, respectively, with $\bar{\mu}_k:=((k-2)/2)^2$ is the best constant in the Hardy inequality for the cylindrical case [13], by which the norm $\|u\|_\mu$ is equivalent to $\|u\|_0$. More explicitly, we have for $k\neq 2$,

$$\left(1-\mu^+/\bar{\mu}_k\right)^{1/2}\|u\|_0\leq\|u\|_\mu\leq\left(1-\mu^-/\bar{\mu}_k\right)^{1/2}\|u\|_0\text{ for }u\in\mathcal{H}_\mu,$$

where $\mu^+=\max(\mu,0)$ and $\mu^-=\min(\mu,0)$.

We know that the weighted Sobolev space $\mathcal{D}:=\mathcal{H}_\mu\cap L^p\left(\mathbb{R}^N,|y|^{-b}\,dx\right)$ is a Banach space with respect to the norm $\mathcal{N}(u):=\|u\|_\mu+(\int_{\mathbb{R}^N}|y|^{-b}\,|u|^p\,dx)^{1/p}$.

Several existence results are available in the case $k=N$, we quote for example [1, 4, 6, 7] and the references therein. For more details, when $N\geq 3$, $b=0$, $p<2^*$, $\mu<0$ and $h\equiv 1$, Badiale et al. [2] and Terracini [15] studied $(\mathcal{P}_{0,\mu})$. In [15], the author proves that no positive solutions for $(\mathcal{P}_{0,\mu})$. When $p=2^*$, the regular problem $(\mathcal{P}_{1,0})$ has been considered, on a bounded domain Ω, by Tarantello [14] with $h\equiv 1$ and $\mu=0$. She proved that for $g\in\left(H_0^1(\Omega)\right)'$ not identically zero and satisfying a suitable condition, the problem considered admits two solutions. The problem $(\mathcal{P}_{\lambda,\mu})$ has been studied by Bouchekif and Matallah in [4], by using Ekeland's variational principle [8] and mountain pass theorem, they established the existence of two nontrivial solutions when $0<\mu\leq\bar{\mu}_N$, $\lambda\in(0,\Lambda_*)$ and under sufficient conditions on functions g and h, with Λ_* a positive constant.

For the cylindrical case i.e., $k<N$, there are much less studies in the literature at our knowledge. We cite for example [3, 9, 10, 12, 13] and the references therein. As noticed in [12], Musina has considered the problem $(\mathcal{P}_{0,\mu})$ with $h\equiv 1$. She established the existence of ground state solution when $0<\mu<\bar{\mu}_k$ and $2<k\leq N$ and the support of the ground state solution is a half-space when $k=1$ and $N\geq 4$. When $b=0$, $h\equiv 1$ and $p=2^*$, she shows that $(\mathcal{P}_{0,\mu})$ does not admits ground state solutions.

Since our approach is variational, we define the functional $I_{\lambda,\mu}$ on $\mathcal{D}$ by

$$I_{\lambda,\mu}(u) := (1/2)\|u\|_{\mu}^{2} - (1/p)\int_{\mathbb{R}^N} h(y)|y|^{-b}|u|^{p}dx - \lambda\int_{\mathbb{R}^N} g(x)\,udx.$$

We say that $u \in \mathcal{D}$ is a weak solution of the problem $(\mathcal{P}_{\lambda,\mu})$ if it satisfies

$$\int_{\mathbb{R}^N}\left(\nabla u\nabla v - \mu|y|^{-2}uv - h(y)|y|^{-b}|u|^{p-2}uv - \lambda g(x)v\right) = 0, \text{ for } v \in \mathcal{D}.$$

Throughout this work, we consider the following assumption

$$(H) \qquad \lim_{|y|\longrightarrow 0} h(y) = \lim_{|y|\longrightarrow \infty} h(y) = h_0 > 0,\ h(y) \geq h_0,\ y \in \mathbb{R}^k.$$

In our work, we prove the existence of at least two distinct critical points of $I_{\lambda,\mu}$. One by the Ekeland variational principle with negative energy, and the other by mountain pass theorem without Palais-Smale conditions with positive energy.

Remark 2.1 *Note that all solutions of* $(\mathcal{P}_{\lambda,\mu})$ *are nontrivial.*

Our main result is given as follows

Theorem 2.1 *Suppose that* $2 < k \leq N$, $b = N - p(N-2)/2$ *with* $p \in (2, 2^*]$, $\mu < \bar{\mu}_k$ *and hypothesis* (H) *holds. Then, there exists* $\Lambda_* > 0$ *such that the problem* $(\mathcal{P}_{\lambda,\mu})$ *has at least two solutions for any* $\lambda \in (0, \Lambda_*)$.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorem 2.1.

2.2 Preliminaries

We start by recalling the following definition and properties from the paper [12].

The first inequality that we need is the Hardy inequality

$$\bar{\mu}_k\int_{\mathbb{R}^N}|y|^{-2}v^2dx \leq \int_{\mathbb{R}^N}|\nabla v|^2dx,\ \forall v \in \mathcal{H}_{\mu}. \tag{2.1}$$

Next, assume $N \geq 3$, $p \in (2, 2^*]$ and $b = N - p(N-2)/2$. The starting point for studying $(\mathcal{P}_{0,\mu})$ is the Hardy-Sobolev-Maz'ya inequality that is particular to the cylindrical case $k < N$ and that was proved by Maz'ya in [12]. It states that there exists positive constant C_p such that

$$C_p \left(\int_{\mathbb{R}^N} |y|^{-b} |v|^p \, dx \right)^{2/p} \leq \int_{\mathbb{R}^N} \left(|\nabla v|^2 - \mu |y|^{-2} v^2 \right) dx, \tag{2.2}$$

for any $v \in C_c^\infty((\mathbb{R}^k \backslash \{0\}) \times \mathbb{R}^{N-k})$.

Definition 2.1 *An entire solution v to $(\mathcal{P}_{\lambda,\mu})$ is a ground state solution if it achieves the best constant*

$$S_{\mu,p} = S_{\mu,p}(k, N) := \inf_{v \in \mathcal{H}_\mu\left(\left(\mathbb{R}^k \backslash \{0\}\right) \times \mathbb{R}^{N-k}\right)} \frac{\int_{\mathbb{R}^N} \left(|\nabla v|^2 - \mu |y|^{-2} v^2 \right) dx}{\left(\int_{\mathbb{R}^N} |y|^{-b} |v|^p \, dx \right)^{2/p}}, \tag{2.3}$$

for $k \geq 2$.

Lemma 2.1 [12] *Assume $2 \leq k < N$, $\mu \leq \bar{\mu}_k$ and $2 < p < 2^*_{N-k+1}$. Then the infimum $S_{\mu,p}$ is achieved on $\mathcal{H}_\mu\left(\left(\mathbb{R}^k \backslash \{0\}\right) \times \mathbb{R}^{N-k}\right)$.*

Lemma 2.2 *Let $(u_n) \subset \mathcal{D}$ be a Palais-Smale sequence ($(PS)_c$ for short) of $I_{\lambda,\mu}$ i.e.*

$$I_{\lambda,\mu}(u_n) \longrightarrow c \text{ and } I'_{\lambda,\mu}(u_n) \longrightarrow 0 \text{ in } \mathcal{D}' (\text{dual of } \mathcal{D}) \text{ as } n \longrightarrow \infty, \tag{2.4}$$

for some $c \in \mathbb{R}$. Then, $u_n \rightharpoonup u$ in $\mathcal{D}$ and $I'_{\lambda,\mu}(u) = 0$.

Proof: From (2.4), we have

$$(1/2) \|u_n\|_\mu^2 - (1/p) \int_{\mathbb{R}^N} h(y) |y|^{-b} |u_n|^p \, dx - \lambda \int_{\mathbb{R}^N} g(x) u_n dx = c + o_n(1)$$

and

$$\|u_n\|_\mu^2 - \int_{\mathbb{R}^N} h(y) |y|^{-b} |u_n|^p \, dx - \lambda \int_{\mathbb{R}^N} g(x) u_n dx = o_n(1),$$

where $o_n(1)$ denotes $o_n(1) \longrightarrow 0$ as $n \longrightarrow \infty$. Then,

$$\begin{aligned} c + o_n(1) &= I_{\lambda,\mu}(u_n) - (1/p)\left\langle I'_{\lambda,\mu}(u_n), u_n \right\rangle \\ &\geq ((p-2)/2p)\|u_n\|_\mu^2 - \lambda((p-1)/p)\|g\|_{\mathcal{H}'_\mu}\|u_n\|_\mu, \end{aligned}$$

(u_n) is bounded in $\mathcal{D}$.

If $p \in (2, 2^*)$ or $p = 2^*$, then we can find $u \in \mathcal{D}$ such that

$$\begin{aligned} u_n &\rightharpoonup u \text{ weakly in } \mathcal{D}, \\ u_n &\rightharpoonup u \text{ weakly in } L_p\left(\mathbb{R}^N; |y|^{-b}\right), \\ u_n &\rightarrow u \text{ a.e in } \mathbb{R}^N. \end{aligned} \tag{2.5}$$

Thus, we deduce that for all $v \in C_0^\infty\left(\left(\mathbb{R}^k \setminus \{0\}\right) \times \mathbb{R}^{N-k}\right)$,

$$\int_{\mathbb{R}^N}\left(\nabla u \nabla v - \mu|y|^{-2}uv - h(y)|y|^{-b}|u|^{p-2}uv + \lambda g(x)v\right) = 0,$$

i.e.,

$$I'_{\lambda,\mu}(u) = 0.$$

■

Lemma 2.3 *Let $(u_n) \subset \mathcal{D}$ be a $(PS)_c$ sequence of $I_{\lambda,\mu}$ for some $c \in \mathbb{R}$. Then,*

$$u_n \rightharpoonup u \text{ in } \mathcal{D}$$

and either

$$u_n \longrightarrow u \quad \text{or } c \geq I_{\lambda,\mu}(u) + ((p-2)/2p)\left(h_0^{-2/p}S_{\mu,p}\right)^{p/(p-2)},$$

for all $p \in (2, 2^*]$.

Proof: We know that (u_n) is bounded in $\mathcal{D}$. Up to a subsequence if necessary, we have that

$$\begin{array}{rcl} u_n & \rightharpoonup & u \text{ in } \mathcal{D} \\ u_n & \rightarrow & u \text{ a.e in } \mathbb{R}^N. \end{array}$$

Denote $v_n = u_n - u$, then $v_n \rightharpoonup 0$. As in Brézis and Lieb [5], we have

$$|v_n|_p^2 = |u_n|_p^2 - |u|_p^2$$

and

$$\lim_{n\longrightarrow\infty} \int_{\mathbb{R}^N} h(y) \left(|y|^{-b} |u_n|^p - |y|^{-b} |u_n - u|^p \right) dx = \int_{\mathbb{R}^N} h(y) |y|^{-b} |u|^p \, dx.$$

On the other hand, by using the assumption (H), we obtain

$$\lim_{n\longrightarrow\infty} \int_{\mathbb{R}^N} h(y) |y|^{-b} |v_n|^p \, dx = h_0 \lim_{n\longrightarrow\infty} \int_{\mathbb{R}^N} |y|^{-b} |v_n|^p \, dx. \tag{2.6}$$

Then, we get

$$I_{\lambda,\mu}(u_n) = I_{\lambda,\mu}(u) + (1/2) \|v_n\|_\mu^2 - (h_0/p) \int_{\mathbb{R}^N} |y|^{-b} |v_n|^p + o_n(1)$$

and

$$\left\langle I'_{\lambda,\mu}(u_n), u_n \right\rangle = \|v_n\|_\mu^2 - h_0 \int_{\mathbb{R}^N} |y|^{-b} |v_n|^p + o_n(1).$$

Then we can assume that

$$\lim_{n\longrightarrow\infty} \|v_n\|_\mu^2 = h_0 \lim_{n\longrightarrow\infty} \int_{\mathbb{R}^N} |y|^{-b} |v_n|^p = l \geq 0.$$

Assume $l > 0$, we have by definition of $S_{\mu,p}$

$$l \geq S_{\mu,p} \left(l h_0^{-1} \right)^{2/p},$$

and so that

$$l \geq \left(h_0^{-2/p} S_{\mu,p} \right)^{p/(p-2)}.$$

Thus we get

$$c = I_{\lambda,\mu}(u) + ((p-2)/2p)\, l \geq I_{\lambda,\mu}(u) + ((p-2)/2p)\left(h_0^{-2/p} S_{\mu,p}\right)^{p/(p-2)}.$$

■

2.3 Existence results

The proof of Theorem 2.1 is given in two parts.

2.3.1 Existence of a local minimizer

We prove that there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$, $I_{\lambda,\mu}$ can achieve a local minimizer.

First, we establish the following result.

Proposition 2.1 *Suppose* $2 < k \leq N$, $b = N - p(N-2)/2$ *with* $p \in (2, 2^*]$, $\mu < \bar{\mu}_k$, *and the hypothesis* (H) *holds. Then there exist* λ_*, ϱ *and* δ *positive constants such that for all* $\lambda \in (0, \lambda_*)$ *we have*

$$I_{\lambda,\mu}(u) \geq \delta > 0 \text{ for } \|u\|_\mu = \varrho. \tag{2.7}$$

Proof: By the Holder inequality and the definition of $S_{\mu,p}$, we get for all $u \in \mathcal{D}\setminus\{0\}$ and $\varepsilon > 0$

$$\begin{aligned} I_{\lambda,\mu}(u) \quad : \quad &= (1/2)\|u\|_\mu^2 - (1/p)\int_{\mathbb{R}^N} h(y)|y|^{-b}|u|^p\, dx - \lambda \int_{\mathbb{R}^N} g(x)\, u dx, \\ &\geq (1/2 - \varepsilon)\|u\|_\mu^2 - (|h|_\infty / p)\, S_{\mu,p}\|u\|_\mu^p - C_\varepsilon \|\lambda g\|_{\mathcal{H}'_\mu}. \end{aligned}$$

Taking $\varepsilon < 1/2$ and $\varrho = \|u\|_\mu$. Then there exist $\varrho > 0$ small enough and a positive constant λ_* such that

$$I_{\lambda,\mu}(u) \geq \delta > 0 \text{ for } \|u\|_\mu = \varrho \text{ and } \lambda \in (0, \lambda_*). \tag{2.8}$$

■

Since g is a continuous function on $\mathbb{R}^N$, not identically zero, we can choose $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} g(x)\,\phi dx > 0$. It follows that for $t > 0$ small enough,

$$I_{\lambda,\mu}(t\phi) := \left(t^2/2\right)\|\phi\|_\mu^2 - (t^p/p)\int_{\mathbb{R}^N} h(y)\,|y|^{-b}\,|\phi|^p\,dx - \lambda t\int_{\mathbb{R}^N} g(x)\,\phi dx < 0. \tag{2.9}$$

and $\|t\phi\|_\mu < \varrho$.

Thus, we have

$$c_1 = \inf\{I_{\lambda,\mu}(u) : u \in B_\varrho\} < 0, \text{ with } B_\varrho = \{u \in \mathcal{D},\ \mathcal{N}(u) \le \varrho\}. \tag{2.10}$$

Using the Ekeland's variational principle, for the complete metric space $\overline{B}_\rho$ with respect to the norm of $\mathcal{D}$, we can prove that there exists a $(PC)_{c_1}$ sequence $(u_n) \subset \overline{B}_\rho$ such that $u_n \rightharpoonup u_1$ for some u_1 with $\mathcal{N}(u_1) \le \rho$.

Now, we claim that $u_n \longrightarrow u_1$. If not, by Lemma 2.3, we have

$$\begin{aligned} c_1 &\ge I_{\lambda,\mu}(u_1) + ((p-2)/2p)\left(h_0^{-2/p}S_{\mu,p}\right)^{p/(p-2)} \\ &\ge c_1 + ((p-2)/2p)\left(h_0^{-2/p}S_{\mu,p}\right)^{p/(p-2)} \\ &> c_1, \end{aligned}$$

which is a contradiction. Then we obtain a critical point u_1 of $I_{\lambda,\mu}$ for all $\lambda \in (0, \lambda_*)$ satisfying

$$c_1 = I_{\lambda,\mu}(u_1) < 0.$$

On the other hand we have

$$\begin{aligned} c_1 &= ((p-2)/2p)\,\|u_1\|_\mu^2 - ((p-1)/p)\int_{\mathbb{R}^N} \lambda g(x) u_1 dx \\ &\ge -(1/2p)\,(p-1)^2\,(p-2)^{-1}\,\lambda^2\,\|g\|_{\mathcal{H}'_\mu}^2. \end{aligned}$$

Thus u_1 is a solution of our problem with negative energy.

2.3.2 Existence of mountain pass type solution

We use the mountain pass theorem without Palais-Smale conditions to prove the existence of a solution with positive energy.

Let $c^*_{\lambda,p} := ((p-2)/2p)\left(h_0^{-2/p} S_{\mu,p}\right)^{p/(p-2)} - (1/2p)(p-1)(p-2)^{-1/2}\lambda^2 \|g\|^2_{\mathcal{H}'_\mu}$

Before completing the proof of the Theorem 2.1, we need the following Lemma.

Lemma 2.4 *Let $\lambda^* > 0$ such that*

$$c^*_{\lambda,p} > 0, \text{ for all } \lambda \in (0, \lambda^*).$$

Then, there exist $\Lambda \in (0, \lambda^*)$ and $\varphi_\varepsilon \in \mathcal{D}$ for $\varepsilon > 0$ such that

$$\sup_{t \geq 0} I_{\lambda,\mu}(t\varphi_\varepsilon) < c^*_{\lambda,p}, \text{ for all } \lambda \in (0, \Lambda).$$

Proof: Let

$$\varphi_\varepsilon(x) = \begin{cases} \omega_\varepsilon(x) & \text{if } g(x) \geq 0 \text{ for all } x \in \mathbb{R}^N, \\ \omega_\varepsilon(x - x_0) & \text{if there is an } x_0 \in \mathbb{R}^N \text{ such that } g(x_0) > 0, \\ -\omega_\varepsilon(x) & \text{if } g(x) \leq 0 \text{ for all } x \in \mathbb{R}^N, \end{cases}$$

where ω_ε achieves $S_{\mu,p}$ defined in (3.1).

Then, we claim that there is an ε_0 such that

$$\int_{\mathbb{R}^N} g(x)\varphi_\varepsilon(x) > 0, \text{ for any } \varepsilon \in (0, \varepsilon_0). \tag{2.11}$$

In fact, $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \in \mathbb{R}^N$, and (2.11) holds obviously. If there exists an $x_0 \in \mathbb{R}^N$ such that $g(x_0) > 0$, by the continuity of $g(x)$ there is an $\eta > 0$ such that $g(x) > 0$ for all $x \in B_\eta(x_0)$. Then, by the definition of $\omega_\varepsilon(x - x_0)$, it is easy to see that there exists an ε_0 small enough such that

$$\int_{\mathbb{R}^N} g(x)\omega_\varepsilon(x - x_0) > 0, \text{ for any } \varepsilon \in (0, \varepsilon_0). \tag{2.12}$$

Now, we consider the following functions

$$f\left(t\right)=I_{\lambda,\mu}\left(t\varphi_{\varepsilon}\right)\text{ and }\bar{f}\left(t\right)=\left(t^{2}/2\right)\left\|\varphi_{\varepsilon}\left(x\right)\right\|_{\mu}^{2}-\left(t^{p}/p\right)h_{0}\int_{\mathbb{R}^{N}}\left|y\right|^{-b}\left|\varphi_{\varepsilon}\left(x\right)\right|^{p}dx,$$

Then, we get for all $\lambda\in(0,\lambda^{*})$

$$0=f\left(0\right)<c_{\lambda,p}^{*}.$$

By the continuity of $f\left(t\right)$, there exists t_1 a sufficiently small positive number such that

$$f\left(t\right)<c_{\lambda,p}^{*},$$

for all $t\in(0,t_1)$. On the other hand, we have

$$\max_{t\geq0}\bar{f}\left(t\right)=\left(\left(p-2\right)/2p\right)\left(h_{0}^{-2/p}S_{\mu,p}\right)^{p/(p-2)},$$

then, we obtain

$$\sup_{t\geq0}I_{\lambda,\mu}\left(t\varphi_{\varepsilon}\right)\leq\left(\left(p-2\right)/2p\right)\left(h_{0}^{-2/p}S_{\mu,p}\right)^{p/(p-2)}-\lambda t_{1}\int_{\mathbb{R}^{N}}\left|y\right|^{-b}g\left(x\right)\varphi_{\varepsilon}dx.$$

Taking $\lambda>0$ such that

$$\lambda t_{1}\int_{\mathbb{R}^{N}}g\left(y\right)\varphi_{\varepsilon}dx>\left(1/2p\right)\left(p-1\right)\left(p-2\right)^{-1/2}\lambda^{2}\left\|g\right\|_{\mathcal{H}_{\mu}'}^{2}.$$

By (2.11), we get

$$0<\lambda<\Lambda_{**}.$$

where

$$\Lambda_{**}:=(2p\left(p-2\right)^{1/2}\left(p-1\right)^{-1})t_{1}\left(\int_{\mathbb{R}^{N}}g\left(x\right)\varphi_{\varepsilon}dx\right)\left\|g\right\|_{\mathcal{H}_{\mu}'}^{-2}.$$

Set

$$\Lambda=\min\left\{\lambda^{*},\Lambda_{**}\right\}.$$

We deduce that

$$\sup_{t\geq0}I_{\lambda,\mu}\left(t\varphi_{\varepsilon}\right)<c_{\lambda,p}^{*},\text{ for all }\lambda\in\left(0,\Lambda\right).$$

■

Now, we complete the proof of the Theorem 2.1.

Since $\lim_{t\longrightarrow\infty} I_{\lambda,\mu}(t\varphi_\varepsilon) = -\infty$, we can choose $T > 0$ large enough such that $I_{\lambda,\mu}(T\varphi_\varepsilon) < 0$. From Proposition 2.1, we have $I_{\lambda,\mu|\partial B_\varrho} \geq \delta > 0$ for all $\lambda \in (0,\lambda_*)$. By mountain pass theorem without the Palais-Smale condition, there exists a $(PC)_{c_2}$ sequence (u_n) in $\mathcal{D}$ which is characterized by

$$c_2 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1],\mathcal{D}), \gamma(0) = 0, \gamma(1) = T\varphi_\varepsilon\}.$$

Then, (u_n) has a subsequence, still denoted by (u_n) such that $u_n \rightharpoonup u_2$ in $\mathcal{D}$. By Lemma 2.3, if u_n doesn't converge to u_2, we get

$$c_2 \geq I_{\lambda,\mu}(u_2) + ((p-2)/2p)\left(h_0^{-2/p} S_{\mu,p}\right)^{p/(p-2)} \geq c^*_{\lambda,p},$$

what contradicts the fact that, by Lemma 2.4, we have

$$\sup_{t\geq 0} I_{\lambda,\mu}(t\varphi_\varepsilon) < c^*_{\lambda,p}, \text{ for all } \lambda \in (0,\Lambda).$$

Then

$$u_n \longrightarrow u_2 \text{ in } \mathcal{D}.$$

Thus, we obtain a critical point u_2 of $I_{\lambda,\mu}$ for all $\lambda \in (0,\Lambda_*)$ with

$$\Lambda_* := \min\{\lambda_*, \Lambda\}$$

satisfying

$$I_{\lambda,\mu}(u_2) > 0.$$

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Chapter 3

On nonhomogeneous elliptic equations with decaying cylindrical potential and critical exponent

In this part, we study the existence and multiplicity of solutions for elliptic equations involving decaying cylindrical potentials and critical exponents by using the Nehari manifold and Hardy-Sobolev-Maz'ya inequality.

3.1 Introduction

In this chapter we consider the following proble

$$\begin{cases} -\operatorname{div}\left(|y|^{-2a}\nabla u\right) - \mu |y|^{-2(a+1)} u = & h|y|^{-2_* b}|u|^{2_*-2} u + \lambda g \text{ in } \mathbb{R}^N, \\ & y \neq 0 \\ u \in \mathcal{D}_0^{1,2}, \end{cases} \quad (1.1)$$

where each point x in $\mathbb{R}^N$ is written as a pair $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ where k and N are integers such that $N \geq 3$ and k belongs to $\{1,...,N\}$, $-\infty < a < (k-2)/2$, $a \leq b < a+1$, $2_* = 2N/(N-2+2(b-a))$, $-\infty < \mu < \bar{\mu}_{a,k} := ((k-2(a+1))/2)^2$, $g \in \mathcal{H}_\mu' \cap C(\mathbb{R}^N)$, h is a bounded positive function on $\mathbb{R}^k$ and λ is real parameter. $\mathcal{H}_\mu'$ is the dual of $\mathcal{H}_\mu$, where $\mathcal{H}_\mu$ and

$\mathcal{D}_0^{1,2}$ will be defined later.

Some results are already available for (1.1) in the case $k = N$, see for example [10, 11] and the references therein. Wang and Zhou [10] proved that there exist at least two solutions for (1.1) with $a = 0$, $0 < \mu \leq \bar{\mu}_{0,N} = ((N-2)/2)^2$ and $h \equiv 1$, under certain conditions on g. Bouchekif and Matallah [2] showed the existence of two solutions of (1.1) under certain conditions on functions g and h, when $0 < \mu \leq \bar{\mu}_{0,N}$, $\lambda \in (0, \Lambda_*)$, $-\infty < a < (N-2)/2$ and $a \leq b < a+1$, with Λ_* a positive constant.

Concerning existence results in the case $k < N$, we cite [6, 7] and the references therein. Musina [7] considered (1.1) with $-a/2$ instead of a and $\lambda = 0$, also (1.1) with $a = 0$, $b = 0$, $\lambda = 0$, with $h \equiv 1$ and $a \neq 2-k$. She established the existence of a ground state solution when $2 < k \leq N$ and $0 < \mu < \bar{\mu}_{a,k} = ((k-2+a)/2)^2$ for (1.1) with $-a/2$ instead of a and $\lambda = 0$. She also showed that (1.1) with $a = 0$, $b = 0$, $\lambda = 0$ does not admit ground state solutions. Badiale et al. [1] studied (1.1) with $a = 0$, $b = 0$, $\lambda = 0$ and $h \equiv 1$. They proved the existence of at least a nonzero nonnegative weak solution u, satisfying $u(y,z) = u(|y|, z)$ when $2 \leq k < N$ and $\mu < 0$. Bouchekif and El Mokhtar [3] proved that (1.1) admits two distinct solutions when $2 < k \leq N$, $b = N - p(N-2)/2$ with $p \in (2, 2^*]$, $\mu < \bar{\mu}_{0,k}$, and $\lambda \in (0, \Lambda_*)$ where Λ_* is a positive constant. Terracini [9] proved that there is no positive solutions of (1.1) with $b = 0$, $\lambda = 0$ when $a \neq 0$, $h \equiv 1$ and $\mu < 0$. The regular problem corresponding to $a = b = \mu = 0$ and $h \equiv 1$ has been considered on a regular bounded domain Ω by Tarantello [8]. She proved that, for $g \in H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, not identically zero and satisfying a suitable condition, the problem considered admits two distinct solutions.

Before formulating our results, we give some definitions and notation.

We denote by $\mathcal{D}_0^{1,2} = \mathcal{D}_0^{1,2}\left(\mathbb{R}^k \backslash \{0\} \times \mathbb{R}^{N-k}\right)$ and $\mathcal{H}_\mu = \mathcal{H}_\mu\left(\mathbb{R}^k \backslash \{0\} \times \mathbb{R}^{N-k}\right)$, the closure of $C_0^\infty\left(\mathbb{R}^k \backslash \{0\} \times \mathbb{R}^{N-k}\right)$ with respect to the norms

$$\|u\|_{a,0} = \left(\int_{\mathbb{R}^N} |y|^{-2a} |\nabla u|^2 \, dx\right)^{1/2}$$

and

$$\|u\|_{a,\mu} = \left(\int_{\mathbb{R}^N} \left(|y|^{-2a} |\nabla u|^2 - \mu |y|^{-2(a+1)} |u|^2\right) dx\right)^{1/2},$$

respectively, with $\mu < \bar{\mu}_{a,k} = ((k-2(a+1))/2)^2$ for $k \neq 2(a+1)$.

From the Hardy-Sobolev-Maz'ya inequality, it is easy to see that the norm $\|u\|_{a,\mu}$ is equivalent to $\|u\|_{a,0}$.

Since our approach is variational, we define the functional $I_{a,b,\lambda,\mu}$ on $\mathcal{H}_\mu$ by

$$I(u) := I_{a,b,\lambda,\mu}(u) := (1/2)\|u\|_{a,\mu}^2 - (1/2_*)\int_{\mathbb{R}^N} h|y|^{-2_*b}|u|^{2_*}dx - \lambda\int_{\mathbb{R}^N} gudx.$$

We say that $u \in \mathcal{H}_\mu$ is a weak solution of the problem $(\mathcal{P})$ if it satisfies

$$\begin{aligned}\left\langle I'(u), v\right\rangle &= \int_{\mathbb{R}^N}\left(|y|^{-2a}\nabla u\nabla v - \mu|y|^{-2(a+1)}uv - h|y|^{-2_*b}|u|^{2_*-2}uv - \lambda gv\right)dx \\ &= 0, \text{ for } v \in \mathcal{H}_\mu.\end{aligned}$$

Here $\langle .,. \rangle$ denotes the product in the duality $\mathcal{H}_\mu'$, $\mathcal{H}_\mu$.

Throughout this work, we consider the following assumptions:

(G) There exist $\nu_0 > 0$ and $\delta_0 > 0$ such that $g(x) \geq \nu_0$, for all x in $B(0, 2\delta_0)$.

(H) $\lim_{|y|\longrightarrow 0} h(y) = \lim_{|y|\longrightarrow\infty} h(y) = h_0 > 0$, $h(y) \geq h_0$, $y \in \mathbb{R}^k$.

Here, $B(a,r)$ denotes the ball centered at a with radius r.

Under some sufficient conditions on coefficients of equation of (1.1), we split $\mathcal{N}$ in two disjoint subsets $\mathcal{N}^+$ and $\mathcal{N}^-$, thus we consider the minimization problems on $\mathcal{N}^+$ and $\mathcal{N}^-$ respectively.

Remark 3.1 *Note thataAll solutions of* (1.1)*, are nontrivial.*

We shall state our main results:

Theorem 3.1 *Assume that* $3 \leq k \leq N$, $-1 < a < (k-2)/2$, $0 \leq \mu < \bar{\mu}_{a,k}$, *and* (G) *holds, then there exists* $\Lambda_1 > 0$ *such that the problem* (1.1)*, has at least one solution on* $\mathcal{H}_\mu$ *for all* $\lambda \in (0, \Lambda_1)$.

Theorem 3.2 *In addition to the assumptions of the Theorem* 3.1*, if* (H) *holds, then there exists* $\Lambda_2 > 0$ *such that the problem* (1.1)*, has at least two solutions on* $\mathcal{H}_\mu$ *for all* $\lambda \in (0, \Lambda_2)$.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 3.1 and 3.2.

3.2 Preliminaries

We list here a few integral inequalities. The first one that we need is the Hardy inequality with cylindrical weights [7]. It states that

$$\bar{\mu}_{a,k} \int_{\mathbb{R}^N} |y|^{-2(a+1)} v^2 dx \leq \int_{\mathbb{R}^N} |y|^{-2a} |\nabla v|^2 \, dx, \text{ for all } v \in \mathcal{H}_\mu,$$

The starting point for studying (1.1), is the Hardy-Sobolev-Maz'ya inequality that is particular to the cylindrical case $k < N$ and that was proved by Maz'ya in [6]. It states that there exists positive constant $C_{a,2_*}$ such that

$$C_{a,2_*} \left(\int_{\mathbb{R}^N} |y|^{-2_* b} |v|^{2_*} \, dx \right)^{2/2_*} \leq \int_{\mathbb{R}^N} \left(|y|^{-2a} |\nabla v|^2 - \mu \, |y|^{-2(a+1)} v^2 \right) dx,$$

for any $v \in C_c^\infty((\mathbb{R}^k \backslash \{0\}) \times \mathbb{R}^{N-k})$.

Proposition 3.1 *(see* [6]*). The value*

$$S_{\mu,2_*} = S_{\mu,2_*}(k, 2_*) := \inf_{v \in \mathcal{H}_\mu \backslash \{0\}} \frac{\int_{\mathbb{R}^N} \left(|y|^{-2a} |\nabla v|^2 - \mu \, |y|^{-2(a+1)} v^2 \right) dx}{\left(\int_{\mathbb{R}^N} |y|^{-2_* b} |v|^{2_*} \, dx \right)^{2/2_*}}, \tag{3.1}$$

is achieved on $\mathcal{H}_\mu$*, for* $2 \leq k < N$ *and* $\mu \leq \bar{\mu}_{a,k}$*.*

Definition 3.1 *Let* $c \in \mathbb{R}$*,* E *a Banach space and* $I \in C^1(E, \mathbb{R})$*.*

(i) $(u_n)_n$ *is a Palais-Smale sequence at level c (in short* $(PS)_c$*) in E for I if*

$$I(u_n) = c + o_n(1) \text{ and } I'(u_n) = o_n(1),$$

where $o_n(1) \longrightarrow 0$ *as* $n \longrightarrow \infty$*.*

(ii) We say that I satisfies the $(PS)_c$ *condition if any* $(PS)_c$ *sequence in E for I has a convergent subsequence.*

3.2.1 Nehari manifold

It is well known that I is of class C^1 in $\mathcal{H}_\mu$ and the solutions of (1.1) are the critical points of I which is not bounded below on $\mathcal{H}_\mu$. Consider the following Nehari manifold

$$\mathcal{N} = \left\{ u \in \mathcal{H}_\mu \backslash \{0\} : \ \left\langle I^{'}(u), u \right\rangle = 0 \right\},$$

Thus, $u \in \mathcal{N}$ if and only if

$$\|u\|_{a,\mu}^2 - \int_{\mathbb{R}^N} h\,|y|^{-2_* b}\,|u|^{2_*}\,dx - \lambda \int_{\mathbb{R}^N} gudx = 0. \tag{3.2}$$

Note that $\mathcal{N}$ contains every nontrivial solution of the problem (1.1) Moreover, we have the following results.

Lemma 3.1 *The functional I is coercive and bounded from below on $\mathcal{N}$.*

Proof: If $u \in \mathcal{N}$, then by (4.2) and the Hölder inequality, we deduce that

$$\begin{aligned} I(u) &= ((2_* - 2)/2_* 2)\,\|u\|_{a,\mu}^2 - \lambda\,(1 - (1/2_*)) \int_{\mathbb{R}^N} gudx \qquad (3.3) \\ &\geq ((2_* - 2)/2_* 2)\,\|u\|_{a,\mu}^2 - \lambda\,(1 - (1/2_*))\,\|u\|_{a,\mu}\,\|g\|_{\mathcal{H}_\mu'} \\ &\geq -\lambda^2 C_0, \end{aligned}$$

where

$$C_0 := C_0\left(\|g\|_{\mathcal{H}_\mu'}\right) = \left[(2_* - 1)^2 / 2_* 2\,(2_* - 2)\right] \|g\|_{\mathcal{H}_\mu'}^2 > 0.$$

Thus, I is coercive and bounded from below on $\mathcal{N}$. ■

Define

$$\Psi_\lambda(u) = \left\langle I^{'}(u), u \right\rangle.$$

Then, for $u \in \mathcal{N}$

$$\begin{aligned} \left\langle \Psi_\lambda'(u), u \right\rangle &= 2\|u\|_{a,\mu}^2 - 2_* \int_{\mathbb{R}^N} h|y|^{-2_* b}|u|^{2_*}dx - \lambda \int_{\mathbb{R}^N} gudx \\ &= \|u\|_{a,\mu}^2 - (2_* - 1)\int_{\mathbb{R}^N} h|y|^{-2_* b}|u|^{2_*}dx \\ &= \lambda(2_* - 1)\int_{\mathbb{R}^N} gudx - (2_* - 2)\|u\|_{a,\mu}^2 . \end{aligned} \tag{3.4}$$

Now, we split $\mathcal{N}$ in three parts:

$$\mathcal{N}^+ = \left\{ u \in \mathcal{N} : \left\langle \Psi_\lambda'(u), u \right\rangle > 0 \right\},\ \mathcal{N}^0 = \left\{ u \in \mathcal{N} : \left\langle \Psi_\lambda'(u), u \right\rangle = 0 \right\},$$

and $\mathcal{N}^- = \left\{ u \in \mathcal{N} : \left\langle \Psi_\lambda'(u), u \right\rangle < 0 \right\}$.

We have the following results.

Lemma 3.2 *Suppose that there exists a local minimizer u_0 for I on $\mathcal{N}$ and $u_0 \notin \mathcal{N}^0$. Then, $I'(u_0) = 0$ in $\mathcal{H}_\mu'$.*

Proof: If u_0 is a local minimizer for I on $\mathcal{N}$, then there exists $\theta \in \mathbb{R}$ such that

$$\left\langle I'(u_0), \varphi \right\rangle = \theta \left\langle \Psi_\lambda'(u_0), \varphi \right\rangle$$

for any $\varphi \in \mathcal{H}_\mu$.

If $\theta = 0$, then the lemma is proved. If not, taking $\varphi \equiv u_0$ and using the assumption $u_0 \in \mathcal{N}$, we deduce

$$0 = \left\langle I'(u_0), u_0 \right\rangle = \theta \left\langle \Psi_\lambda'(u_0), u_0 \right\rangle .$$

Thus,

$$\left\langle \Psi_\lambda'(u_0), u_0 \right\rangle = 0,$$

which contradicts the fact that $u_0 \notin \mathcal{N}^0$. ■

Let be

$$\Lambda_1 := (2_* - 2)(2_* - 1)^{-(2_*-1)/(2_*-2)} \left[(h_0)^{-1} S_{\mu,2_*} \right]^{2_*/2(2_*-2)} \|g\|_{\mathcal{H}_\mu'}^{-1} . \tag{3.5}$$

Lemma 3.3 *We have $\mathcal{N}^0 = \emptyset$ for all $\lambda \in (0, \Lambda_1)$.*

Proof: Let us reason by contradiction.

Suppose $\mathcal{N}^0 \neq \emptyset$ for some $\lambda \in (0, \Lambda_1)$. Then, by (3.4) and for $u \in \mathcal{N}^0$, we have

$$\begin{aligned} \|u\|_{a,\mu}^2 &= (2_* - 1) \int_{\mathbb{R}^N} h\,|y|^{-2_* b}\,|u|^{2_*}\,dx \\ &= \lambda\left((2_* - 1)\,/\,(2_* - 2)\right) \int_{\mathbb{R}^N} gudx. \end{aligned} \tag{3.6}$$

Moreover, by (G), the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\left[\left((h_0)^{-1} S_{\mu,2_*}\right)^{2_*/2} / \,(2_* - 1)\right]^{1/(2_*-2)} \leq \|u\|_{a,\mu} \leq \left[\lambda\left((2_* - 1)\,\|g\|_{\mathcal{H}'_\mu} / \,(2_* - 2)\right)\right]. \tag{3.7}$$

This implies that $\lambda \geq \Lambda_1$, which is a contradiction with the fact that $\lambda \in (0, \Lambda_1)$. ■

Thus $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ for $\lambda \in (0, \Lambda_1)$.

Define

$$c := \inf_{u \in \mathcal{N}} I(u)\,,\ c^+ := \inf_{u \in \mathcal{N}^+} I(u) \text{ and } c^- := \inf_{u \in \mathcal{N}^-} I(u)\,.$$

For the sequel, we need the following Lemma.

Lemma 3.4 (i) *If $\lambda \in (0, \Lambda_1)$, then one has $c \leq c^+ < 0$.*

(ii) *If $\lambda \in (0, (1/2)\,\Lambda_1)$, then $c^- > C_1$, where*

$$\begin{aligned} C_1 &= C_1\left(\lambda, S_{\mu,2_*}\,\|g\|_{\mathcal{H}'_\mu}\right) = \left((2_* - 2)\,/2_* 2\right)(2_* - 1)^{2/(2_*-2)}\,(S_{\mu,2_*})^{2_*/(2_*-2)} + \\ &\qquad - \lambda\left(1 - (1/2_*)\right)(2_* - 1)^{2/(2_*-2)}\,\|g\|_{\mathcal{H}'_\mu}\,. \end{aligned}$$

Proof: (i) Let $u \in \mathcal{N}^+$. By (3.4), we have

$$\left[1/\,(2_* - 1)\right]\|u\|_{a,\mu}^2 > \int_{\mathbb{R}^N} h\,|y|^{-2_* b}\,|u|^{2_*}\,dx$$

and so

$$\begin{aligned} I(u) &= (-1/2)\|u\|_{a,\mu}^2 + (1-(1/2_*))\int_{\mathbb{R}^N} h|y|^{-2_*b}|u|^{2_*}dx \\ &< [(-1/2)+(1-(1/2_*))(1/(2_*-1))]\|u\|_{a,\mu}^2 \\ &= -((2_*-2)/2_*2)\|u\|_{a,\mu}^2, \end{aligned}$$

we conclude that $c \leq c^+ < 0$.

(ii) Let $u \in \mathcal{N}^-$. By (3.4), we get

$$[1/(2_*-1)]\|u\|_{a,\mu}^2 < \int_{\mathbb{R}^N} h|y|^{-2_*b}|u|^{2_*}dx.$$

Moreover, by Sobolev embedding theorem, we have

$$\int_{\mathbb{R}^N} h|y|^{-2_*b}|u|^{2_*}dx \leq (S_{\mu,2_*})^{-2_*/2}\|u\|_{a,\mu}^{2_*}.$$

This implies

$$\|u\|_{a,\mu} > [(2_*-1)]^{-1/(2_*-2)}(S_{\mu,2_*})^{2_*/2(2_*-2)}, \text{ for all } u \in \mathcal{N}^-.$$

By (3.3), we get

$$I(u) \geq ((2_*-2)/2_*2)\|u\|_{a,\mu}^2 - \lambda(1-(1/2_*))\|u\|_{a,\mu}\|g\|_{\mathcal{H}'_\mu}.$$

Thus, for all $\lambda \in (0,(1/2)\Lambda_1)$, we have $I(u) \geq C_1$. ■

For each $u \in \mathcal{H}_\mu$, we write

$$t_m := t_{\max}(u) = \left[\frac{\|u\|_{a,\mu}}{(2_*-1)\int_{\mathbb{R}^N} h|y|^{-2_*b}|u|^{2_*}dx}\right]^{1/(2_*-2)} > 0.$$

Lemma 3.5 *Let $\lambda \in (0,\Lambda_1)$. For each $u \in \mathcal{H}_\mu$, one has the following:*

(i) If $\int_{\mathbb{R}^N} g(x)udx \leq 0$, then there exists a unique $t^- > t_m$ such that $t^-u \in \mathcal{N}^-$ and

$$I(t^-u) = \sup_{t\geq 0} I(tu).$$

(ii) If $\int_{\mathbb{R}^N} g(x)\, u dx > 0$*, then there exist unique* t^+ *and* t^- *such that* $0 < t^+ < t_m < t^-$*,* $t^+u \in \mathcal{N}^+$*,* $t^-u \in \mathcal{N}^-$*,*

$$I\left(t^+u\right) = \inf_{0 \le t \le t_m} I(tu) \ \text{ and } I\left(t^-u\right) = \sup_{t \ge 0} I(tu).$$

Proof: With minor modifications, we refer to [5]. ■

3.3 Proof of Theorem 3.1

For the proof we get, firstly, the following results:

Proposition 3.2 *(see* [5]*)*

(i) If $\lambda \in (0, \Lambda_1)$*, then there exists a minimizing sequence* $(u_n)_n$ *in* $\mathcal{N}$ *such that*

$$I(u_n) = c + o_n(1) \ \text{ and } I'(u_n) = o_n(1) \ \text{ in } \mathcal{H}_\mu', \tag{3.8}$$

where $o_n(1)$ *tends to* 0 *as* n *tends to* ∞*.*

(ii) if $\lambda \in (0, (1/2)\Lambda_1)$*, then there exists a minimizing sequence* $(u_n)_n$ *in* $\mathcal{N}^-$ *such that*

$$I(u_n) = c^- + o_n(1) \ \text{ and } I'(u_n) = o_n(1) \ \text{ in } \mathcal{H}_\mu'.$$

Now, taking as a starting point the work of Tarantello [8], we establish the existence of a local minimum for I on $\mathcal{N}^+$.

Proposition 3.3 *If* $\lambda \in (0, \Lambda_1)$*, then* I *has a minimizer* $u_1 \in \mathcal{N}^+$ *and it satisfies*

(i) $I(u_1) = c = c^+ < 0$*,*

(ii) u_1 *is a solution of* (1.1)*.*

Proof: (i) By Lemma 4.2, I is coercive and bounded below on $\mathcal{N}$. We can assume that there exists $u_1 \in \mathcal{H}_\mu$ such that

$$\begin{aligned} u_n &\rightharpoonup u_1 \text{ weakly in } \mathcal{H}_\mu, \\ u_n &\rightharpoonup u_1 \text{ weakly in } L^{2_*}\left(\mathbb{R}^N, |y|^{-2_* b}\right), \\ u_n &\to u_1 \text{ a.e in } \mathbb{R}^N. \end{aligned} \tag{3.9}$$

Thus, by (3.8) and (3.9), u_1 is a weak solution of (1.1) since $c < 0$ and $I(0) = 0$. Now, we show that u_n converges to u_1 strongly in $\mathcal{H}_\mu$. Suppose otherwise. Then $\|u_1\|_{a,\mu} < \liminf_{n \longrightarrow \infty} \|u_n\|_{a,\mu}$ and we obtain

$$\begin{aligned} c &\leq I(u_1) = ((2_* - 2)/2_* 2) \|u_1\|_{a,\mu}^2 - \lambda (1 - (1/2_*)) \int_{\mathbb{R}^N} g u_1 dx \\ &< \liminf_{n \longrightarrow \infty} I(u_n) = c. \end{aligned}$$

We get a contradiction. Therefore, u_n converges to u_1 strongly in $\mathcal{H}_\mu$. Moreover, we have $u_1 \in \mathcal{N}^+$. If not, then by Lemma 4.6, there are two numbers t_0^+ and t_0^-, uniquely defined so that $t_0^+ u_1 \in \mathcal{N}^+$ and $t_0^- u_1 \in \mathcal{N}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} I(tu_1)_{|t=t_0^+} = 0 \text{ and } \frac{d^2}{dt^2} I(tu_1)_{|t=t_0^+} > 0,$$

there exists $t_0^+ < t^- \leq t_0^-$ such that $I\left(t_0^+ u_1\right) < I(t^- u_1)$. By Lemma 4.6,

$$I\left(t_0^+ u_1\right) < I\left(t^- u_1\right) < I\left(t_0^- u_1\right) = I(u_1),$$

which is a contradiction. ■

3.4 Proof of Theorem 3.2

In this section, we establish the existence of a second solution of (1.1). For this, we require the following Lemmas with C_0 is given in (3.3).

Lemma 3.6 *Assume that (G) holds and let $(u_n)_n \subset \mathcal{H}_\mu$ be a $(PS)_c$ sequence for I for some*

$c \in \mathbb{R}$ with $u_n \rightharpoonup u$ in $\mathcal{H}_\mu$. Then, $I'(u) = 0$ and

$$I(u) \geq -C_0\lambda^2.$$

Proof: It is easy to prove that $I'(u) = 0$, which implies that $\left\langle I'(u), u \right\rangle = 0$, and

$$\int_{\mathbb{R}^N} h\,|y|^{-2_*b}\,|u|^{2_*}\,dx = \|u\|^2_{a,\mu} - \lambda \int_{\mathbb{R}^N} gudx.$$

Therefore, we get

$$I(u) = ((2_* - 2)/2_*2)\,\|u\|^2_{a,\mu} - \lambda\,(1 - (1/2_*)) \int_{\mathbb{R}^N} gudx.$$

Using (3.3), we obtain that

$$I(u) \geq -C_0\lambda^2.$$

■

Lemma 3.7 *Assume that (G) holds and for any $(PS)_c$ sequence with c is a real number such that $c < c^*_\lambda$. Then, there exists a subsequence which converges strongly.*

Here $c^*_\lambda := ((2_* - 2)/2_*2)\,(h_0)^{-2/(2_*-2)}\,(S_{\mu,2_*})^{2_*/(2_*-2)} - C_0\lambda^2$.

Proof: Using standard arguments, we get that $(u_n)_n$ is bounded in $\mathcal{H}_\mu$. Thus, there exist a subsequence of $(u_n)_n$ which we still denote by $(u_n)_n$ and $u \in \mathcal{H}_\mu$ such that

$$\begin{aligned} u_n &\rightharpoonup u \text{ weakly in } \mathcal{H}_\mu, \\ u_n &\rightharpoonup u \text{ weakly in } L^{2_*}\left(\mathbb{R}^N, |y|^{-2_*b}\right). \\ u_n &\to u \text{ a.e in } \mathbb{R}^N. \end{aligned}$$

Then, u is a weak solution of (1.1). Let $v_n = u_n - u$, then by Brézis-Lieb [4], we obtain

$$\|v_n\|^2_{a,\mu} = \|u_n\|^2_{a,\mu} - \|u\|^2_{a,\mu} + o_n(1) \tag{3.10}$$

and

$$\int_{\mathbb{R}^N} h\,|y|^{-2_*b}\,|v_n|^{2_*}\,dx = \int_{\mathbb{R}^N} h\,|y|^{-2_*b}\,|u_n|^{2_*}\,dx - \int_{\mathbb{R}^N} h\,|y|^{-2_*b}\,|u|^{2_*}\,dx + o_n(1)\,. \tag{3.11}$$

On the other hand, by using the assumption (H), we obtain

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N} h(x)\,|y|^{-2_*b}\,|v_n|^{2_*}\,dx = h_0 \lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N} |y|^{-2_*b}\,|v_n|^{2_*}\,dx. \tag{3.12}$$

Since $I(u_n) = c + o_n(1)$, $I^{'}(u_n) = o_n(1)$ and by (3.10), (3.11), and (3.12) we can deduce that

$$(1/2)\,\|v_n\|_{a,\mu}^2 - (1/2_*)\int_{\mathbb{R}^N} h\,|y|^{-2_*b}\,|v_n|^{2_*}\,dx = c - I(u) + o_n(1)\,, \tag{3.13}$$

$$\|v_n\|_{a,\mu}^2 - \int_{\mathbb{R}^N} h\,|y|^{-2_*b}\,|v_n|^{2_*}\,dx = o_n(1)\,.$$

Hence, we may assume that

$$\|v_n\|_{a,\mu}^2 \longrightarrow l, \quad \int_{\mathbb{R}^N} h\,|y|^{-2_*b}\,|v_n|^{2_*}\,dx \longrightarrow l. \tag{3.14}$$

Sobolev inequality gives $\|v_n\|_{a,\mu}^2 \geq (S_{\mu,2_*})\int_{\mathbb{R}^N} h\,|y|^{-2_*b}\,|v_n|^{2_*}\,dx$. Combining this inequality with (3.14), we get

$$l \geq S_{\mu,2_*}\left(l^{-1}h_0\right)^{-2/2_*}.$$

Either $l = 0$ or $l \geq (h_0)^{-2/(2_*-2)}\,(S_{\mu,2_*})^{2_*/(2_*-2)}$. Suppose that $l \geq (h_0)^{-2/(2_*-2)}\,(S_{\mu,2_*})^{2_*/(2_*-2)}$. Then, from (3.13), (3.14) and Lemma 4.7, we get

$$c \geq ((2_*-2)/2_*2)\,l + I(u) \geq c_\lambda^*,$$

which is a contradiction. Therefore, $l = 0$ and we conclude that u_n converges to u strongly in $\mathcal{H}_\mu$. ■

Lemma 3.8 *Assume that (G) and (H) hold. Then, there exist $v \in \mathcal{H}_\mu$ and $\Lambda_* > 0$ such that* *for* $\lambda \in (0, \Lambda_*)$, *one has*

$$\sup_{t \geq 0} I(tv) < c_\lambda^*,$$

In particular,

$$c^- < c_\lambda^*, \text{ for all } \lambda \in (0, \Lambda_*).$$

Proof: Let φ_ε be such that

$$\varphi_\varepsilon(x) = \begin{cases} \omega_\varepsilon(x) & \text{if } g(x) \geq 0 \text{ for all } x \in \mathbb{R}^N \\ \omega_\varepsilon(x - x_0) & \text{if } g(x_0) > 0 \text{ for } x_0 \in \mathbb{R}^N \\ -\omega_\varepsilon(x) & \text{if } g(x) \leq 0 \text{ for all } x \in \mathbb{R}^N \end{cases}$$

where ω_ε verifies (3.1). Then, we claim that there exists $\varepsilon_0 > 0$ such that

$$\lambda \int_{\mathbb{R}^N} g(x)\, \varphi_\varepsilon(x)\, dx > 0 \text{ for any } \varepsilon \in (0, \varepsilon_0). \tag{3.15}$$

In fact, if $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \in \mathbb{R}^N$, (3.15) obviously holds. If there exists $x_0 \in \mathbb{R}^N$ such that $g(x_0) > 0$, then by the continuity of $g(x)$, there exists $\eta > 0$ such that $g(x) > 0$ for all $x \in B(x_0, \eta)$. Then by the definition of $\omega_\varepsilon(x - x_0)$, it is easy to see that there exists an ε_0 small enough such that

$$\lambda \int_{\mathbb{R}^N} g(x)\, \omega_\varepsilon(x - x_0)\, dx > 0, \text{ for any } \varepsilon \in (0, \varepsilon_0).$$

Now, we consider the following functions

$$f(t) = I(t\varphi_\varepsilon) \text{ and } \tilde{f}(t) = \left(t^2/2\right) \|\varphi_\varepsilon\|_{a,\mu}^2 - \left(t^{2_*}/2_*\right) \int_{\mathbb{R}^N} h\, |y|^{-2_* b} \, |\varphi_\varepsilon|^{2_*}\, dx.$$

Then, we get for all $\lambda \in (0, \Lambda_1)$

$$f(0) = 0 < c_\lambda^*.$$

By the continuity of f, there exists $t_0 > 0$ small enough such that

$$f(t) < c_\lambda^*, \text{ for all } t \in (0, t_0).$$

On the other hand, we have

$$\max_{t \geq 0} \tilde{f}(t) = ((2_* - 2)/2_* 2)(h_0)^{-2/(2_*-2)} (S_{\mu,2_*})^{2_*/(2_*-2)}.$$

Then, we obtain

$$\sup_{t \geq 0} I(t\varphi_\varepsilon) < ((2_* - 2)/2_* 2)(h_0)^{-2/(2_*-2)} (S_{\mu,2_*})^{2_*/(2_*-2)} - \lambda t_0 \int_{\mathbb{R}^N} g\varphi_\varepsilon dx.$$

Now, taking $\lambda > 0$ such that

$$-\lambda t_0 \int_{\mathbb{R}^N} g\varphi_\varepsilon dx < -C_0 \lambda^2,$$

and by (3.15), we get

$$0 < \lambda < (t_0/C_0) \left(\int_{\mathbb{R}^N} g\varphi_\varepsilon \right), \text{ for } \varepsilon << \varepsilon_0.$$

Set

$$\Lambda_* = \min \left\{ \Lambda_1, \ (t_0/C_0) \left(\int_{\mathbb{R}^N} g\varphi_\varepsilon \right) \right\}.$$

We deduce that

$$\sup_{t \geq 0} I(t\varphi_\varepsilon) < c_\lambda^*, \ \text{for all } \lambda \in (0, \Lambda_*). \tag{3.16}$$

Now, we prove that

$$c^- < c_\lambda^*, \ \text{for all } \lambda \in (0, \Lambda_*).$$

By (G) and the existence of ψ_n satisfying (3.1), we have

$$\lambda \int_{\mathbb{R}^N} g\psi_n dx > 0.$$

Combining this with Lemma 4.6 and from the definition of c^- and (3.16), we obtain that there exists $t_n > 0$ such that $t_n\psi_n \in \mathcal{N}^-$ and for all $\lambda \in (0, \Lambda_*)$,

$$c^- \leq I(t_n\psi_n) \leq \sup_{t \geq 0} I(t\psi_n) < c_\lambda^*.$$

■

Now we establish the existence of a local minimum of I on $\mathcal{N}^-$.

Proposition 3.4 *There exists $\Lambda_2 > 0$ such that for $\lambda \in (0, \Lambda_2)$, the functional I has a minimizer u_2 in $\mathcal{N}^-$ and satisfies*

(i) $I(u_2) = c^-$,

(ii) u_2 is a solution of (1.1) *in $\mathcal{H}_\mu$, where $\Lambda_2 = \min\{(1/2)\Lambda_1, \Lambda_*\}$ with Λ_1 defined as in* (3.5) *and Λ_* defined as in the proof of Lemma* 3.8*.*

Proof: By Proposition 4.1 *(ii)*, there exists a $(PS)_{c^-}$ sequence for I, $(u_n)_n$ in $\mathcal{N}^-$ for all $\lambda \in (0, (1/2)\Lambda_1)$. From Lemmas 3.7, 3.8 and 4.5 *(ii)*, for $\lambda \in (0, \Lambda_*)$, I satisfies $(PS)_{c^-}$ condition and $c^- > 0$. Then, we get that $(u_n)_n$ is bounded in $\mathcal{H}_\mu$. Therefore, there exist a subsequence of $(u_n)_n$ still denoted by $(u_n)_n$ and $u_2 \in \mathcal{N}^-$ such that u_n converges to u_2 strongly in $\mathcal{H}_\mu$ and $I(u_2) = c^-$ for all $\lambda \in (0, \Lambda_2)$. Finally, by using the same arguments as in the proof of the Proposition 4.2, for all $\lambda \in (0, \Lambda_1)$, we have that u_2 is a solution of (1.1). ■

Now, we complete the proof of Theorem 3.2. By Propositions 4.2 and 3.4, we obtain that (1.1) has two solutions u_1 and u_2 such that $u_1 \in \mathcal{N}^+$ and $u_2 \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, this implies that u_1 and u_2 are distinct.

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Chapter 4

On nonhomogeneous singular elliptic systems involving a singular weakly coupled potential and the Caffarelli-Kohn-Nirenberg critical exponent

In this paper, we are interested in the existence and multiplicity results of nontrivial solutions to nonhomogeneous singular elliptic systems involving a singular weakly coupled potential and the Caffarelli-Kohn-Nirenberg critical exponent $(\mathcal{S}_{\lambda_1,\lambda_2})$. With the help of the Nehari manifold and under sufficient conditions on the parameters λ_1 and λ_2, we prove some existence results.

4.1 Introduction

This paper deals with the existence and multiplicity of nontrivial solutions to the following system $(\mathcal{S}_{\lambda_1,\lambda_2})$

$$\begin{cases} -\operatorname{div}\left(|x|^{-2a}\nabla u\right)-\mu|x|^{-2(a+1)}u= & (\alpha+1)|x|^{-2_*b}|u|^{\alpha-1}u|v|^{\beta+1}+\lambda_1 f_1 \text{ in } \Omega \\ -\operatorname{div}\left(|x|^{-2a}\nabla v\right)-\mu|x|^{-2(a+1)}v= & (\beta+1)|x|^{-2_*b}|u|^{\alpha+1}|v|^{\beta-1}v+\lambda_2 f_2 \text{ in } \Omega \\ u=v=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$ $(N\geq 3)$ containing 0 in its interior, $-\infty < a < (N-2)/2$, $a \leq b < a+1$, $2_* = 2N/(N-2+2(b-a))$ is the critical Caffarelli-Kohn-Nirenberg exponent, $-\infty < \mu < \bar{\mu}_a := ((N-2(a+1))/2)^2$, α, β are positive real such that $\alpha+\beta = 2_*-2$, λ_1, λ_2 are real parameters and f_1, f_2 are functions defined on $\bar{\Omega}$.

The degeneracy and singularity occur in the system $(\mathcal{S}_{\lambda_1,\lambda_2})$, thus standard variational methods do not apply.

In recent years much attention has been paid to the existence of nontrivial solutions for problems $(\mathcal{P}_{a,\lambda,\mu})$ of the type

$$\begin{cases} -\operatorname{div}\left(|x|^{-2a}\nabla u\right)-\mu|x|^{-2(a+1)}u= & h(x)|x|^{-2_*b}|u|^{2_*-2}u+\lambda f(x) \text{ in } \Omega \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Wang and Zhou [10] have proved that $(\mathcal{P}_{0,\mu,1})$, for $h(x)\equiv 1$ and $a=0$, has at least two distinct solutions when $0\leq\mu<\bar{\mu}_0 := ((N-2)/2)^2$ and under some sufficient conditions on f. In [2], Bouchekif and Matallah have showed the existence of two nontrivial solutions of $(\mathcal{P}_{a,\lambda,\mu})$ when $0<\mu\leq\bar{\mu}_a$, $-\infty<a<(N-2)/2$, $a\leq b<a+1$, $\lambda\in(0,\Lambda_*)$ with Λ_* a positive constant and under some appropriate conditions on functions f and h.

Many existence results are available for regular systems which derive from potential, we quote for example [1] and [6]. However, to our knowledge there are few results for singular systems, we can cite for example [8].

By $\mathcal{H}_\mu := \mathcal{H}_\mu(\Omega)$, we denote the completion of the space $\mathcal{C}_0^\infty\left(\mathbb{R}^N\right)$ with respect to the norm

$$\|u\|_{\mu,a} = \left(\int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu|y|^{-2(a+1)}|u|^2\right)dx\right)^{1/2}, \text{ for } -\infty<\mu<\bar{\mu}_a.$$

Using the Hardy inequality, this norm is equivalent to $\|u\|_{0,a}$. More explicitly, we have

$$\left(1-\max\left(\mu,0\right)/\bar{\mu}_a\right)^{1/2}\|u\|_{0,a}\leq\|u\|_{\mu,a}\leq\left(1-\min\left(\mu,0\right)/\bar{\mu}_a\right)^{1/2}\|u\|_{0,a}\,.$$

The space $\mathcal{H}:=\mathcal{H}_\mu\times\mathcal{H}_\mu$ is endowed with the norm

$$\|(u,v)\|_{\mu,a}=\left(\|u\|_{\mu,a}^2+\|v\|_{\mu,a}^2\right)^{1/2}.$$

Since our approach is variational, we define the functional $J:=J_{\lambda_1,\lambda_2}$ on $\mathcal{H}$ by

$$J\left(u,v\right):=\left(1/2\right)\|(u,v)\|_{\mu,a}^2-P\left(u,v\right)-Q\left(u,v\right),$$

where

$$P\left(u,v\right):=\int_\Omega|u|^{\alpha+1}|v|^{\beta+1}|x|^{-2_*b}\,dx \text{ and } Q\left(u,v\right):=\int_\Omega\left(\lambda_1f_1u+\lambda_2f_2v\right)dx.$$

A couple $(u,v)\in\mathcal{H}$ is a weak solution of the system $(\mathcal{S}_{\lambda_1,\lambda_2})$ if it satisfies

$$\left\langle J'\left(u,v\right),\left(\varphi,\psi\right)\right\rangle:=R\left(u,v\right)\left(\varphi,\psi\right)-S\left(u,v\right)\left(\varphi,\psi\right)-T\left(u,v\right)\left(\varphi,\psi\right)=0,\ \text{for all}\ \left(\varphi,\psi\right)\in\mathcal{H},$$

with

$$\begin{aligned}R\left(u,v\right)\left(\varphi,\psi\right)\ &:=\int_\Omega\left(|x|^{-2a}\left(\nabla u\nabla\varphi+\nabla v\nabla\psi\right)-\mu|x|^{-2(a+1)}\left(u\varphi+v\psi\right)\right)\\ S\left(u,v\right)\left(\varphi,\psi\right)\ &:=\int_\Omega|x|^{-2_*b}\left[\left(\alpha+1\right)|u|^\alpha|v|^{\beta+1}\varphi+\left(\beta+1\right)|u|^{\alpha+1}|v|^\beta\psi\right]\\ T\left(u,v\right)\left(\varphi,\psi\right)\ &:=\int_\Omega\left(\lambda_1f_1\varphi+\lambda_2f_2\psi\right).\end{aligned}$$

Here $\langle .,.\rangle$ denotes the product in the duality $\mathcal{H}'$, $\mathcal{H}$.

Let

$$S_\mu:=\inf_{u\in\mathcal{H}_\mu\backslash\{0\}}\frac{\|u\|_{\mu,a}^2}{\left(\int_\Omega|x|^{-2_*b}|u|^{2_*}\,dx\right)^{2/2_*}}$$

and

$$\tilde{S}_\mu := \inf_{(u,v)\in\mathcal{H}\setminus\{(0,0)\}} \frac{\|(u,v)\|^2_{\mu,a}}{\left(\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}|x|^{-2_*b}\,dx\right)^{2/2_*}}.$$

From [7], S_μ is achieved.

Lemma 4.1 *Let Ω be a domain (not necessarily bounded), $-\infty < \mu < \bar{\mu}_a$ and $\alpha + \beta \leq 2_* - 2$. Then we have*

$$\tilde{S}_\mu := \left[\left(\frac{\alpha+1}{\beta+1}\right)^{(\beta+1)/2_*} + \left(\frac{\beta+1}{\alpha+1}\right)^{(\alpha+1)/2_*}\right] S_\mu.$$

For simplicity of writing, let us note the quantity $\left[\left(\frac{\alpha+1}{\beta+1}\right)^{(\beta+1)/2_*} + \left(\frac{\beta+1}{\alpha+1}\right)^{(\alpha+1)/2_*}\right]$ by $K(\alpha,\beta)$.

Proof: The proof is essentially given in [1] with minor modifications. ■

In our work, we research the critical points as the minimizers of the energy functional associated to the problem $(\mathcal{S}_{\lambda_1,\lambda_2})$ on the constraint defined by the Nehari manifold, which are solutions of our system, under some sufficient conditions on the parameters α, β, μ, λ_1 and λ_2.

Let Λ_0 be positive number such that

$$\Lambda_0 := 2_*(2_*-2)\left[2_*(2_*-1)\right]^{-\frac{(2_*-1)}{(2_*-2)}}\left[K(\alpha,\beta)\right]^{\frac{2_*}{2(2_*-2)}}(S_\mu)^{\frac{2_*}{2(2_*-2)}}.$$

Then, we obtain the following results.

Theorem 4.1 *Let be $f_1,\ f_2 \in \mathcal{H}'_\mu$ (dual of $\mathcal{H}_\mu$). Assume that $-\infty < a < (N-2)/2$, $-\infty < \mu < \bar{\mu}_a$, $\alpha+\beta+2 = 2_*$ and λ_1, λ_2 real parameters satisfying $0 < |\lambda_1|\,\|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2|\,\|f_2\|_{\mathcal{H}'_\mu} < \Lambda_0$, then $(\mathcal{S}_{\lambda_1,\lambda_2})$ has at least one solution.*

Theorem 4.2 *In addition to the assumptions of the Theorem 4.1, λ_1, λ_2 verifying $0 < |\lambda_1|\,\|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2|\,\|f_2\|_{\mathcal{H}'_\mu} < (1/2)\,\Lambda_0$, then $(\mathcal{S}_{\lambda_1,\lambda_2})$ has at least two nontrivial solutions.*

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proofs of Theorems 4.1 and 4.2.

4.2 Preliminaries

We list here a few integral inequalities. The first one that we need is the Caffarelli-Kohn-Nirenberg inequality [4], which ensures the existence of a positive constant $C_{a,b}$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-2_*b} |v|^{2_*} dx\right)^{2/2_*} \leq C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla v|^2 dx, \text{ for all } v \in \mathcal{C}_0^\infty \left(\mathbb{R}^N\right). \tag{4.1}$$

In (4.1), as $b = a + 1$, then $2_* = 2$ and we have the following weighted Hardy inequality [5]:

$$\int_{\mathbb{R}^N} |x|^{-2(a+1)} v^2 dx \leq \frac{1}{\bar{\mu}_a} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla v|^2 dx, \text{ for all } v \in \mathcal{C}_0^\infty \left(\mathbb{R}^N\right).$$

Definition 4.1 *Let $c \in \mathbb{R}$, E a Banach space and $I \in C^1(E, \mathbb{R})$.*

(i) $(u_n, v_n)_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for I if

$$I(u_n, v_n) = c + o_n(1) \text{ and } I'(u_n, v_n) = o_n(1),$$

where $o_n(1)$ tends to 0 as n goes at infinity.

(ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for I has a convergent subsequence.

4.2.1 Nehari manifold

It is well known that J is of class C^1 in $\mathcal{H}$ and the solutions of $(\mathcal{S}_{\lambda_1,\lambda_2})$ are the critical points of J which is not bounded below on $\mathcal{H}$. Consider the following Nehari manifold

$$\mathcal{N} = \left\{(u, v) \in \mathcal{H}\backslash\{0, 0\} : \left\langle J'(u, v), (u, v)\right\rangle = 0\right\}.$$

Thus, $(u, v) \in \mathcal{N}$ if and only if

$$\|(u, v)\|_{\mu,a}^2 - 2_* P(u, v) - Q(u, v) = 0. \tag{4.2}$$

Note that $\mathcal{N}$ contains every nontrivial solution of the problem $(\mathcal{S}_{\lambda_1,\lambda_2})$. Moreover, we have the following results.

Lemma 4.2 *J is coercive and bounded from below on $\mathcal{N}$.*

Proof: If $(u,v)\in\mathcal{N}$, then by (4.2), the Hölder and Young inequalities, we deduce that

$$
\begin{aligned}
J(u,v) &= ((2_*-2)/2_*2)\,\|(u,v)\|_{\mu,a}^2-(1-(1/2_*))\,Q(u,v)\\
&\geq ((2_*-2)/2_*2)\,\|(u,v)\|_{\mu,a}^2\\
&\quad -(1-(1/2_*))\left(|\lambda_1|\,\|f_1\|_{\mathcal{H}'_\mu}+|\lambda_2|\,\|f_2\|_{\mathcal{H}'_\mu}\right)\|(u,v)\|_{\mu,a}\\
&\geq -C_0,
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
C_0 \quad &: \;= C_0\left(\lambda_1,\ \lambda_2,\ \|f_1\|_{\mathcal{H}'_\mu},\ \ \|f_2\|_{\mathcal{H}'_\mu}\right)\\
&= \left[2\,(2_*-1)^2/2_*\,(2_*-2)\right]\left(|\lambda_1|\,\|f_1\|_{\mathcal{H}'_\mu}+|\lambda_2|\,\|f_2\|_{\mathcal{H}'_\mu}\right)^2>0.
\end{aligned}
$$

Thus, J is coercive and bounded from below on $\mathcal{N}$. ■

Define

$$\phi(u,v)=\left\langle J'(u,v),(u,v)\right\rangle.$$

Then, for $(u,v)\in\mathcal{N}$

$$
\begin{aligned}
\left\langle \phi'(u,v),(u,v)\right\rangle &= 2\,\|(u,v)\|_{\mu,a}^2-(2_*)^2\,P(u,v)-Q(u,v) && (4.4)\\
&= \|(u,v)\|_{\mu,a}^2-2_*\,(2_*-1)\,P(u,v) && (4.5)\\
&= (2_*-1)\,Q(u,v)-(2_*-2)\,\|(u,v)\|_{\mu,a}^2.
\end{aligned}
$$

Now, we split $\mathcal{N}$ in three parts:

$\mathcal{N}^+=\left\{(u,v)\in\mathcal{N}:\ \left\langle \phi'(u,v),(u,v)\right\rangle>0\right\}$, $\mathcal{N}^0=\left\{(u,v)\in\mathcal{N}:\ \left\langle \phi'(u,v),(u,v)\right\rangle=0\right\}$,

and $\mathcal{N}^-=\left\{(u,v)\in\mathcal{N}:\ \left\langle \phi'(u,v),(u,v)\right\rangle<0\right\}$.

We have the following results.

Lemma 4.3 *Suppose that* (u_0, v_0) *is a local minimizer for* J *on* $\mathcal{N}$*. Then, if* $(u_0, v_0) \notin \mathcal{N}^0$*,* (u_0, v_0) *is a critical point of* J*.*

Proof: If (u_0, v_0) is a local minimizer for J on $\mathcal{N}$, then (u_0, v_0) is a solution of the optimization problem

$$\min_{\{(u,v)/\ \phi(u,v)=0\}} J(u, v).$$

Hence, there exists a Lagrange multipliers $\theta \in \mathbb{R}$ such that

$$J'(u_0, v_0) = \theta \phi'(u_0, v_0) \text{ in } \mathcal{H}' \text{ (dual of } \mathcal{H})$$

Thus,

$$\left\langle J'(u_0, v_0), (u_0, v_0) \right\rangle = \theta \left\langle \phi'(u_0, v_0), (u_0, v_0) \right\rangle,$$

But $\left\langle \phi'(u_0, v_0), (u_0, v_0) \right\rangle \neq 0$, since $(u_0, v_0) \notin \mathcal{N}^0$. Hence $\theta = 0$. This completes the proof. ■

Lemma 4.4 *There exists a positive number* Λ_0 *such that, for all* λ_1*,* λ_2 *verifying*

$$0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < \Lambda_0,$$

we have $\mathcal{N}^0 = \emptyset$*.*

Proof: Let us reason by contradiction.

Suppose $\mathcal{N}^0 \neq \emptyset$ such that $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < \Lambda_0$. Then, by (4.4) and for $(u, v) \in \mathcal{N}^0$, we have

$$\begin{aligned} \|(u, v)\|^2_{\mu,a} &= 2_* (2_* - 1) P(u, v) \qquad (4.6) \\ &= ((2_* - 1)/(2_* - 2)) Q(u, v). \end{aligned}$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\|(u,v)\|_{\mu,a} \geq [K(\alpha,\beta)]^{\frac{2_*}{2(2_*-2)}} (S_\mu)^{\frac{2_*}{2(2_*-2)}} [2_*(2_*-1)]^{\frac{-1}{(2_*-2)}} \tag{4.7}$$

and

$$\|(u,v)\|_{\mu,a} \leq \left[\left((2_*-1)\left(|\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu}\right)(2_*-2)^{-1}\right)\right]. \tag{4.8}$$

From (4.7) and (4.8), we obtain $|\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} \geq \Lambda_0$, which contradicts our hypothesis. ■

Thus $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$. Define

$$c := \inf_{u\in\mathcal{N}} J(u,v), c^+ := \inf_{u\in\mathcal{N}^+} J(u,v) \text{ and } c^- := \inf_{u\in\mathcal{N}^-} J(u,v).$$

For the sequel, we need the following Lemma.

Lemma 4.5 *(i) For all* λ_1, λ_2 *such that* $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < \Lambda_0$*, one has* $c \leq c^+ < 0$.

(ii) For all λ_1, λ_2 *such that* $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < (1/2)\Lambda_0$*, one has*

$$c^- > C_1 = C_1\left(\lambda_1,\ \lambda_2,\ S_\mu,\ \|f_1\|_{\mathcal{H}'_\mu},\ \|f_2\|_{\mathcal{H}'_\mu}\right),$$

where

$$\begin{aligned} C_1 \ : \ &= ((2_*-2)/2_*2)[2_*(2_*-1)]^{-2/(2_*-2)} [K(\alpha,\beta)]^{2_*/(2_*-2)} (S_\mu)^{2_*/(2_*-2)} + \\ &\quad -((2_*-2)/2_*)\left(|\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu}\right). \end{aligned}$$

Proof: (i) Let $(u,v) \in \mathcal{N}^+$. By (4.4), we have

$$[1/2_*(2_*-1)] \|(u,v)\|^2_{\mu,a} > P(u,v)$$

and so

$$\begin{aligned} J(u,v) &= (-1/2)\|(u,v)\|^2_{\mu,a} + (2_*-1)P(u,v) \\ &< -((2_*-1)/2_*2)\|(u,v)\|^2_{\mu,a}. \end{aligned}$$

We conclude that $c \leq c^+ < 0$.

(ii) Let $(u,v) \in \mathcal{N}^-$. By (4.4), we get

$$[1/2_* (2_* - 1)] \|(u,v)\|_{\mu,a}^2 < P(u,v).$$

Moreover, by Sobolev embedding theorem, we have

$$P(u,v) \leq [K(\alpha,\beta)]^{-2_*/2} (S_\mu)^{-2_*/2} \|(u,v)\|_{\mu,a}^{2_*}.$$

This implies

$$\|(u,v)\|_{\mu,a} > [2_* (2_* - 1)]^{\frac{-1}{(2_*-2)}} [K(\alpha,\beta)]^{\frac{2_*}{2(2_*-2)}} (S_\mu)^{\frac{2_*}{2(2_*-2)}}, \text{ for all } u \in \mathcal{N}^-. \tag{4.9}$$

By (4.3), we get

$$J(u,v) \geq ((2_* - 2)/2_* 2) \|(u,v)\|_{\mu,a}^2 - (1 - (1/2_*)) \left(|\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} \right) \|(u,v)\|_{\mu,a}.$$

Thus, for all λ_1, λ_2 such that $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < (1/2)\Lambda_0$, we have $J(u,v) \geq C_1$. ■

For each $(u,v) \in \mathcal{H}$, we write

$$t_m := t_{\max}(u,v) = \left[\frac{\|(u,v)\|_{\mu,a}}{2_* (2_* - 1) \int_\Omega |u|^{\alpha+1} |v|^{\beta+1} |x|^{-2_* b} dx} \right]^{1/(2_*-2)} > 0.$$

Lemma 4.6 *Let λ_1, λ_2 such that $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < \Lambda_0$. For each $(u,v) \in \mathcal{H}$, one has the following:*

(i) If $Q(u,v) \leq 0$, then there exists a unique $t^- > t_m$ such that $(t^- u, t^- v) \in \mathcal{N}^-$ and

$$J\left(t^- u, t^- v\right) = \sup_{t \geq 0} (tu, tv).$$

(ii) If $Q(u,v) > 0$, then there exist unique t^+ and t^- such that $0 < t^+ < t_m < t^-$,

$(t^+u, t^+v) \in \mathcal{N}^+$, $(t^-u, t^-v) \in \mathcal{N}^-$,

$$J\left(t^+u, t^+v\right) = \inf_{0 \leq t \leq t_m} J(tu, tv) \ \ \textit{and} \ J\left(t^-u, t^-v\right) = \sup_{t \geq 0} J(tu, tv).$$

Proof: With minor modifications, we refer to [3]. ■

Taking the idea of the work of Brown-Zhang [3], we prove the following result

Proposition 4.1 (i) *For all* λ_1, λ_2 *such that* $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < \Lambda_0$, *there exists a* $(PS)_{c^+}$ *sequence in* $\mathcal{N}^+$.

(ii) *For all* λ_1, λ_2 *such that* $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < (1/2)\Lambda_0$, *there exists a* $(PS)_{c^-}$ *sequence in* $\mathcal{N}^-$.

4.3 Proof of Theorem 4.1

Drawing on the works of [3] and [9], we establish the existence of a local minimum for J on $\mathcal{N}^+$.

Proposition 4.2 *For all* λ_1, λ_2 *such that* $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < \Lambda_0$, *the functional* J *has a minimizer* $\left(u_0^+, v_0^+\right) \in \mathcal{N}^+$ *and it satisfies*

(i) $J\left(u_0^+, v_0^+\right) = c = c^+$,

(ii) $\left(u_0^+, v_0^+\right)$ *is a nontrivial solution of* $(\mathcal{S}_{\lambda_1,\lambda_2})$.

Proof: If $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < \Lambda_0$, then by Proposition 4.1 (i) there exists a $(u_n, v_n)_n$ $(PS)_{c^+}$ sequence in $\mathcal{N}^+$, thus it bounded by Lemma 4.2. Then, there exists $\left(u_0^+, v_0^+\right) \in \mathcal{H}$ and we can extract a subsequence which will denoted by $(u_n, v_n)_n$ such that

$$\begin{aligned} (u_n, v_n) &\rightharpoonup \left(u_0^+, v_0^+\right) \text{ weakly in } \mathcal{H} \\ (u_n, v_n) &\rightharpoonup \left(u_0^+, v_0^+\right) \text{ weakly in } \left(L^{2_*}\left(\Omega, |x|^{-2_* b}\right)\right)^2 \\ u_n &\to u_0^+ \text{ a.e in } \Omega, \\ v_n &\to v_0^+ \text{ a.e in } \Omega. \end{aligned} \tag{4.10}$$

Thus, by (4.10), $\left(u_0^+, v_0^+\right)$ is a weak nontrivial solution of $(\mathcal{S}_{\lambda_1,\lambda_2})$. Now, we show that (u_n, v_n) converges to $\left(u_0^+, v_0^+\right)$ strongly in $\mathcal{H}$. Suppose otherwise. By the lower semi-continuity of the norm, then either $\left\|u_0^+\right\|_{\mu,a} < \liminf_{n\longrightarrow\infty} \|u_n\|_{\mu,a}$ or $\left\|v_0^+\right\|_{\mu,a} < \liminf_{n\longrightarrow\infty} \|v_n\|_{\mu,a}$ and we obtain

$$\begin{aligned} c &\leq J\left(u_0^+, v_0^+\right) = \left(\left(2_* - 2\right)/2_*2\right)\left\|\left(u_0^+, v_0^+\right)\right\|_{\mu,a}^2 - \left(1 - \left(1/2_*\right)\right) Q\left(u_0^+, v_0^+\right) \\ &< \liminf_{n\longrightarrow\infty} J\left(u_n, v_n\right) = c. \end{aligned}$$

We get a contradiction. Therefore, (u_n, v_n) converge to $\left(u_0^+, v_0^+\right)$ strongly in $\mathcal{H}$. Moreover, we have $\left(u_0^+, v_0^+\right) \in \mathcal{N}^+$. If not, then by Lemma 4.6, there are two numbers t_0^+ and t_0^-, uniquely defined so that $\left(t_0^+ u_0^+, t_0^+ v_0^+\right) \in \mathcal{N}^+$ and $\left(t^- u_0^+, t^- v_0^+\right) \in \mathcal{N}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} J\left(tu_0^+, tv_0^+\right)_{\downarrow t=t_0^+} = 0 \text{ and } \frac{d^2}{dt^2} J\left(tu_0^+, tv_0^+\right)_{\downarrow t=t_0^+} > 0,$$

there exists $t_0^+ < t^- \leq t_0^-$ such that $J\left(t_0^+ u_0^+, t_0^+ v_0^+\right) < J\left(t^- u_0^+, t^- v_0^+\right)$. By Lemma 4.6, we get

$$J\left(t_0^+ u_0^+, t_0^+ v_0^+\right) < J\left(t^- u_0^+, t^- v_0^+\right) < J\left(t_0^- u_0^+, t_0^- v_0^+\right) = J\left(u_0^+, v_0^+\right),$$

which is a contradiction. ■

4.4 Proof of Theorem 4.2

Next, we establish the existence of a local minimum for J on $\mathcal{N}^-$. For this, we require the following Lemma.

Lemma 4.7 *For all λ_1, λ_2 such that $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < (1/2)\Lambda_0$, the functional J has a minimizer $\left(u_0^-, v_0^-\right)$ in $\mathcal{N}^-$ and it satisfies*

(i) $J\left(u_0^-, v_0^-\right) = c^- > 0$,

(ii) $\left(u_0^-, v_0^-\right)$ *is a nontrivial solution of $(\mathcal{S}_{\lambda_1,\lambda_2})$ in $\mathcal{H}$.*

Proof: If $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < (1/2)\Lambda_0$, then by Proposition 4.1 *(ii)* there exists a $(u_n, v_n)_n$, $(PS)_{c^-}$ sequence in $\mathcal{N}^-$, thus it bounded by Lemma 4.2. Then, there exists $\left(u_0^-, v_0^-\right) \in$

$\mathcal{H}$ and we can extract a subsequence which will denoted by $(u_n, v_n)_n$ such that

$$\begin{aligned}
(u_n, v_n) &\rightharpoonup \left(u_0^-, v_0^-\right) \text{ weakly in } \mathcal{H} \\
(u_n, v_n) &\rightharpoonup \left(u_0^-, v_0^-\right) \text{ weakly in } \left(L^{2_*}\left(\Omega, |x|^{-2_* b}\right)\right)^2 \\
u_n &\to u_0^- \text{ a.e in } \Omega \\
v_n &\to v_0^- \text{ a.e in } \Omega.
\end{aligned}$$

This implies

$$P\left(u_n, v_n\right) \to P\left(u_0^-, v_0^-\right), \text{ as } n \to \infty.$$

Moreover, by (4.4) we obtain

$$P\left(u_n, v_n\right) > \left[2_*\left(2_* - 1\right)\right]^{-1} \left\|u_n, v_n\right\|_{\mu,a}^2, \tag{4.11}$$

thus, by (4.7) and (4.11) there exists a positive number

$$C_2 := \left[12_*\left(2_* - 1\right)\right]^{-2_*/(2_*-2)} \left[K\left(\alpha, \beta\right)\right]^{\frac{2_*}{(2_*-2)}} \left(S_\mu\right)^{\frac{2_*}{(2_*-2)}},$$

such that

$$P\left(u_n, v_n\right) > C_2. \tag{4.12}$$

This implies that

$$P\left(u_0^-, v_0^-\right) \geq C_2.$$

Now, we prove that $(u_n, v_n)_n$ converges to $\left(u_0^-, v_0^-\right)$ strongly in $\mathcal{H}$. Suppose otherwise. Then, either $\left\|u_0^-\right\|_{\mu,a} < \liminf_{n\longrightarrow\infty} \|u_n\|_{\mu,a}$ or $\left\|v_0^-\right\|_{\mu,a} < \liminf_{n\longrightarrow\infty} \|v_n\|_{\mu,a}$. By Lemma 4.6 there is a unique t_0^- such that $\left(t_0^- u_0^-, t_0^- v_0^-\right) \in \mathcal{N}^-$. Since

$$(u_n, v_n) \in \mathcal{N}^-,\ J\left(u_n, v_n\right) \geq J\left(t u_n, t v_n\right), \text{ for all } t \geq 0,$$

we have

$$J\left(t_0^- u_0^-, t_0^- v_0^-\right) < \lim_{n\longrightarrow\infty} J\left(t_0^- u_n, t_0^- v_n\right) \leq \lim_{n\longrightarrow\infty} J\left(u_n, v_n\right) = c^-,$$

and this is a contradiction. Hence,

$$(u_n, v_n)_n \to \left(u_0^-, v_0^-\right) \text{ strongly in } \mathcal{H}.$$

Thus,

$$J\left(u_n, v_n\right) \text{ converges to } J\left(u_0^-, v_0^-\right) = c^- \text{ as } n \text{ tends to } +\infty.$$

By (4.12) and Lemma 4.3, we may assume that $\left(u_0^-, v_0^-\right)$ is a solution of $(\mathcal{S}_{\lambda_1,\lambda_2})$. ■

Now, we complete the proof of Theorem 4.2. By Propositions 4.2 and Lemma 4.7, we obtain that $(\mathcal{S}_{\lambda_1,\lambda_2})$ has two nontrivial solutions $\left(u_0^+, v_0^+\right) \in \mathcal{N}^+$ and $\left(u_0^-, v_0^-\right) \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, this implies that $\left(u_0^+, v_0^+\right)$ and $\left(u_0^-, v_0^-\right)$ are distinct.

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Perspectives

A) Badiale et al. [3], showed that the problem

$$(P_{\alpha,\gamma})\begin{cases} -\Delta u-\mu\,|y|^{-\alpha}\,u= & |u|^{\gamma-2}\,u \text{ in } \mathbb{R}^N, \\ & y\neq 0 \\ u\geq 0, \end{cases}$$

does not admit solutions in the region $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3$ where

$$\begin{aligned} \mathcal{A}_1 \ &: \ =\left\{(\alpha,\gamma)\in\mathbb{R}^2:\alpha\in(0,2)\,,\gamma\notin(2_\alpha,2^*)\,,\gamma\geq 2\right\}\setminus\{(2,2^*)\}\,, \\ \mathcal{A}_2 \ &: \ =\left\{(\alpha,\gamma)\in\mathbb{R}^2:\alpha\in(2,N)\,,\gamma\notin(2^*,2_\alpha)\,,\gamma\geq 2\right\}, \\ \mathcal{A}_3 \ &: \ =\left\{(\alpha,\gamma)\in\mathbb{R}^2:\alpha\in[N,+\infty),\gamma\in[2,2^*]\right\}, \end{aligned}$$

with $2_\alpha := 2N/\left(N-\alpha\right)$

Thus, if one considers the perturbed problem

$$(P_{\alpha,\gamma,\lambda})\begin{cases} -\Delta u-\mu\,|y|^{-\alpha}\,u= & |u|^{\gamma-2}\,u+\lambda g\left(x\right) \ \text{ in } \mathbb{R}^N, \\ & y\neq 0 \\ u\geq 0, \end{cases}$$

to find the necessary assumptions that it is to pose on the parameters α, γ, λ and the function g so that one has solutions in the region $\mathcal{A}$?.

$B)$ We consider the following problem

$$(\mathcal{P}_{\alpha,\beta,\lambda}) \begin{cases} -\operatorname{div}\left(|x|^{\alpha}|\nabla u|^{p-2}\nabla u\right) = & |x|^{\beta} u^{p(\alpha,\beta)-1} + \lambda u^{q-1} \text{ in } \Omega \\ 0 < u \in H_0^{1,p}(\Omega), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $1 < p < N$, $0 \leq q < 2 \leq p < p(\alpha,\beta) = p(N+\beta)/(N-p+\alpha)$ with $p(\alpha,\beta) < p^* - 2 = [pN/(N-p)] - 2$.

The problem $(\mathcal{P}_{\alpha,\beta,0})$ has been sdudied by Thomas et al. [24]. They have obtained existence and nonexistence results.

The main perspective here is that the possibility of to prove the nonexistence result and the existence at least four positive solutions for $(\mathcal{P}_{\alpha,\beta,\lambda})$ by exploiting a Pohozaev-type identity, the Nehari manifold and the mountain pass theorem as in [26].

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