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Existence and multiplicity of solutions for elliptics systems

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Dedication

I dedicate this thesis to :

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My dear sisters, for their constant encouragement and moral support.

To all my family and to all the people who have known how to be present when I needed them.

Contents

Notation

Symbols	**Definition**
$x = (x_1, x_2, x_3, .., x_N)$	Element of $\mathbb{R}^N$
$\|x\| = \sqrt{(x_1^2 + x_2^2 + x_3^2 + .. + x_N^2)}$	Norm of x
$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\right)$	The gradient of u
$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$	The Laplacian of u
q	Conjugate exponent of p, $\frac{1}{p} + \frac{1}{q} = 1$
$m^{'}$	Conjugate exponent of m, $\frac{1}{m} + \frac{1}{m^{'}} = 1$
$mes(A) = \|A\|$	Lebesgue measure of $A \subset \mathbb{R}^N$
$\|\|u\|\|_s$	Norm of u in $L^s(\Omega)$
$\|\|u\|\|_X$	Norm of u in the space X
$X^{'}$	The dual space of X
$s^* = \frac{Ns}{N-s}$	Critical Sobolev exponent
Ω	Non-empty bounded open set of $\mathbb{R}^N$
$\partial\Omega$	Boundary of Ω
f^+	$f^+ = \max(f, 0)$

Symbols	**Definition**
$\langle\cdot,\cdot\rangle$ *or* $(\cdot,\cdot)$	The scalar product in $\mathbb{R}^N$
$C(\Omega)$ *or* $C^0(\Omega)$	Space of continuous function on Ω
$C_0(\Omega)$	Space of continuous function on Ω with compact support
$C^k(\Omega)$	Set of function on Ω, for which the k-th partial derivatives are continuous
$C_0^k(\Omega)$	Space of $C^k(\Omega)$ with compact support
$C^\infty(\Omega)$	Space of infinitely differentiable functions on Ω
$C_0^\infty(\Omega)$	Space of $C^\infty(\Omega)$ with compact support
$L^p(\Omega)$	$\left\{u:\Omega\to\mathbb{R}^N \mid u \text{ } measurable \text{ }, \int_\Omega \lvert u\rvert^p<\infty\right\};\ 1\leq p<\infty$
$L^\infty(\Omega)$	$\{u:\Omega\to\mathbb{R}^N \mid u \text{ } measurable \text{ }, \exists C$, such that $\lvert u\rvert\leq C, \forall x\in\Omega\}\ ;\ 1\leq p<\infty$
$L^q(\Omega)$	Dual space of $L^p(\Omega)$
$W^{k,p}(\Omega)$	Sobolev space , with derivatives up to order k in $L^p(\Omega)$
$W_0^{k,p}(\Omega)$	Sobolev space , with zero trace
$W^{-k,q}(\Omega)$	Dual space of $W_0^{k,p}(\Omega)$

Introduction

Elliptic differential equations in bounded domains under different boundary conditions are used to describe many engineering or physical phenomena and play a role in modeling in applied sciences. A large number of these models are often written in the form of a Dirichlet boundary value problem of the type

$$\begin{cases} L[u] = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \ , \end{cases}$$

where Ω is non empty bounded open set in $\mathbb{R}^N$, f is a real value function, and L is a quasilinear elliptic operator.
In the literature, some classical tools have been used to study these problems: the fixed point theorems [15], the method of upper and lower solutions [35], the theory of mixed monotone operators [23], the a priori estimation method with Leray-Schauder fixed point theorem [36].
Another very powerful tool in the study of partial differential equations is the variational method. This method consists of seeking the solutions to an elliptic PDE as a critical points of a functional J defined on appropriately chosen functional space. The solutions obtained in this manner are called weak solutions or sometimes solutions in the sense of duality.
The study of elliptic equations, by variational methods, have received wide attention in recent years. This is due to the fact that they have many applications to a various range of phenomena including elastic mechanics [38], electro-rheological and thermo-rheological viscous flows of non-Newtonian fluids [6], image restoration [19] and mathematical biology [21].

The aim of the present work is to apply a variational approach for elliptic problems by using Three critical points Theorem of Ricceri.

Presentation

This work consists of three chapters organized as follows :
Chapter 1 :

Within this chapter, we provide a brief review of functional analysis principles pertinent to Sobolev spaces, follow by an exposition of some elements of convergence criteria with some properties of the p-Laplacian operator.

Chapter 2 :

In this chapter, we start by recalling the basic tools to introduce variational methods then we give an overview of these methods and the different existence and multiplicity results for problems having a variational structure, in particular The Three critical point theorem of Ricceri and its variants.

Chapter 3 :

In this chapter, we focus on studying the existence and multiplicity of solutions to two elliptic boundary value problems using variational method based on " Three Critical Points Theorem" of Ricceri.
We list here the first Dirichlet problem studies involving the p-Laplacian defined such that:

$$\begin{cases} -\Delta(u) = \lambda_0(f(u) + \mu g(u)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where :

- Ω is non empty bounded open set in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $\delta > 0$, $\mu \in [-\delta, \delta]$, λ_0 is a positive real parameter.
- $\Delta(u) = div(\nabla u)$ is the Laplacian operator.
- $f, g : R \to R$ are a continuous, differentiable and carathéodory functions, verifying other hypotheses presented in the chapter.

The next problem is the following Dirichlet boundary system involving (p, q)-Laplacian :

$$\begin{cases} \Delta_p(u) + \lambda f(x, u, v) = a(x)|u|^{p-2}\, u & \text{in } \Omega \\ \Delta_q(v) + \lambda g(x, u, v) = b(x)|v|^{q-2}\, v & \text{in } \Omega \\ u = \; v = 0 & \text{on } \partial\Omega \end{cases}$$

where :

- Ω is non empty bounded open set in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, p,q $>$ N , λ is a positive real parameter
- $\Delta_p(u) = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator.
- $f, g : \Omega \times R^2 \to R$ are a continuous and differentiable functions, satisfying some supplementary hypotheses presented in the chapter.
- a and b are two positives weight functions such that a, b in $C(\overline{\Omega})$.

Finally, we achieve our work by a general conclusion.

Chapter 1

Background

In this chapter, we review some foundational concepts that we use in our study. This concepts are typically covered in some works such as H.Brezis [16], or R.A.Adamas [1], O.Kavian [22].

1.1 Operators on Banach spaces

1.1.1 Definitions and properties

Definition 1.1.1 (**Reflexive space**). *Let E be a Banach space, and E^* is the dual space with the norm*

$$||f||_{E^*} = \sup_{\substack{x\in E \\ ||x||_E\leq 1}} |\langle f, x\rangle| .$$

The bidual E^{**} is the dual of E^* with the norm :

$$||g||_{E^{**}} = \sup_{\substack{f\in E^* \\ ||f||_{E^*}\leq 1}} |\langle g, f\rangle| .$$

We define a canonical injection $J : E \to E^{**}$ as follows : given $x \in E$ fixed, $f \mapsto \langle f, x\rangle$ from E^* to $\mathbb{R}$ constitutes a continuous linear form on E^*, i.e an element of E^{**} denoted by Jx. Thus

$$\langle Jx, f\rangle_{E^{**},E^*} = \langle f, x\rangle_{E^*,E} \qquad \forall x \in E, \forall f \in E^*.$$

J is an isometry, that's means

$$||Jx||_{E^{**}} = ||x||_E \quad \forall x \in E,$$

and we have that J is linear, in fact

$$||Jx||_{E^{**}} = \sup_{\substack{f\in E^* \\ ||f||_{E^*}\leq 1}} |\langle Jx, f\rangle| = \sup_{\substack{f\in E^* \\ ||f||^*_E\leq 1}} |\langle f, x\rangle| = ||x||_E.$$

When J is surjective, we say that E is a reflexive space.

Definition 1.1.2 (**Separable space**). *The Banach space E is separable if there exists a subset D of E that is countable and dense within E.*

Definition 1.1.3 (**Compact Operator**). *Let E and F be two Banach spaces, and let $A : E \to F$ be a continuous operator (not necessarily linear).*

We say that A is a compact operator if the image of every bounded set in E under A is relatively compact in F.
In other words, if $(u_n)_n \subset E$ is a bounded sequence, then the sequence $(v_n = A(u_n))_n \subset F$ has a convergent subsequence in F.

Proposition We recall the following definitions.
Let E be a reflexive Banach space, E^* is its dual space.

- Operator $T : E \to E^*$ is called monotone if

$$\langle Tu - Tv, u - v\rangle \ \geq 0 \qquad\qquad \forall u, v \in E. \tag{1.1}$$

- Operator T is called Strictly monotone if

$$\langle Tu - Tv, u - v\rangle \ > 0 \qquad\qquad u \neq v. \tag{1.2}$$

- Operator T is hemicontinuous

$$\lim_{t\to 0} < T(u + tv), w > = < Tu, w > \qquad\qquad \forall u, v, w \in E. \tag{1.3}$$

- Operator T is Coercive

$$\lim_{||u||\to\infty} \frac{\langle Tu, u\rangle}{||u||} = +\infty. \tag{1.4}$$

Definition 1.1.4. ***(Linear form)***
Let $f : E \to \mathbb{R}$. We say that f is a linear form on E if and only if:

$$\forall u, v \in E, \forall \alpha, \beta \in \mathbb{R} : f(\alpha u + \beta v) = \alpha f(u) + \beta f(v).$$

Definition 1.1.5. ***(Bilinear Form)***
Let $a : E \times E \to \mathbb{R}$. We say that a is a bilinear form on E if for every fixed $u \in E$, the following mappings are linear:

$$a(u, \cdot) : v \in E \to a(u, v) \in \mathbb{R},$$

$$a(\cdot, u) : v \in E \to a(v, u) \in \mathbb{R},$$

Recall that if a is a continuous bilinear form on E, then there exists $c > 0$ such that

$$|a(u, v)| \leq c\|u\|_E\|v\|_E \quad \forall u, v \in E.$$

Definition 1.1.6. ***(Coercive Bilinear Form)***
Let V be a Hilbert space and a a bilinear form on V. a is coercive on V if there exists $\alpha > 0$ such that

$$a(u, u) \geq \alpha\|u\|_V^2 \quad \forall u \in V.$$

Theorem 1.1 ([37],Theroem 26 A). *Let $A : X \to X^*$ be a monotone , coercive and hemicontinuous operator on the real ,separable , reflexive Banach space X.Then*

- *If A is strictly monotone, then the inverse operator A^{-1} exist. This operator is strictly monotone , semicontinuous and bounded.*
- *if A is strictly monotone, then A^{-1} is continuous.*
- *A is strongly monotone, then A^{-1} is Lipschitz continuous.*

1.1.2 Some convergence results

Definition 1.1.7 (**Weak convergence**). *Let E be a Banach space, and E^* is the dual space and $< .,. >$ the duality Bracket on $E \times E^*$.*
We say that the sequence $(x_n)_n$ in E weakly convergence to $x \in E$ if :

$$< f, x_n > \to < f, x > \quad \forall f \in E^*,$$

and we write :

$$x_n \underset{n \to +\infty}{\rightharpoonup} x \text{ weakly in } E.$$

Theorem 1.2. *Let E be a Banach space, E^* is the dual space, let $(x_n)_n$ be a sequence in E.*

- *if $x_n \underset{n \to +\infty}{\rightharpoonup} x$ (weakly in E), we have :*

$$\begin{cases} \exists k > 0, \forall n \in \mathbb{N} : ||x_n||_E \leq k \\ ||x||_E \leq \lim_{n \to +\infty} ||x_n||_E. \end{cases}$$

- *if $x_n \underset{n \to +\infty}{\to} x$ (Strongly in E), so we have : $x_n \underset{n \to +\infty}{\rightharpoonup} x$ weakly in E.*

Theorem 1.3. *Let E be a reflexive Banach space , and let $(x_n)_n$ be a bounded sequences in E so there exists a subsequence $(x_{\sigma(n)})$ of $(x_n)_n$, and $x \in E$, such that*

$$x_{\sigma(n)} \underset{n \to +\infty}{\rightharpoonup} x \text{ weakly in } E.$$

If every subsequence converges weakly to the same limit x, then :

$$x_n \underset{n \to +\infty}{\rightharpoonup} x \text{ weakly in } E.$$

1.2 Sobolev Spaces

Sobolev spaces play a fundamental role in variational calculus. They owe their name to the Russian mathematician Sergei Lvovitch Sobolev (1908-1989). It is therefore wise to give a brief presentation before discussing variational methods. We start by giving some definitions and notations necessary for the introduction of these spaces. For a more complete presentation of Sobolev spaces refer to [2].

1.2.1 The space of continuous functions

We will denote by $C(\Omega)$ the space of continuous functions defined in Ω with values in $\mathbb{R}$, we equip it with the norm :

$$||u||_\infty = sup_{x\in\Omega}|u(x)|.$$

$(C(\Omega))^m$ is whole continuous functions defined $\Omega \to \mathbb{R}^m$.
$C_b(\overline{\Omega})$ the set of continuous functions and bounded on $\overline{\Omega}$.
For $k > 1$ integer, $C^k(\Omega)$ is the space of functions u which are k times differentiable and whose derivative of order k is continuous on Ω.
$C_0^k(\Omega)$ is the set of functions in $C^k(\Omega)$, whose support is compact and contained in Ω.
We also define $C^k(\overline{\Omega})$ like all the restrictions on $\overline{\Omega}$ elements of $C^k(\mathbb{R}^{\mathbb{N}})$ or as being the set of functions of $C^k(\Omega)$, such that for all $0 \leq j \leq k$, and $\forall x_0 \in \partial\Omega$, the limit $lim_{x\to x_0} D^j u(x)$ exists and depends only on x_0.
C_0^∞ or $D(\Omega)$is the space of indefinitely differentiable functions, with compact supports which we call the space of test functions.

1.2.2 L^p Spaces

Let p be a real with $1 \leq p < \infty$, and $\Omega \subset \mathbb{R}^{\mathbb{N}}$ a set measurable in the Lebesgue sense. We design by :

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \quad \text{is measurable and} \quad \int_\Omega |f|^p < \infty \right\},$$

and we define the norm of f in L^p with the p-norm :

$$||f||_{L^p} = (\int_\Omega |f(x)|^p \, dx)^{1/p} \ < \ +\infty.$$

If $p = \infty$,

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \quad \text{measurable,exist} \quad C > 0 : \int_\Omega |f(x)| \leq C \quad \text{every where on} \quad \Omega \right\},$$

with the norm :

$$||f||_\infty = inf \left\{ M \geq 0 : |f(x)| \leq M \quad \text{every where on } \Omega \right\}.$$

$L^\infty(\Omega)$ is a Banach space.
For $p = 2$, the space $L^2(\Omega)$ is a Hilbert space for the scalar product :

$$< f, g > = \int_\Omega f(x)g(x)dx.$$

We denote by $L^1_{Loc}(\Omega)$ the set of locally integrable functions on Ω and we write :

$$L^1_{Loc}(\Omega) = \left\{ u : u \in L^1(K) \quad \text{for every compact} \quad K \quad \text{on } \Omega \right\}.$$

Remarks:

1. $L^p(\Omega) \subset L^1_Loc(\Omega)$ for all $1 \leq p \leq \infty$
2. The space $(L^p(\Omega), ||.||_p)$ is Banach for $1 \leq p \leq \infty$, separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.
3. Let $1 \leq p < \infty$, The dual space of $L^p(\Omega)$ is $L^q(\Omega)$, such that $\frac{1}{q} + \frac{1}{p} = 1$.

Definition 1.2.1. *let $1 \leq p \leq +\infty$, we define the conjugate q of p by :*

$$\frac{1}{q} + \frac{1}{p} = 1 \qquad if\ 1 < p < \infty,$$

- *if $p = 1$ we have $q = \infty$,*
- *if $p = \infty$ we have $q = 1$.*

Theorem 1.4 (**Green′s Formula**). *Consider Ω a bounded open set in R^N of boundary Γ, Let u and v be functions mapping from Ω to R , such that $\forall u \in C^2(\bar{\Omega})$ and $\forall v \in C^1(\bar{\Omega})$ we have :*

$$\int_\Omega (\Delta u) v \, dx = \int_\Gamma v(\nabla u) \cdot \nu \, d\Gamma - \int_\Omega \nabla u \cdot \nabla v \, dx, \tag{1.5}$$

where ν represents the unit outward normal vector for Γ, and $d\Gamma$ represents the surface measure on Γ.

Theorem 1.5 (**Holder′s inequality**). *If Ω is an open subset of $\mathbb{R}^N$, f, $g : \Omega \to \mathbb{R}$ such that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p < +\infty$, then we have :*

$$\int_\Omega |f(x)g(x)| \, dx \leq ||f||_{L^p(\Omega)} ||g||_{L^q(\Omega)}.$$

Theorem 1.6 (**Young′s inequality**). *If a and b are non-negative real numbers , p and q are real numbers greater than 1 , with $\frac{1}{q} + \frac{1}{p} = 1$, then we have :*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem 1.7 (**Minkowski inequality**). *Let $1 \leq p < \infty$, and let f and g be element of $L^p(\Omega)$, then $f + g \in L^p(\Omega)$, and we have :*

$$\left(\int |f+g|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \leq \left(\int |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} + \left(\int |g|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

Some convergence criteria

Theorem 1.8 (**Dominated Convergence**). *[16]*
Let $(f_n)_{n\in N}$ be a sequence of measurable functions on a measured space (E, A, μ), with values in the set of real or complex numbers when :

- *the sequence of functions $(f_n)_{n\in N}$ converges pointwise to a function f,*
- *it exists an integrable function g such that:*

$$\forall n \in N \;\; \forall x \in E \;\; |f_n(x)| \leq g(x).$$

Then f is integrable and

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0.$$

proposition 1.1. *[16]*
Let (f_n) be a sequence in $L^p(\Omega)$ and $f \in L^p(\Omega)$, such that $\|f_n - f\|_{L^p(\Omega)} \to 0$; then there exists a function $g \in L^p(\Omega)$ and a subsequence (f_{n_i}) such that:

- $f_{n_i}(x) \to f(x)$ *almost everywhere on* Ω*;*
- $|f_{n_i}(x)| \leq g(x)$ *for all* i *and* p *and everywhere on* Ω*.*

Weak convergence in $L^p(\Omega)$ space

Theorem 1.9. *Let (f_n) be a bounded sequence in $L^p(\Omega)$. Then, it admits a subsequence that converges weakly. The concept of weak convergence in $L^p(\Omega;\mathbb{R}^N)$ is defined as follows:*

- *If $1 \leq p < +\infty$, then $f_n \underset{n \to +\infty}{\rightharpoonup} f$ weakly in $L^p(\Omega;\mathbb{R}^N)$ if:*

$$\int_\Omega \langle f_n(x), g(x)\rangle \, dx \underset{n \to +\infty}{\to} \int_\Omega \langle f(x), g(x)\rangle \, dx \quad \forall g \in L^q(\Omega;\mathbb{R}^N)$$

- *If $p = +\infty$, then $f_n \underset{n \to +\infty}{\rightharpoonup} f$ weakly in $L^\infty(\Omega;\mathbb{R}^N)$ if:*

$$\int_\Omega \langle f_n(x), g(x)\rangle \, dx \underset{n \to +\infty}{\to} \int_\Omega \langle f(x), g(x)\rangle \, dx \quad \forall g \in L^1(\Omega;\mathbb{R}^N).$$

Theorem 1.10. *The space $L^p(\Omega;\mathbb{R}^N)$ is reflexive for $1 < p < +\infty$. Furthermore, $L^2(\Omega;\mathbb{R}^N)$ is a Hilbert space with the inner product defined by:*

$$\langle f, g\rangle_{L^2(\Omega;\mathbb{R}^N)} = \int_\Omega \langle f(x), g(x)\rangle \, dx.$$

1.2.3 Sobolev Spaces

Let p be a real with $1 \leq p \leq \infty$ and Ω open set of $\mathbb{R}^{\mathbb{N}}$.

Definition 1.2.2. *The Sobolev space $W^{1,p}(\Omega)$ is defined as:*

$$W^{1,p}(\Omega) = \left\{u \in L^p(\Omega) : \nabla u \in L^p(\Omega, \mathbb{R}^N)\right\},$$

where

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\right)$$

represents the first derivative in the sense of distributions of the real-valued function u.
In this space, we define the following norm :

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)},$$

or sometimes an equivalent norm:

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^N)}\right)^{1/p} \quad if \ (\quad 1 \leq p < +\infty).$$

Definition 1.2.3. *Let $1 \leq p < +\infty$.*
The space $W_0^{1,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. The dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,q}(\Omega)$ with $\frac{1}{p}+\frac{1}{q}=1$.

Remark 1.1. *If $p=2$, the space $W^{1,2}(\Omega)$ is denoted by $H^{1,2}(\Omega)$ or simply $H^1(\Omega)$. Similarly, $W_0^{1,2}(\Omega)$ is denoted by $H_0^{1,2}(\Omega)$ or simply $H_0^1(\Omega)$.*

proposition 1.2.

1. *The space $W^{1,p}(\Omega)$ is a Banach space for $1 \leq p \leq +\infty$.*
2. *The space $W^{1,p}(\Omega)$ is a reflexive space for $1 < p < +\infty$.*
3. *The space $W^{1,p}(\Omega)$ is a separable space for $1 \leq p < +\infty$.*
4. *The space $W_0^{1,p}(\Omega)$ is a separable Banach space and is also reflexive for $1 < p < +\infty$.*
5. *The spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ are Hilbert spaces equipped with the following inner product:*
$$(u,v)_{H^1(\Omega)} = (u,v)_{L^2(\Omega)} + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)}.$$

Theorem 1.11 (**Poincare inequality**). *Let Ω be a subset with at least one bound. Then there exists a constant $k(\Omega)$, depending only on Ω and p, then $\forall u \in W_0^{1,p}(\Omega)$ we have*

$$||u||_{L^p(\Omega)} \leq k(\Omega)||\nabla u||_{L^p(\Omega)}. \tag{1.6}$$

Remark 1.2. *Poincare's inequality allows us to establish the equivalence on $W_0^{1,p}(\Omega)$, between the norm $\|u\|_{W^{1,p}(\Omega)}$ and $\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)}$, which is denoted by $\|u\|_{W_0^{1,p}(\Omega)}$.*

Theorem 1.12 (**Sobolev inequality**). *let Ω be a regular open of $\mathbb{R}^N$ and $1 \leq p < \infty$, so there exists a constant k, depending only on N and p, then $\forall u \in W_0^{1,p}(\Omega)$, such that*

$$||u||_{L^{p^*}(\Omega)} \leq k||\nabla u||_{L^p(\Omega)}.$$

1.2.4 Embeddings Theorem

Definition 1.2.4. *An open set Ω in $\mathbb{R}^{\mathbb{N}}$ is said to have a Lipschitz boundary, if for some $L, a, r \in (0,\infty)$,for any $x_0 \in \partial\Omega$, there exist an orthogonal coordinate system with origin at $x_0 = 0$, a cylinder $K = K' \times (-a,a)$ centered at the origin, with K' open ball in $\mathbb{R}^{(N-1)}$ of radius r,and a function $\varphi : K' \to (-a,a)$, L-Liptshitz continuous with $\varphi(0)=0$, and*

$$\partial\Omega \cap K = \{(x', \varphi(x'); x' \in K')\},$$

$$\Omega \cap K = \{(x', x_N); x' \in K', x_N > \varphi(x')\}.$$

Theorem 1.13 (**Compact Embedding**). *Let Ω be a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$.*

- *if $1 \leq p < +\infty$, then $W^{1,p}(\Omega) \subset L^q(\Omega)$ $\forall q \in [1, \frac{Np}{N-p}]$ with compact embedding for $q \in [1, \frac{Np}{N-p}]$.*
- *if $p = N$, then $W^{1,p}(\Omega) \subset L^q(\Omega)$ $\forall q \in [1, +\infty]$ with compact embedding.*
- *if $p > N$, then $W^{1,p}(\Omega) \subset C(\bar{\Omega})$ with compact embedding.*

Remark 1.3. *Compact embedding allows us to pass from weak convergence to strong convergence as follows: let $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$.*

- *If* $1 \leq p < +\infty$ *, then* $u_{\sigma(n)} \to u$ *strongly in* $L^q(\Omega)$*,* $1 \leq q < \frac{Np}{N-p}$*.*
- *if* $p = N$*, then* $u_{\sigma(n)} \to u$ *strongly in* $L^q(\Omega)$*,* $1 \leq q < +\infty$*.*
- *If* $p > N$*, then* $u_{\sigma(n)} \to u$ *strongly in* $L^\infty(\Omega)$*.*
- *The previous embeddings hold true for* $W_0^{1,p}(\Omega)$ *without any condition on the boundedness of the domain* Ω*.*
- *Therefore, it is important to remember the following very useful fact for later: if* $\{u_n\}$ *denotes a bounded sequence in* $W^{1,p}(\Omega)$ $(1 \leq p < N)$*, then we can extract a subsequence* $\{u_{n_k}\}$ *from* $\{u_n\}$ *such that* $\{u_{n_k}\}$ *converges strongly in* $L^q(\Omega)$ *for all* $q \in [1, p^*[$*, where* $p^* = \frac{Np}{N-p}$*.*

1.3 The p-Laplacian operator

Definition 1.3.1. *The p-Laplace operator is a second-order quasi-linear elliptic partial differential operator defined by:*

$$\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right) = |\nabla u|^{p-4}\left(|\nabla u|^2\Delta u + (p-2)\sum_{i,j=1}^{N}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)$$

where $1 < p < +\infty$*. This operator in divergence form is degenerate when* $p \neq 2$*, and for* $p = 2$*, the p-Laplace operator coincides with the Laplacian* Δ*.*

1.3.1 Properties of the p-Laplacian operator

Proposition 1
Let Ω be a bounded open set in $\mathbb{R}^N$; let p and q be real numbers, with $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the operator defined on $L^p(\Omega)$ by $u \mapsto |u|^{p-2}u$ is in $L^q(\Omega)$; moreover, it is continuous.

Now let's consider the p-Laplacian operator defined from the Sobolev space $W_0^{1,p}(\Omega)$ into its dual $W^{-1,q}(\Omega)$, where Ω is a bounded open set in $\mathbb{R}^N$, p and q are real numbers, $1 < p < +\infty$, and $\frac{1}{p} + \frac{1}{q} = 1$.
For any $u \in W_0^{1,p}(\Omega)$ and for all i, $1 \leq i \leq N$, we deduce from Proposition (1) that $|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i} \in L^q(\Omega)$, from which we can define the following application on $(W_0^{1,p}(\Omega))^2$:

$$(u, v) \mapsto a(u, v) = \int_X \sum_{i=1}^{N} |\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx$$

Theorem 1.14. *For any* u *in* $W_0^{1,p}(\Omega)$*, the application :* $W_0^{1,p}(\Omega) \to \mathbb{R}$*;* $\quad v \mapsto a(u, v)$*; is a continuous linear form. Hence, there exists a unique element, denoted* $A(u)$ *in* $W^{-1,q}(\Omega)$*, such that:* $a(u, v) = \langle A(u), v\rangle, \quad \forall v \in W_0^{1,p}(\Omega)$*. The application* $A : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$*,* $u \mapsto A(u)$*, is denoted:*

$$-\Delta_p(u) = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i}\right).$$

Definition 1.3.2. *Let* $r \mapsto \varphi(r)$ *be a strictly increasing continuous function from* $\mathbb{R}^+ \to \mathbb{R}^+$*, with* $\varphi(0) = 0$ *and* $\varphi(r) \to 1$ *as* $r \to \infty$*. An application* $J : X \to X^*$ *with* X *a*

Banach space, is called a "duality application" relative to φ if the following conditions hold:

$$\langle J(u), u\rangle = \|J(u)\|_* \|u\|, \quad \forall u \in X,$$
$$\|J(u)\|_* = \varphi(\|u\|), \forall u \in X.$$

Remark
Naturally, this notion depends on the norm chosen on X.
For example:

- If $X = L^p(\Omega)$; $\|u\| = \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}} = \|u\|_{L^p(\Omega)}$, $\varphi(r) = r^{p-1}$ then $J(u) = |u|^{p-2}u$.

- If $X = W_0^{1,p}(\Omega)$; $\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{1}{p}}$; $\varphi(r) = r^{p-1}$ then

$$J(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i}\right) = -\Delta_p(u)$$

  where $\| -\Delta_p(u)\|_* = \varphi(\|u\|) = \|u\|^{p-1}_{W_0^{1,p}(\Omega)}$.

Proposition 1.3 The operator $-\Delta_p$ is bounded from $W_0^{1,p}(\Omega)$ into $W^{-1,q}(\Omega)$.
Proposition 1.4 The operator $-\Delta_p$ is monotone $W_0^{1,p}(\Omega)$ into $W^{-1,q}(\Omega)$.
Proposition 1.5 The operator $-\Delta_p$ is coercive from $W_0^{1,p}(\Omega)$ into $W^{-1,q}(\Omega)$.
Corollary:
Let Ω be a bounded open set in $\mathbb{R}^N$, p a real number satisfying $1 \leq p \leq +\infty$; then the operator $-\Delta_p$ is surjective from $W_0^{1,p}(\Omega)$ into $W^{-1,q}(\Omega)$; where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.15. *Let Ω be a bounded open set in $\mathbb{R}^N$.*

a) $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ is uniformly continuous on any bounded set in $W_0^{1,p}(\Omega)$.

b) $(-\Delta_p)^{-1} : W^{-1,q}(\Omega) \to W_0^{1,p}(\Omega)$ is continuous.

c) The composite operator $(-\Delta_p)^{-1} : W^{-1,q}(\Omega) \to W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, is compact if $1 \leq q < \frac{Np}{N-p}$

Theorem 1.16. *Let Ω be a bounded open set in $\mathbb{R}^N$, where $N \geq 3$. For $p \in [1, +\infty[$, we define the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by:*

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx.$$

Then J is differentiable on $W_0^{1,p}(\Omega)$ and

$$\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = < -\Delta_p u, v > .$$

Chapter 2

Some elements of critical point theory

In this section we present a brief introduction to the critical point theory for functionals of class C^1 on a Banach space.
Let's consider the following differential equation

$$Au = 0,$$

where $A : X \to Y$, X and Y are Banach spaces. This equation has a variational structure, if there exists a functional $J : X \to \mathbb{R}$ such that

$$(A(u), v) = \lim_{t \to 0} \frac{J(u + tv) - J(u)}{t}, \forall v \in X.$$

Where $Y = X^*$,$(.,.)$ is the duality pair between X and X^* In this case, we can write $A = J'$ and the equation $Au = 0$, becomes

$$(J'(u), v) = 0 \quad \forall v \in X.$$

Thus, we have expressed equation $Au = 0$, in weak form. The problem that we must solve then transforms into the search for critical points of J.

In the following, we will discuss the arguments to prove the existence of critical points of real functionals defined on a Banach space X, see [22].

2.1 Differentiability and Critical Points

Definition 2.1.1 (**Directional Derivative**). *Let E be a subset of a Banach space X and $J : E \to \mathbb{R}$ a real-valued function. If $u \in E$ and $v \in X$ are such that for sufficiently small $t > 0$, $u + tv \in E$, we say that J has (at point u) a derivative in the direction v if*

$$\lim_{t \to 0^+} \frac{J(u + tv) - J(u)}{t}$$

exists. We denote this limit by $J'_v(u)$.

Definition 2.1.2. *(Differentiability in the Sense of Gâteaux) We say that the function J defined on an open set E of a Banach space X, with real values, is Gâteaux differentiable (or G-differentiable) at $u \in E$ if there exists $\xi \in X^*$ such that in every direction $v \in X$ where $J(u + tv)$ exists for sufficiently small $t > 0$, the directional derivative $J'_v(u)$ exists and we have:*

$$\lim_{t \to 0^+} \frac{J(u + tv) - J(u)}{t} = \langle \xi, v \rangle.$$

The function ξ is called the Gâteaux differential of J at point u, denoted by $J'_G(u) = \xi$.

Definition 2.1.3. *(Differentiability in the Sense of Frechet) J is Frechet differentiable at $x_0 \in U$ if and only if there $\exists L_{x_0} \in L(X,Y)$ such that*

$$\frac{\|J(x_0+h) - J(x_0) - L_{x_0}(h)\|_Y}{\|h\|_X} \to 0 \text{ as } \|h\| \to 0.$$

Properties:

1. *If J is Frechet differentiable, then it is Gateaux differentiable.*
2. *If J is Gâteaux differentiable and J' is continuous, then J is Frechet differentiable.*

Definition 2.1.4. *Let X be a Banach space, $E \subset X$ an open set, and $J \in C^1(E;\mathbb{R})$. We say that $u \in E$ is a critical point of J if $J'_G(u) = 0$, where $J'_G(u)$ is the Gateaux differential of J at point u.*
If u is not a critical point, then we say that u is a regular point of J.
If $c \in \mathbb{R}$, we say that c is a critical value of J if there exists $u \in E$ such that $J(u) = c$ and $J'_G(u) = 0$. If c is not a critical value, then we say that c is a regular value of J.

Definition 2.1.5. *Let X be a Banach space, $F \in C^1(X;\mathbb{R})$, and a set of constraints:*

$$S = \{v \in X : F(v) = 0\}.$$

We assume that for all $u \in S$, we have $F'_G(u) \neq 0$. If $J \in C^1(X;\mathbb{R})$, we say that $c \in \mathbb{R}$ is a critical value of J on S if there exists $u \in S$ and $\lambda \in \mathbb{R}$ such that

$$J(u) = c \quad \text{and} \quad J'_G(u) = \lambda F'_G(u)$$

The point u is a critical point of J on S and the real number λ is called the Lagrange multiplier for the critical value c (or the critical point u).
When X is a functional space and the equation $J'_G(u) = \lambda F'_G(u)$ corresponds to a partial differential equation, we say that $J'_G(u) = \lambda F'_G(u)$ is the Euler-Lagrange equation (or Euler equation) satisfied by the critical point u on the constraint S.

Proposition 2.1 Under the assumptions and notations of Definition (2.1.5), we suppose that $u_0 \in S$ is such that

$$J(u_0) = \inf_{v \in S} J(v).$$

Then there exists $\lambda \in \mathbb{R}$ such that:

$$J'_G(u_0) = \lambda F'_G(u_0).$$

2.1.1 Convexity and Lower semi-continuity

Definition 2.1.6. *A set C is said to be convex if, for every x and y in C, the segment $[x,y]$ is entirely contained in C. In other words, for all x and y in C, for all t in $[0,1]$:*

$$(tx + (1-t)y) \in C.$$

Definition 2.1.7. *A function $f : C \to \mathbb{R}$, where C is a convex set, is convex if, for all x and y in C, for all t in $[0,1]$:*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Definition 2.1.8. *A function f is concave if $(-f)$ is convex, that is, for all x and y in C, for all t in $[0,1]$:*

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y).$$

Definition 2.1.9. *Let X be a Banach space.*
A minimizing sequence for a function $J : X \to]-\infty; +\infty[$ is a sequence (x_k) such that

$$J(x_k) \to \inf J \quad \text{when} \quad k \to +\infty.$$

Definition 2.1.10. *Let X be a Banach space and E be a subset of X. A function $J : E \to \mathbb{R}$ is said to be weakly sequentially lower semicontinuous if for any sequence (u_n) in E converging weakly to $u \in E$, we have:*

$$J(u) \leq \liminf_{n\to\infty} J(u_n).$$

Proposition 2.2. Let $f : C \to \mathbb{R}$ be a convex function, then f is weakly lower semicontinuous if and only if it is lower semicontinuous.

Proposition 2.3. [[37], Proposition 25.26] Let $f : M \subseteq X \to \mathbb{R}$ be a functional on the convex closed set M of the Banach space X. Then f is weakly lower semicontinuous if one of the following conditions is satisfied :

- f is convex continuous.
- f is Gâteaux derivative and f' is monotone on M.

2.2 Existence results

A theorem which plays an important role in establishing the existence and uniqueness of the weak solution for linear problems is the Lax-Milgram theorem:

Theorem 2.1. ***Lax-Milgram Theorem:***
Let H be a Hilbert space and $a : H \times H \to \mathbb{R}$ be a bilinear, continuous and coercive form. Then for any $\varphi \in H^*$, there exists unique $u \in H$ such that

$$a(u, v) = \langle \varphi, v\rangle \quad \forall v \in H.$$

Furthermore, if a is symmetric, then u is characterized by the following property:

$$u \in H \quad \text{and} \quad J(u) = \frac{1}{2}a(u, u) - \langle \varphi, u\rangle = \min_{v\in H} J(v), \quad \text{where} \quad J(v) = \left\{\frac{1}{2}a(v, v) - \langle \varphi, v\rangle\right\}.$$

Another very important theorem used for direct minimization with or without constraints is the following:

Theorem 2.2. *(see Theorem 1.1 in [26]) Let X be a reflexive Banach space, E a weakly closed subset of X, and $J : E \to \mathbb{R}$ be weakly lower semi-continuous, then J has a minimum on E if and only if it admits a bounded minimizing sequence on E.*
Remark *The existence of a bounded minimizing sequence is ensured when J is coercive, meaning J such that $J(x) \to +\infty$ as $\|x\|_X \to +\infty$.*

In the case where the function J is lower bounded (upper bounded), it is reasonable to try to show that the minimum (respectively the maximum) is reached.
For convex functionals, a classic result is given by the theorem (see [16] page 46).
If J is not convex, it does not need to reach its infimum. However, Ekeland's result (see [33] page 51) shows the existence of points which are almost minimums.
A compactness condition which is usually used to prove the existence of stationary points is the Palais-Smale (PS) condition, for a function J of class C^1.

Definition 2.2.1. *(The Palais-Smale condition)*
Let X be a Banach space, and $J : X \to \mathbb{R}$ a C^1 function. If $c \in \mathbb{R}$, we say that J satisfies the Palais-Smale condition at level c if every sequence (u_n) in X such that:

$$J(u_n) \to c \text{ in } \mathbb{R} \quad \text{and} \quad J'(u_n) \to 0 \text{ in } X^*$$

contains a convergent subsequence $(u_{n_k})_{n_k}$.

Theorem 2.3. *Let J be a real function of class C^1 defined on a Banach space X satisfying the condition (PS) and bounded lower. Then J attains a minimum at a certain point x_0 of X.*

For a function which is not bounded, looking for its critical points amounts to looking for saddle points of the functional associated with the problem studied. These points are determined by minimax type arguments. Which brings us back to the use of the Mountain Pass Theorem and its variants.

Theorem 2.4. *Mountain Pass Theorem* *[see Theorem 4.10 in [26]]*
Let X be a Banach space, and $J \in C^1(X, \mathbb{R})$ satisfying the Palais-Smale condition. Suppose $J(0) = 0$ and:

- *There exist $R > 0$ and $\alpha > 0$ such that $\|u\| = R$ implies $J(u) \geq \alpha$.*
- *There exists $u_0 \in X$ such that $\|u_0\| > R$ implies $J(u) < \alpha$.*

Then J has a critical value c such that $c \geq \alpha$. More precisely, if we define:

- $P := \{p \in C([0, 1], X), p(0) = 0, p(1) = u_0\}$
- $c := \inf_{p \in P} \max_{t \in [0,1]} J(p(t))$

Then c is a critical value of J, and $c \geq \alpha$.

2.3 Existence and multiplicity results

To show the existence and multiplicity of solutions we can apply Clark's theorem (see Theorem 9.1 in [29]) which is a variant of the Mountain Pass Theorem, as well as other important theorems such as Ricceri's theorem and its variants which allows us to obtain the existence of three solutions.

Let ϕ be a continuously differentiable real function defined on $\mathbb{R}^m$. Assuming ϕ is coercive, denoted by $\phi(x) \to \infty$ as $\|x\| \to \infty$, it is well-established that under these conditions, ϕ attains a minimum at some point x_0. Now, consider x_1 as a critical point of ϕ which does not represent a global minimum.In 1968 M.A. Krasnosel'ski [27] observed the following: if x_1 is a nondegenerate singular point of the vector field $\nabla\phi$ (i.e., the topological index $\text{ind}(\nabla\phi, x_1)$ is non-zero), then ϕ possesses a third critical point. Subsequently, this proposition became renowned as the "Three Critical Points Theorem" (TCPT).
In the following many authors was extended the above result of Krasnosel'ski to the contex of Banach space (see [4] , [17]).
In 1998, V. Moroz et al. [28] obtained a version of TCPT as follows:

Theorem 2.5. *Let X be a Banach space, we denote by $m = \inf_X \phi$, if ϕ has an essential critical value $c > m$, then either ϕ admits at least three distinct critical values, or the set of minimum points M is not contractible in itself. In particular, ϕ has at least a three critical points.*

After several improvements of this concept, in 2000, Ricceri [30] has formulated a theorem regarding the existence of multiple critical points for a continuously differentiable, convex functional defined over a Banach space, which states :

Theorem 2.6. ***Three critical point theorem*** *Let X be a separable and reflexive real Banach space , the $\phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and convex functional whose Gâteaux derivative admits a continuous inverse on X^*,*
$\psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbb{R}$ an interval, $\mu_0 \in I$.
Assume that :

$$(h_1) \qquad \lim_{||u|| \to \infty} (\phi(u) + (\lambda - \mu_0)\psi(u)) = +\infty$$

$\forall \lambda \in I$, and that there exists a continuous concave function ; $h : I \to \mathbb{R}$ such that

$$(h_2) \quad \sup_{\lambda \geq 0} \inf_{u \in X} (\phi(u) + (\lambda - \mu_0)\psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\phi(u) + (\lambda - \mu_0)\psi(u) + h(\lambda))$$

Then, there exists $\lambda^* \in I \setminus \{\mu_0\}$ such that the equation

$$(\phi'(u) + (\lambda - \mu_0)\psi'(u)) = 0$$

has at least three solution in X.

Specifically, in [31] **B. Ricceri** has enhanced this previous finding by demonstrating that convexity can be substituted with sequential weak lower semi continuity. Additionally, the conclusion remains valid for each each λ^* in an open sub-interval of I, which allows us to cite that the preceding theorem can be derived from the subsequent enhancement by taking in the latter $I \setminus \{\mu_0\}$ instead of I and $h(\lambda + \mu_0)$ instead of $h(\lambda)$. Therefore, Theorem 2.6 is a special instance of the next Theorem 2.7 , as follow:

Theorem 2.7. *:* ***Improved three critical point theorem***
Let X be a separable and reflexive real Banach space , the $\phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semi continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, $I \subseteq \mathbb{R}$ an interval.
Assume that :

$$(D1) \qquad \lim_{||u|| \to \infty} (\phi(u) + \lambda\psi(u)) = +\infty$$

$\forall \lambda \in I$, and that there exists a continuous concave function $h : I \to \mathbb{R}$ such that

$$(D2) \quad \sup_{\lambda \geq 0} \inf_{u \in X} (\phi(u) + \lambda\psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\phi(u) + \lambda\psi(u) + h(\lambda)).$$

Then , there exists an open interval $\Lambda \subseteq I$ and a positive real number ρ such that, $\forall$ $\lambda \in \Lambda$, the equation

$$(\phi'(u) + \lambda\psi'(u)) = 0$$

has at least three solution in X whose norms are less than ρ.

In particular, in the last years, the result of B. Ricceri's Theorem 2.7 has been widely used for non linear boundary value problems in in [8], [9], [10], [11], [12], (see also [25] for the non smooth case), establishing multiplicity results for equations depending on a parameter λ.

In the settings of the mentioned theorem, the typical assumption is that the following minimax inequality

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\phi(u)+\lambda\psi(u)+h(\lambda)) < \inf_{u\in X}\sup_{\lambda\geq 0}(\phi(u)+\lambda\psi(u)+h(\lambda))$$

has to be satisfied by some continuous and concave function $h : I \to \mathbb{R}$. When $I = [0, +\infty[$, it has been proven by G. Cordaro [18], that the problem of finding such function h is equivalent to looking for a linear one.
Very recently, another three critical point theorem was established (Theorem 2.1 of [13]) by giving some remarks on a strict minimax inequality, which plays a fundamental role in Ricceri's three critical points theorem. As a consequence, some recent applications of Ricceri's theorem to nonlinear boundary value problems are revisited by obtaining more precise conclusions.
Our work is based to apply Theorem 2.7 to two elliptic problems in the next chapter.

2.4 Variational structure of elliptic problems

In this part, we will deal with two simple examples of elliptic problems, the first is nonlinear and the last one is nonlinear.

2.4.1 Linear problem

Consider the following problem (1):

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $f \in L^q(\Omega)$ for $q \geq \frac{2N}{N+2}$.
Let $X = H_0^1(\Omega)$, which is a separable Hilbert space. Define

$$\forall u, v \in X, \quad a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx,$$

by Holder's and Poincaré's inequalities, we have

$$\forall u, v \in X, \quad |a(u, v)| \leq \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} = \|u\|_X\|v\|_X,$$

and

$$\forall u \in X, \quad a(u, u) = \int_\Omega |\nabla u|^2 \, d\Omega = \|u\|_X^2.$$

We just need to demonstrate the coercivity of a to be able to apply the Lax-Milgram theorem and conclude the well-posedness of the problem.
clearly, a is a continuous coercive bilinear form on X.
Define $\varphi(v) = \int_\Omega fv \, dx$. By using Holder's and Sobolev's inequalities, it is easy to show that

$$|\varphi(v)| \leq C\|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\|v\|_X.$$

So, φ is a continuous linear form on X.
Therefore, by the Lax-Milgram Theorem, there exists a unique $u \in X$ solution of the problem

$$a(u, v) = \varphi(v) \quad \forall v \in X,$$

moreover, u minimizes the following energy functional:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega fu\,dx.$$

In conclusion, u is a weak solution of problem (1).

2.4.2 Nonlinear problem

Now consider the following nonlinear problem (2):

$$\begin{cases} -\Delta u &= u^q \quad \text{in } \Omega, \\ u &\geq 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0, \end{cases}$$

where $0 < q < 1$ and Ω is a bounded domain in $\mathbb{R}^N$.
It is clear that weak solutions of (2) are critical points of J defined on $X = H_0^1(\Omega)$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{q+1}\int_\Omega u_+^{q+1}\,dx,$$

where $u_+ = \max\{0, u\}$, the positive part of u.
By Holder's and Poincaré's inequalities we have

$$\int_\Omega u_+^{q+1}\,dx \leq C\left(\int_\Omega u^2\,dx\right)^{\frac{q+1}{2}} \leq C(\Omega)\|u\|_X^{q+1},$$

and since $q < 1$,

$$J(u) \geq \frac{1}{2}\|u\|_X^2 - C(\Omega)\|u\|_X^{q+1} \to +\infty \text{ as } \|u\|_X \to +\infty.$$

Then, we obtain that J is coercive and thus inferiorly bounded.
Let $m = \inf_{u\in X} J(u) > -\infty$. Note that J is of class C^1 on X. To conclude, it suffices to show that m is attained.

Let $\{u_n\}_n$ be a minimizing sequence of J, then $J(u_n) \to m$ as $n \to \infty$. Since J is coercive, we conclude that $\{u_n\}_n$ is a bounded sequence in X, a reflexive space. Then, there exists a subsequence, denoted by $\{u_n\}$, such that $u_n \rightharpoonup u$ weakly in $X = H_0^1(\Omega)$. According to the compact embeddings of Sobolev, we obtain $u_n \to u$ strongly in $L^s(\Omega)$ for all $s < 2^*$.

In particular, $u_n \to u$ strongly in $L^{q+1}(\Omega)$. Therefore, we conclude that

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{q+1}\int_\Omega u_+^{q+1}\,dx \\ &\leq \liminf_{n\to\infty}\left(\frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx - \frac{1}{q+1}\int_\Omega (u_n)^{q+1}\,dx\right) \\ &= m. \end{aligned}$$

Thus $J(u) = m$ and then $J'(u) = 0$ and u is a weak solution of (2).

Chapter 3

Three solutions for quasi-linear Dirichlet elliptic problems

In this chapter, we were inspired by the work of B. Ricceri [31] and that of G.A. Afrouzi and all [3].

3.1 Introduction :

Recently, a number of researches on the numbers of the existence of weak solutions to quasilinear elliptic systems via variational methods have received wide attention (see, for example [24] and [20]). In particular, many papers deal with problems related to the p-Laplacian we can cite, among others,the articles [12], [5] and [7] refer to the references therein for details.
In this part, we are concerned with two quasi-linear Dirichlet elliptic problems, the first is associated with p-Laplacian operator and the second is a system involving the (p, q)-Laplacian operator.
Under some appropriate conditions and using variational appoach, we prove the existence of at least, three solutions using Ricceri's three critical point theorem.

3.1.1 Preliminaries

In this step we cite for readers some propositions which need it in the proof of our main results.

proposition 3.1 ([30],Proposition 3.1)**.** *Let X be non empty set and ϕ, ψ two real functions on X. Assume that are $r > 0$ and $x_0, x_1 \in X$ such that*

$$(A_1) : \phi(x_0) = -\psi(x_0) = 0, \quad \phi(x_1) > r,$$

$$(A_2) : sup_{x\in\phi^{-1}(]-\infty,r])} - \psi(x) < r\frac{-\psi(x_1)}{\phi(x_1)},$$

Then for each ρ satisfying

$$sup_{x\in\phi^{-1}(]-\infty,r])} - \psi(x) < \rho < r\frac{-\psi(x_1)}{\phi(x_1)}, \tag{3.1}$$

One has

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\phi(u) + \lambda(\rho + \psi(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\phi(u) + \lambda(\rho + \psi(u)))$$

Proof. First of all, observe that :

$$\inf_{u\in X}\sup_{\lambda\geq 0}(\phi(u)+\lambda(\rho+\psi(u))) = inf_{x\in-\psi^{-1}([\rho,+\infty[)}\phi(x)$$

Next, note that by (3.1), one has

$$r \leq inf_{x\in-\psi^{-1}([\rho,+\infty[)}\phi(x)$$

Moreover, since $\phi(x_1) > r$, from (3.1), we infer that $-\psi(x_1) > \rho$. This implies that the function $\lambda \to \inf_{x\in X}(\phi(u)+\lambda(\rho+\psi(u)))$tends to $-\infty$ as $\lambda \to +\infty$. But, this function is upper semicontinuous in $[0,+\infty[$, and hence, it attains its supremum at a point $\overline{\lambda}$. We now distinguish two cases.
If $0 \leq \overline{\lambda} < \frac{r}{\rho}$ (note that $\rho > 0$ since $\phi(x_0) = -\psi(x_0) = 0$), then

$$\phi(u)+\lambda(\rho+\psi(u)) = \overline{\lambda}\rho < r.$$

If $\frac{r}{\rho} \leq \overline{\lambda}$, then since (by(3.1) again) $\frac{(r-\phi(x_1))}{(\rho+\psi(x_1))} < \frac{r}{\rho}$, we have

$$\phi(x_1)+\overline{\lambda}(\rho+\psi(x_1)) < r,$$

and the proof is complete. □

Proposition 3.2[14] Let $T : X \to X^*$ be an operator defined by

$$T(u,v)(h_1,h_2) = \int_\Omega (|\nabla u|^{p-2}\nabla u\nabla h_1 + a(x)|u|^{p-2}uh_1)dx + \int_\Omega (|\nabla v|^{q-2}\nabla v\nabla h_2 + b(x)|v|^{q-2}vh_2)dx$$
$$\forall (u,v),(h_1,h_2) \in X.$$

Then T admits a continuous inverse on X^*.

Proof. Denoting by $<.,.>$ the usual inner product in $\mathbb{R}^N$, for $p \geq 2$ there exists a positive constant C_p such that the following inequality (see (2,2) in [32])

$$< |x|^{p-2}x - |y|^{p-2}y, x-y > \geq C_p|x-y|^p,$$

holds $\forall x,y \in \mathbb{R}^N$. Thus it easy to see that :

$$< T(u_1,v_1)-T(u_2,v_2),(u_1-u_2,v_1-v_2) > \geq min(C_p,C_q)(||u_1-u_2||_1^p + ||v_1-v_2||_2^q), \forall (u_1,u_2),(v_1,v_2) \in X.$$

This means that T is uniformly monotone operator in X. Moreover T is coercive and hemi-continuous in X. Therefore, by Theorem 1.1 we conclude that T admits a continuous inverse on X^*. □

3.2 The Laplacian elliptic problem

In this section, we consider the following elliptic Dirichlet problem involving the Laplacian operator:

$$\begin{cases} -\Delta(u) = \lambda_0(f(u)+\mu g(u)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

where :

- Ω is non empty bounded open set in $\mathbb{R}^N(N \geq 2)$ with smooth boundary $\partial\Omega$, $\delta > 0$, $\mu \in [-\delta, \delta]$, λ_0 is a positive real parameter.
- $f, g : R \to R$ are a continuous , differentiable and carathéodory functions.
- $\Delta(u) = div(\nabla u)$ is the Laplacian operator.

we recall that a weak solution of problem 3.2 is any $u \in W_0^{1,2}(\Omega)$ such that :

$$\int_\Omega \nabla u \nabla h dx - \lambda_0 \int_\Omega (f(u) + \mu g(u)) h dx = 0, \quad \forall h \in W_0^{1,2}(\Omega).$$

Now, we state the following theorem.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set, with smooth boundary, and $f, g : \mathbb{R} \to \mathbb{R}$ two continuous functions, with $sup_{\epsilon\in\mathbb{R}}(\int_0^\epsilon f(t)\,dt) > 0$. Assume that there are four positive constants a,q,s,γ, with $q < \frac{N+2}{N-2}$ (if $N > 2$), $s < 2$, and $\gamma > 2$, such that :*

1. *max* $\{|f(\epsilon)|, |g(\epsilon)|\} \leq a(1 + |\epsilon|^q), \quad \forall \epsilon \in \mathbb{R}$,
2. *max* $\left\{\int_0^\epsilon f(t)\,dt, |\int_0^\epsilon g(t)\,dt|\right\} \leq a(1 + |\epsilon|^s), \quad \forall \epsilon \in \mathbb{R}$
3. *lim* $sup_{\epsilon\to 0} \frac{\int_0^\epsilon f(t)\,dt,}{|\epsilon|^\gamma} < +\infty$

Then, there exists $\delta > 0$ such that , for each $\mu \in [-\delta, \delta]$,there exists λ_0 such that the problem:

$$\begin{cases} -\Delta(u) = \lambda_0(f(u) + \mu g(u)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least a three distinct weak solutions in $W_0^{1,2}(\Omega)$.

3.2.1 Variational formulation of problem 3.2

We put :

$$I = [0, +\infty[, \quad X = W_0^{1,2}(\Omega),$$

with the norm

$$||u|| = (\int_\Omega |\nabla u(x)|^2\,dx)^{1/2}, \forall u \in X.$$

We take :

$$I_1(u) = \int_\Omega (\int_0^{u(x)} f(t)\,dt) dx,$$

$$I_2(u) = \int_\Omega (\int_0^{u(x)} g(t)\,dt) dx.$$

By (3) of Theorem 3.1), there are $\eta \in]0, 1]$ such that :

$$\int_0^\epsilon f(t)\,dt \leq c|\epsilon|^\gamma, \quad \forall \epsilon \in [-\eta, \eta].$$

Clearly, it is not restrictive to assume that $\gamma < \frac{2N}{N-2}(if N > 2)$.
In view of (2) of Theorem 3.1, let's be :

$$c_1 = max\left\{c, sup_{|\epsilon|>\eta}\frac{a(1+|\epsilon|^s)}{|\epsilon|^\gamma}\right\}.$$

One has

$$\int_0^\epsilon f(t)\,dt \le c_1|\epsilon|^\gamma, \quad \forall\epsilon\in\mathbb{R}.$$

So, if $r > 0$ and $||u||^2 \le 2r$, by the Sobolev embedding theorem, we have

$$I_1(u) \le c_1\int_\Omega |u(x)|^\gamma\,dx \le c_2(\int_\Omega |\nabla u(x)|^2\,dx)^{\gamma/2} \le c_3 r^{\gamma/2} \quad (F_1)$$

Then, we have :

$$\lim_{r\to 0^+}\frac{sup_{||u||^2\le 2r}I_1(u)}{r} = 0.$$

Moreover, by assumption , $sup_{\epsilon\in\mathbb{R}}\int_0^\epsilon f(t)\,dt > 0$, we can put $w \in X\setminus\{0\}$ in such a way that $I_1(w) > 0$. At this step fix $r, \epsilon > 0$, with $r < (1/2)||w||^2$, thus we have :

$$sup_{||u||^2\le 2r}I_1(u) \le 2r\frac{|I_1(w)|}{||w||^2} - \epsilon.$$

Then, fix $\delta > 0$ verifed

$$\delta(sup_{||u||^2\le 2r}|I_2(u)| + 2r\frac{|I_2(w)|}{||w||^2}) < \epsilon.$$

Next we put

$$\sigma = \epsilon - \delta(sup_{||u||^2\le 2r}|I_2(u)| + 2r\frac{|I_2(w)|}{||w||^2}).$$

So, we obtain

$$sup_{||u||^2\le 2r}(I_1(u)+\mu I_2(u)) \le 2r\frac{I_1(w)+\mu I_2(w)}{||w||^2} - \sigma, \quad \forall\mu\in[-\delta,\delta],$$

At this step we will define the operators $\phi, \psi : X \to \mathbb{R}$, and $h : [0, +\infty[\to \mathbb{R}$, before fixing $\mu \in [-\delta, \delta]$, with :

$$\begin{aligned}&\phi(u) = 1/2||u||^2,\\&\psi(u) = -(I_1(u)+\mu I_2(u)),\\&h(\lambda) = \rho\lambda,\end{aligned}$$

where $\rho \in \mathbb{R}$ verified :

$$sup_{||u||^2\le 2r}(I_1(u)+\mu I_2(u)) < \rho < 2r\frac{I_1(w)+\mu I_2(w)}{||w||^2}.$$

3.2.2 Proof of main result

For proof of main result, we will verified that conditions of Recerri's Therem are satisfied. By (1) of Theorem 3.1 it follows that the functional ψ is continuously Gâteaux differentiable, with compact Gâteaux derivative, the weak solutions of our
problem 3.2 are the critical points of the functional $\phi + \lambda\psi$, with the help of
(2) of Theorem 3.1 and to the Poincaré inequality we prove that this functional is coercive:

$$\begin{aligned}
\phi(u) + \lambda\psi(u) &= \frac{||u||^2}{2} - \lambda I_1(u) - \lambda\mu I_2(u) \\
&\geq \frac{||u||^2}{2} - \lambda \int_\Omega (\int_0^{u(x)} f(t)\, dt) dx - \lambda\mu \int_\Omega (\int_0^{u(x)} g(t)\, dt) dx \\
&\geq \frac{||u||^2}{2} - \lambda \int_\Omega a(1 + |u|^s) dx - \lambda\delta \int_\Omega a(1 + |u|^s) dx \\
&\geq \frac{||u||^2}{2} - \lambda a(mes(\Omega) + ||u||_s^s) - \lambda\delta a(mes(\Omega) + ||u||_s^s), \\
&\geq \frac{||u||^2}{2} - \lambda a mes(\Omega) - \lambda a||u||_s^s - \lambda\delta a mes(\Omega) - \lambda\delta a||u||_s^s, \\
&\geq \frac{||u||^2}{2} - ||u||_s^s(\lambda a(1-\delta)) - \lambda a mes(\Omega)(1-\mu), \\
&\geq \frac{||u||^2}{2} - ||u||_s^s d_1 - d_2,
\end{aligned}$$

where d_1, d_2 constants, since $s < 2$, we have :

$$lim_{||u|| \to \infty}(\phi(u) + \lambda\psi(u)) = +\infty, \quad \forall\lambda \geq 0.$$

So $D1$ of Theorem2.7 is verified, we will check the next condition.

Let $x_0 = 0, \quad x_1 = w$.
By the definition of I_1, I_2, it easy to verify that :
$\phi(x_0) = -\psi(x_0) = 0, \quad \phi(x_1) = \frac{||w||^2}{2} > r$.
At this point the assumption (A_1) of proposition 3.1 is satisfied.
Then for each $\rho \in \mathbb{R}$ such that

$$sup_{(u)\in\phi^{-1}(]-\infty,r])}(-\psi(u)) < \rho < r\frac{-\psi(u)}{\phi(u)},$$

taking :

$$h(\lambda) = \lambda\rho, \quad \forall\lambda \geq 0.$$

By proposition 3.1 we obtains :

$$\sup_{\lambda\geq 0} \inf_{u\in X}(\phi(u) + \lambda\psi(u) + h(\lambda)) < \inf_{u\in X} \sup_{\lambda\geq 0}(\phi(u) + \lambda\psi(u) + h(\lambda)).$$

So $(D2)$ of Theorem 2.7 holds.
Consequently, we can see that all the hypotheses of Theorem 2.7 are fulfilled. The problem 3.2 has at least three weak solutions whose norms are less than ρ.

3.3 Dirichlet elliptic system

We consider the following boundary value problem of type quasi-linear Dirichlet elliptic system involving (p,q)-Laplacian operator :

$$\begin{cases} \Delta_p(u) + \lambda f(x,u,v) = a(x)|u|^{p-2}\, u & \text{in } \Omega \\ \Delta_q(v) + \lambda g(x,u,v) = b(x)|v|^{q-2}\, v & \text{in } \Omega \\ u = \ v = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

where :

- Ω is non empty bounded open set in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $p, q > N$, λ is a positive real parameter
- $\Delta_p(u) = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator.
- $f, g : \Omega \times R^2 \to R$ are a continuous and differentiable functions.
- a and b are two positives weight functions such that a, b in $C(\overline{\Omega})$.

By using the same Ricceri's Theorem 2.7, we will solve this second problem.

3.3.1 Variational formulation of problem (3.3)

In the sequel, we fix $p, q > N$, $I = [0, +\infty[$, E will denote the Sobolev space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ equipped with norm :

$$||(u,v)|| = ||u|| + ||v||,$$

where

$$||u|| = (\int_\Omega (|\nabla u|^p) dx)^{\frac{1}{p}}.$$

$$||v|| = (\int_\Omega (|\nabla v|^q) dx)^{\frac{1}{q}}.$$

We define

$$||u||_1 = (\int_\Omega (|\nabla u|^p + a(x)|u|^p) dx)^{\frac{1}{p}},$$

$$||v||_2 = (\int_\Omega (|\nabla v|^q + b(x)|v|^q) dx)^{\frac{1}{q}}.$$

Put

$$k = max\{sup_{u \in W_0^{1,p}(\Omega)\backslash\{0\}} \frac{max_{x\in\overline{\Omega}}|u(x)|^p}{||u||^p}, sup_{v \in W_0^{1,q}(\Omega)\backslash\{0\}} \frac{max_{x\in\overline{\Omega}}|v(x)|^q}{||v||^q}\}.$$

Since $p, q > N$, then $k < +\infty$.
In addition, from [34] one has :

$$sup_{u \in W_0^{1,p}(\Omega)\backslash\{0\}} \frac{max_{x\in\overline{\Omega}}|u|^p}{||u||^p} \leq \frac{N^{-1/p}}{\sqrt{\pi}} (\Gamma(1 + \frac{N}{2}))^{\frac{1}{N}} (\frac{p-1}{p-N})^{1-\frac{1}{p}} m(\Omega)^{\frac{1}{N}-\frac{1}{p}},$$

and

$$sup_{v \in W_0^{1,q}(\Omega)\backslash\{0\}} \frac{max_{x\in\overline{\Omega}}|v|^q}{||v||^q} \leq \frac{N^{-1/q}}{\sqrt{\pi}} (\Gamma(1 + \frac{N}{2}))^{\frac{1}{N}} (\frac{q-1}{q-N})^{1-\frac{1}{q}} m(\Omega)^{\frac{1}{N}-\frac{1}{q}},$$

where Γ denotes the Gamma function and m(Ω) is the Lebesgue measure of the set Ω, and equality occurs when Ω is a ball. Clearly,one has

$$||u|| \leq ||u||_1 \leq (1 + (||a||_\infty m(\Omega)k)^{\frac{1}{p}} ||u||,$$

and

$$||v|| \leq ||v||_2 \leq (1 + (||b||_\infty m(\Omega)k)^{\frac{1}{q}} ||v||.$$

Hence, In E the norm

$$||(u,v)|| = ||u||_1 + ||v||_2,$$

is clearly equivalent to the usual one.
Now, we give the definition of solution of problem 3.3.

Definition 3.3.1. *A weak solution of system 3.3 is any* $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ *such that :*

$$\int_\Omega (|\nabla u|^{p-2} \nabla u \nabla h_1 + a|u|^{p-2} u h_1) dx + \int_\Omega (|\nabla v|^{q-2} \nabla v \nabla h_2 + b|v|^{q-2} v h_2) dx - \lambda \int_\Omega (f h_1 + g h_2) dx = 0,$$
$$\forall (h_1, h_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega).$$

Let $c > 0$ we define :

$$K_1(c) = \{(t_1, t_2) \in \mathbb{R}^2\} : \frac{|t_1|^p}{p} + \frac{|t_1|^p}{p} \leq c\}.$$

Next, we put:

$$K(x, t_1, v(x)) = \int_0^{t_1} f(x, \xi, v(x))\, d\xi,$$

$$E(x, u(x), t_2) = \int_0^{t_2} g(x, u(x), \eta) d\eta,$$

$$w(x,u,v) = E(x, u(x), v(x)) + K(x, u(x), v(x)).$$

For each $(u,v) \in E$, we define the operators ϕ and ψ,

$$\phi(u,v) = \frac{||u||_1^p}{p} + \frac{||v||_2^q}{q},$$

and

$$\psi(u,v) = -\int_\Omega w(x,u,v)\, dx.$$

By the definition of ϕ and ψ, they are well defined and continuously Gâteaux differentiable with :

- $\phi'(u,v)(h_1,h_2) = \int_\Omega (|\nabla u|^{p-2} \nabla u \nabla h_1 + a|u|^{p-2} u h_1)\, dx + \int_\Omega (|\nabla v|^{q-2} \nabla v \nabla h_2 + b|v|^{q-2} v h_2)\, dx.$
- $\psi'(u,v)(h_1,h_2) = -\int_\Omega (f(x,u,v) h_1 + g(x,u,v) h_2)\, dx.$

Hence, a critical point for the functional $\psi + \lambda\phi$ is any $u \in E$ verified :

$$(\phi'(u) + \lambda \psi'(u)) = 0.$$

We can observe that each critical point for the functional $\psi + \lambda\phi$ is exactly the weak solution of Problem 3.3.

3.3.2 Main result and Proof

In this step, we state the following main result.

Theorem 3.2. *Let f and $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a functions such that $f(.,t_1,t_2), g(.,t_1,t_2)$ are continuous in $\overline{\Omega}$,$\forall (t_1,t_2) \in \mathbb{R}^2$, $f(x,.,.), g(x,.,.)$ is C^1 in $\mathbb{R}^2$ and exist two positives constants γ, β such that $\gamma < p$, $\beta < q$, and a positive function $\eta \in L^1$ such that:*

(i) $w(x,t_1,t_2) \geq 0, \quad \forall x \in \Omega \quad \text{and} \quad \forall (t_1,t_2) \in \mathbb{R}^2,$

(ii) There exist a positive constant r and a function $(u_1,v_1) \in X$ such that

$$\frac{||u_1||_1^p}{p} + \frac{||v_1||_2^q}{q} > r,$$

and

$$\frac{\int_\Omega sup_{(t_1,t_2)\in K_1(kr)} w(x,t_1,t_2)\, dx}{r} < \frac{\int_\Omega w(x,u,v)\, dx}{\frac{||u||_1^p}{p} + \frac{||v||_2^q}{q}},$$

(iii)

$$w(x,t_1,t_2) \leq \eta(x)(1 + |t_1|^\gamma + |t_2|^\beta), \forall x \in \Omega \quad \text{and} \quad \forall (t_1,t_2) \in \mathbb{R}^2,$$

then there exists an open interval $\Lambda \subseteq [0,+\infty[$ and a positive real nuumber ρ such that,$\forall \lambda \in \Lambda$, problem admits at least three solution in X whose norms are less than ρ.

Proof. To prove Theorem 3.2, we will verified that conditions of Recerri's Theorem 2.7 are satisfied.
First, we will check some proprieties about the operators ϕ and ψ.

- **ϕ' admits an inverse :**
  Indeed, since $\phi(u)$ is well defined and continuously Gâteaux differentiable, it follow that $\phi'(u)$ is continuous, thanks to proposition 3.1.1 which assume that $\phi'(u)$ is continuous on E^*.

- **ϕ' is monotone :**
  Indeed, using the inequality in [7], noting <.,.> the usual product in $\mathbb{R}^2$ (N $\geq$ 2), for p$\geq$ 2 there exists a positive constant $C_{(p,q)}$ such that :

$$< \phi'(u_1,v_1) - \phi'(u_2,v_2), (u_1 - u_2, v_1 - v_2) > \geq C_{(p,q)}(||u_1 - u_2||_1^p + ||v_1 - v_2||_2^q) > 0 ,$$
$$\forall (u_1,u_2),(v_1,v_2) \in E.$$

  Which means that also ϕ' is strictly monotone. Moreover ϕ' is coercive and hemi-continuous on E.

From all these points and proposition 2.1.1, we obtain that ϕ' is sequentially weakly lower semi-continuous and bounded on each bounded subset of E.

- **ψ' is sequentially upper continuous, with a compact derivative**:
  As well as ψ is sequentially upper semi-continuous by the assumptions of f, g.
  We fix $(u,v) \in E$, assume that :

$$(u_n, v_n) \rightharpoonup (u,v) \quad \text{weakly in} \quad E \quad \text{as} \quad n \to +\infty.$$

Then

$$(u_n, v_n) \longrightarrow (u, v) \quad \text{strongly in} \quad C(\overline{\Omega}).$$

Since the functions $f(x,.,.), g(x,.,.)$ are in $C^1(R^2), \forall x \in \Omega$.
So, its continuous in R^2, $\quad \forall x \in \Omega$, and we get that

$$f(x, u_n, v_n) \longrightarrow f(x, u, v),$$

$$g(x, u_n, v_n) \longrightarrow g(x, u, v)$$

strongly as $n \to +\infty$.
By the Lebesgue control convergence theorem,

$$\psi'(u_n, v_n) \longrightarrow \psi'(u, v) \quad \text{strongly as} \quad n \to +\infty,$$

which means that ψ' is strongly continuous, then it is a compact operator.

Consequently, ψ is well defined, sequentially upper continuous, with a compact derivative $\psi' : X \to X^*$.

$\phi(u, v) + \lambda\psi(u, v)$ **is coercive**:
Indeed, thanks to (iii) of Theorem 3.2,

$$\begin{aligned}
\phi(u,v) + \lambda\psi(u,v) &= \frac{||u||_1^p}{p} + \frac{||v||_2^q}{q} - \lambda \int_\Omega w(x,u,v)\,dx \\
&\geq \frac{||u||_1^p}{p} + \frac{||v||_2^q}{q} - \lambda \int_\Omega \eta(x)(1 + |u|^\gamma + |v|^\beta)\,dx \\
&\geq \frac{||u||_1^p}{p} + \frac{||v||_2^q}{q} - \lambda \int_\Omega \eta(x)\,dx - \lambda \int_\Omega \eta(x)|u|^\gamma\,dx - \lambda \int_\Omega \eta(x)|v|^\beta\,dx \\
&\geq \frac{||u||_1^p}{p} + \frac{||v||_2^q}{q} - \lambda||\eta||_{L^1_{(\Omega)}} - \lambda||\eta||_{L^1_{(\Omega)}}||u||_\infty^\gamma - \lambda||\eta||_{L^1_{(\Omega)}}||v||_\infty^\beta \\
&\geq \frac{||u||_1^p}{p} + \frac{||v||_2^q}{q} - \lambda||\eta||_{L^1_{(\Omega)}} - \lambda c_\infty^\gamma||\eta||_{L^1_{(\Omega)}}||u||_1^\gamma - \lambda d_\infty^\beta||\eta||_{L^1_{(\Omega)}}||v||_2^\beta \\
&\geq \frac{||u||_1^p}{p} + \frac{||v||_2^q}{q} - \lambda||\eta||_{L^1_{(\Omega)}} - a||u||_1^\gamma - b||v||_2^\beta,
\end{aligned}$$

where $c_\infty^\gamma, d_\infty^\beta, a$ and b are constants, and since $\gamma < p, \beta < q$, then we have for each $\lambda \geq 0$, on has that :

$$\lim_{||(u,v)|| \to +\infty} (\phi(u,v) + \lambda\psi(u,v)) = +\infty,$$

and so the assumption $(D1)$ of Theorem 2.7 is verified.
Next, we will prove that assumption $(D2)$ 2.7 is also satisfied .
To do that we need the help of proposition 3.1, taken $r > 0$.
We obtain from the definition of k that

$$sup_{x\in\Omega}|u(x)|^p \leq k||u||^p,$$

and

$$sup\ _{x\in\Omega}|v(x)|^p \leq k||v||^q.$$

Then for $(u,v) \in X$, then we have :

$$\phi^{-1}(]-\infty, r]) = \{(u,v) \in X; \phi(u,v) \leq r\} = \left\{(u,v) \in X; \frac{||u||_1^p}{p} + \frac{||v||_2^q}{q} \leq r\right\}$$
$$\subseteq \left\{(u,v) \in X; \frac{|u|^p}{p} + \frac{|v|^q}{q} \leq kr\right\},$$

and it follows that

$$sup_{(u,v)\in\phi^{-1}(]-\infty,r])}(-\psi(u,v)) = sup_{\phi^{-1}(]-\infty,r])}\int_\Omega w(x,u(x),v(x))\,dx$$
$$\leq \int sup_{(t_1,t_2)\in K_1(kr)} w(x,t_1,t_2)\,dx.$$

Therefore from (ii) of Theorem 3.2, we have

$$sup_{(u,v)\in\phi^{-1}(]-\infty,r])}(-\psi(u,v)) = sup_{\phi^{-1}(]-\infty,r])}\int_\Omega w(x,u(x),v(x))\,dx$$
$$\leq \int sup_{(t_1,t_2)\in K_1(kr)} w(x,t_1,t_2)\,dx$$
$$< r\frac{\int_\Omega w(x,u(x),v(x))\,dx}{\frac{|u|^p}{p}+\frac{|v|^q}{q}} = r\frac{-\psi(u,v)}{\phi(u,v)}.$$

Then, one has

$$sup_{(u,v)\in\phi^{-1}(]-\infty,r])}(-\psi(u,v)) < r\frac{-\psi(u,v)}{\phi(u,v)}.$$

The assumption (A_2) of proposition3.1 is satisfied.

Let $x_0 = (0,0)$, $\quad x_1 = (u_1, v_1)$, by the definition of w, it easy to verify that :
$\phi(x_0) = -\psi(x_0) = 0$, $\quad \phi(x_1) = \frac{||u_1||_1^p}{p} + \frac{||v_1||_2^q}{q} > r$.
At this point the assumption (A_1) of proposition 3.1 is satisfied.
Then for each ρ satisfying :

$$sup_{(u,v)\in\phi^{-1}(]-\infty,r])}(-\psi(u,v)) < \rho < r\frac{-\psi(u,v)}{\phi(u,v)},$$

we define :

$$h(\lambda) = \lambda\rho, \quad \forall\lambda \geq 0.$$

By proposition 3.1, we obtains :

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\phi(u) + \lambda\psi(u) + h(\lambda)) < \inf_{u\in X}\sup_{\lambda\geq 0}(\phi(u) + \lambda\psi(u) + h(\lambda)).$$

So $(D2)$ of Theorem 2.7 holds.
At this step, we can see that all the hypotheses of Theorem 2.7 are fulfilled. We can deduce that there exist an open interval $\Lambda \subseteq [0, +\infty]$ and a positive constant ρ such that for any $\lambda \in \Lambda$. The problem 3.3 has at least three weak solutions whose norms are less than ρ. □

Conclusion

In this thesis, we studied two quasilinear elliptic problems involving the Laplacian and (p, q)-Laplacian. By using variational approach based on critical point theorem, we established existence and multiplicity of solutions of studied problems. We used specifically Ricceri's theroem.
In futur works, we plan to investigate the existence and multiplicity for similar problems by applying the latter improved versions of Ricceri's theorem.

Bibliography

[1] R. Adams and J. Fournier, Sobolev Spaces, Pure and Applied Mathematics, Volume 140, 2nd Edition, Academic Press, (2003).

[2] Adams, R. A. (1975). *Sobolev Spaces.* Academic Press.

[3] G.A. Afrouzi, S. Shamlo and M. Mahdavi, Three Solutions for a Class of Quasilinear Dirichlet Elliptic Systems Involving (P, Q)-Laplacian Operator, The Journal of Mathematics and Computer Science Vol. 4 No.3 (2012), 487 - 493.

[4] H. Amann, *A note on degree theory for gradient mappings*, Proc. Amer. Math. Soc. 85 (1982), 591 597.

[5] G. Anello et al., *Existence of solutions of the Neumann problem involving the p-Laplacian via variational principle of Ricceri*, Arch. Math. (Basel), (2002).

[6] S. N. Antontsev and J. F. Rodrigues, *On stationary thermorheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat., vol. 52, pp. 19–36, 2006.

[7] Boccardo, L., & Figueiredo, D. (2002). Some remarks on a system of quasilinear elliptic equations. *Nonlinear Differential Equations Appl.*, 9, 309-323

[8] G. Bonanno, Existence of three solutions for a two point boundary value problem, Appl. Math. Letters 13 (2000), 53-57.

[9] G. Bonanno, A minimax inequality and its applications to ordinary differential equations, J. Math. Anal. Appl. 270 (2002), 210-229.

[10] G. Bonanno, Multiple solutions for a Neumann boundary value problem, J. Nonlinear Convex Anal., 4 (2003), to appear.

[11] G. Bonanno, P. Candito, Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian, Arch. Math. (Basel), 80 (2003), 424-429.

[12] G. Bonanno, R. Livrea, Multiplicity theorems for the Dirichlet problem involving the p-Laplacian, Nonlinear Anal., 54 (2003), 1-7.

[13] G. Bonanno, *Some remarks on a three critical points theorem*, Nonlinear Anal., 54 (2003), 651-665.

[14] G. Bonanno, S. Heidarkhani, and D. O'Regan, "Multiple solutions for a class of Dirichlet quasi-linear elliptic systems driven by a (p, q)-Laplacian operator," *Dynam. Systems Appl.*, vol. 20, pp. 89-100, 2011.

[15] A. Boucherif, *Second-order boundary value problems with integral boundary conditions*, *Nonlinear Analysis*, vol. 70, (2009), pp. 364-371.

[16] Brezis, H. (1999). *Analyse fonctionnelle, th orie et applications.* Masson, Paris.

[17] A. Castro and A. C. Lazer, *Critical point theory and the number of solutions of nonlinear Dirichlet problems*, Ann. Math. Pura Appl. (4) 70 (1979), 113-137.

[18] G. Cordaro, *On a minimax problem of Ricceri*, J. Inequal. App., vol. 6, pp. 261–285, 2001.

[19] Y. Chen, S. Levine, and R. Ran, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., vol. 66, pp. 1383–1406, 2006.

[20] Chun Li and Chun-Lei Tang, *Three solutions for a class of quasilinear elliptic systems involving the* (p, q)*-Laplacian*, Nonlinear Anal., vol. 69, pp. 3322–3329, 2009.

[21] G. Fragnelli, *Positive periodic solutions for a system of anisotropic parabolic equations*, J. Math. Anal. Appl. **73** (2010), 110 121.

[22] O. Kavian, Introduction la Th orie des Points Critiques et Applications aux Probl mes Ellitiques. Springer-Verlag,1993.

[23] L. Kong,*Second order singular boundary value problems with integral boundary conditions*, *Nonlinear Analysis: Theory, Methods & Applications*,vol. 72, no. 5, (2010), pp. 2628-2638.

[24] A. Krist?ly,*Existence of two non-trivial solutions for a class of quasilinear elliptic variational systems on strip-like domain*,Proc. Edinb. Math. Soc.,vol. 48,no. 2,pp. 465–477,2005.

[25] S. A. Marano, D. Motreanu, On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems, Nonlinear Anal. 48 (2002), 37-52.

[26] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74, Springer, New York, NY, USA, 1989.

[27] V. Moroz. *The Operator of Translation along the Trajectories of Differential Equations.* Nauka, Moscow, 1963. (Russian); English transl. in: Translations of Mathematical Monographs, Vol. 19, Amer. Math. Soc., Providence, R.I., 1968.

[28] V. Moroz, A. Vignoli, and P. Zabre??ko, *ON THE THREE CRITICAL POINTS THEOREM*, *Topological Methods in Nonlinear Analysis*, Journal of the Juliusz Schauder Center, Volume 11, 1998, 103 113.

[29] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Application to Differential Equations*; conference board of the mathematical sciences regional conference series in mathematics number 65.

[30] B. Ricceri, *Existence of three solutions for a class of elliptic eigenvalue problems*, Math. Comput. Modelling, vol. 32, pp. 1485–1494, 2000.

[31] B. Ricceri, "On a three critical points theorem," *Arch. Math. (Basel)*, vol. 75, 2000, pp. 220-226.

[32] J. Simon, Regularité de la solution d'une equation non lineaire dans RN . LMN 665, P. Benilan ed. Berlin-Heidelberg-New York, 1978.

[33] M. Struwe, *Variational Methods: Application to nonlinear partial Differential Equations and Hamiltonian systems,* 3. ed., Berlin, Spinger, 2000.

[34] G. Talenti, Some inequalities of Sobolev type on two-dimensional spheres, in: W. Walter (Ed.), General Inequalities, vol. 5, in: Internat. Ser. Numer. Math., Birkhauser, Basel, 80 (1987), 401-408.

[35] Y. Wang, G. Liu, and Y. Hu, *Existence and uniqueness of solutions for a second order differential equation with integral boundary conditions*,

[36] Z. Yang, *Existence and uniqueness of positive solutions for an integral boundary value problem*, *Nonlinear Analysis*, vol. 69, (2008), pp. 3910-3918.

[37] E. Zeidler, Nonlinear functional analysis and its applications, Vol. II/B. Berlin-Heidelberg-New York 1985.

[38] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat., vol. 50, pp. 675–710, 1986.

Abstract

In this thesis, we study the existence and multiplicity of solutions for quasilinear Dirichlet problems associated to Laplacian operators. We use variational approach, based on Three critical point theorem.

Key words : quasilinear Dirichet problem, existence and multiplicity of solutions, Variational method, Three critical point theorem.

Résumé

Ce mémoire traite des questions d'existence et de multiplicité de solutions pour des problèmes de Dirichlet quasi-linéaires associés à des opérateurs Laplaciens. L'approche utilisée est variationnelle, basée sur l'application du théorème des trois points critiques.

Mots clés : Problème de Dirichlet quasi-linéaire, existence et multiplicité de solutions, méthode variationnelle, Théorème des trois points critiques.

ملخص

ناقشنا في هذه المذكرة اشكالية وجود وتعدد الحلول لصنف من المسائل الحدية الشبه خطية المرتبطة بالمؤثرلبلاصيان، وذلك باستخدام أساليب التغيرات معتمدين في ذلك غلى نظرية النقاط الحرجة الثلاثة.

الكلمات المفتاحية: مسالة حدية شبه خطية، وجود وتعدد الحلول، أساليب التغيرات، نظرية النقاط الحرجة الثلاثة.