PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

UNIVERSITY OF ABOU BEKR BELKAID-TLEMCEN

LMD THESIS

Submitted to

FACULTY OF SCIENCE-DEPARTMENT OF MATHEMATICS

In Fulfillment of the Requirements for The Doctorate Degree in

MATHEMATICAL ANALYSIS AND APPLICATIONS

By

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Hardy-Sobolev equations in p-Laplacian on compact Riemannian manifolds.

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Academic year : 2023-2024

Acknowledgments

First of all, I would like to express my deep gratitude and appreciation to my supervisor *Mr. Youssef Maliki* , for his guidance, his availability and all the time and consideration he has given me throughout my entire journey as a PhD student. I am very grateful to him for his constant help.

Sincere thanks should also go to all my examining committee members, *Prof. Hichem Boughazi*, *Prof. Bensid Sabri*, and (chair of committee) *Prof. Boumediene Abdelloui* for accepting to evaluate this thesis. Their time and inputs are greatly appreciated.

Sincere thanks should go to *Prof. Fethi Mahmoudi* for his kind hospitality and availability during my visit to the University of El Manar in Tunis.

I would like to thank my dear parents for their love, advice and unconditional support, both moral and material. I also thank my dear brothers for their help and support. My thanks goes also to all my family members.

I am also very grateful to my friends and colleagues who have helped throughout this long journey.

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Introduction

The topic of this thesis falls within the field of nonlinear analysis on manifolds. It principally deals with the study of a quasilinear elliptic equation containing a Hardy term and a critical Sobolev exponent.

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$, with $Scal_g$ its scalar curvature. One of the well-known equations in the field of partial differential equations, and which has roots in Riemannian geometry, is the Yamabe equation which explicitly is given by:

$$\Delta_g u + \frac{(n-2)Scal_g}{4(n-1)} u = \lambda u^{2^*-1}$$

with $u \in C^\infty(M), u > 0, \Delta_g u$ is the Laplacian on (M, g), $2^* = \frac{2n}{n-2}$ and $\lambda \in \mathbb{R}^*$.

The origin of this equation traces back to the famous Yamabe problem posed in 1960 by Yamabe [51] and is stated as follows: find a conformal metric g' to g (i.e. $g' = fg$, $f \in C^\infty(M), f > 0$), such that the scalar curvature of g' is constant. In 1968, Trudinger [49] showed that there is a significant difficulty in proving this statement and giving rise to one of the major problems in nonlinear analysis on manifolds. It has been shown, see for example the book [25], that the search for this metric is equivalent to the search of a positive and regular solution u of the above equation. The searched conformal metric is then given by $g' = u^{\frac{4}{n-2}} g$ with scalar curvature is the constant λ. This problem is now completely resolved by the works of [51], [49], [4], [41] and [31]. For a compendium on this problem and related topics, we suggest to read the books [4] and [25].

The Yamabe problem extends naturally to the problem of the prescribed scalar curvature problem stated as follows: for a given positive and smooth function f, is there a conformal metric g' to g with scalar curvature the function f? This, in turn, is equivalent to the search of a positive and regular solution of the equation

$$\Delta_g u + \frac{(n-2)Scal_g}{4(n-1)} u = f u^{2^*-1}.$$

This equation has been extensively studied, we cite as example [3], [6], [12], [16], [19], [24], [27], [29], [32] and [41].

The prescribed scalar curvature equation has been generalized to the so called generalized prescribed scalar curvature where the p-Laplacian operator is involved, by O.Druet [17]. The same equation has been studied on complete non-compact Riemannian manifold by M.Benalili and Y.Maliki in [7, 8, 9].

Now, let Inj_g denote the injectivity radius of (M,g) and let x_o be a fixed point in M. Define a distance function on M as follows

$$\rho_{x_o}(x) = \begin{cases} dist_g(x_o,x), & x \in B(x_o, Inj_g), \\ Inj_g, & x \in M \setminus B(x_o, inj_g). \end{cases} \tag{0.0.1}$$

Let $s \in (0,1)$ and consider metrics on M of the form

$$g_s = (1 + (\rho_{x_o}(x))^{2-s})^m g,\ m \in \mathbb{N}^*.$$

It is not difficult to see that scalar curvature of the metrics g_s are of the form $\frac{h(x)}{(\rho_{x_o}(x))^s}$, where h is a smooth function (see [34], page 60). As it is aforesaid for the prescribed scalar curvature problem, finding a conformal metric to g_s with scalar curvature a function f amounts to finding a positive and regular solution of the following equation

$$\Delta_g u - \frac{h(x)}{(\rho_{x_o(x)})^s} u = f u^{2^*-1}, \quad u \in H_1^2(M). \tag{0.0.2}$$

This equation, which issues from a singular Yamabe problem, has been studied by F.Madani [33].

As a remark, when dealing with the problem of existence of solutions of this equation, since $s \in (0,2)$, by compacity of the inclusion $H_1^p(M) \subset L_p(M,(\rho_{x_o})^s)$, the singular term adds no further difficulty more than those faced in studying the prescribed scalar curvature above. In contrast, when $s = 2$, some more serious technical difficulties appear. This case has been already dealt with in F.Z. Terki and Y. Maliki [46].

Now, for a real p, $1 < p < n$, let us consider the $p-$Laplacian operator defined by:

$$\Delta_{g,p} u = -\mathrm{div}(|\nabla_g u|^{p-2} \nabla_g u), \quad u \in H_1^p(M).$$

Let h and f be two regular functions on M. For $0 < s \le p$, we consider the following singular quasi-linear elliptic equation.

$$\Delta_{g,p} u - \frac{h(x)}{(\rho_{x_o}(x))^s} |u|^{p-2} u = f(x) |u|^{p^*-2} u, \tag{E_s}$$

with $p^* = \frac{np}{n-p}$. We notice immediately that when varying $s \in (0,p]$ and $p \in (1,n)$, equation (E_s) covers all equations that we have mentioned so far. In fact, when $s = 0$ and $p = 2$, we fall on the prescribed scalar curvature equation. When, $s \in (0.2]$ and $p = 2$, we meet the equation considered in F.Madani [34] and F.Z. Terki and Y. Maliki [46]. Finally, when $s = 0$ and $p \in (1,n)$, we meet the generalized prescribed scalar curvature equation studied in O.Druet [17].

In this thesis, we consider equations (E_s) with $s \in (0,p]$. We first establish a decomposition result of Struwe type. Then, in a second part, we prove some existence results relying on the decomposition result.

We conclude this introduction by giving a general overview of the content of the thesis. The first chapter is devoted to some reminders of some basics of Riemannian geometry and some results from nonlinear analysis that will be used throughout the thesis.

In the second chapter, we establish a decomposition result. We show that Palais-Smale sequences are submitted to the well-known Struwe decomposition formulas. In fact, in the subcritical case, we prove that Palais-Smale sequences of our equation can be splitted (up to a subsequence) to a sum of a weak solution of (E_s), a term that tends to zero, and a sum of re-scaled non-trivial weak solutions of the Euclidean equation

$$\Delta_{\xi,p} u = |u|^{p^*-2}u, \; u \in D^{1,p}(\mathbb{R}^n) \tag{0.0.3}$$

Where $D^{1,p}(\mathbb{R}^n)$ is the Sobolev space defined as the completion of the space $C_0^\infty(\mathbb{R}^n)$, and ξ is the Euclidean metric on $I\!R^n$.
Note that the existence and classification of positive solutions of (2.0.7) are studied in [11], [50] and [42].
In the critical case $s = p$, another term enters in the decomposition which is the sum of a non trivial weak solution of the Euclidean equation

$$\Delta_{\xi,p} u - \frac{h(x_o)}{|x|^p}|u|^{p-2}u = f(x_o)|u|^{p^*-2}u, \; u \in D^{1,p}(\mathbb{R}^n), \tag{0.0.4}$$

for which the existence of solutions is studied in [1].
In a precise way, we prove the following two theorems.
Denote by η a smooth cut-off function on $\mathbb{R}^n$ such that

$$\begin{cases} \eta(x) = 1, x \in B(\frac{1}{4}) \\ 0 \leq \eta(x) \leq 1, x \in B(\frac{3}{4}) \setminus B(\frac{1}{4}) \\ \eta(x) = 0, x \in \mathbb{R}^n \setminus B(\frac{3}{4}). \end{cases} \tag{0.0.5}$$

For $y \in M$ with $0 \leq \delta \leq \frac{I_g}{2}$, we introduce the cut-off function $\eta_{\delta,y}$ as follows:

$$\eta_{\delta,y}(x) = \eta_\delta(exp_y^{-1}(x)) = \eta(\delta^{-1}(exp_y^{-1}(x))), \tag{0.0.6}$$

where $\exp_y : B(\delta) \subset \mathbb{R}^n \to B(y,\delta) \subset M$ is the exponential map at the point $y \in M$, which defines a diffeomorphism from $B(\delta) \subset \mathbb{R}^n$ to $B(y,\delta) \subset M$.

Theorem 0.1. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Let f and h be two regular functions on M. Let x_o be a point of M as defined in* (0.0.1)*. Assuming that $f(x) > 0, x \in M$.*
Let u_m be a Palais-Smale sequence of the functional $J_{f,h,s}$ at level β_s, $0 < s < p$. Then, there exist $k \in \mathbb{N}$, sequences $R_m^i \geq 0$, $R_m^i \underset{m\to\infty}{\to} 0$, convergent sequences

of points in M, $x_m^i \underset{m\to\infty}{\to} x_o^i$, a weak solution $u \in H_1^p(M)$ of (E_s), $0 < s < p$, non-trivial weak solutions $v_i \in D^{1,p}(\mathbb{R}^n)$ of (0.0.3) *such that, up to a subsequence, for* $0 < s < p$, *we have*

$$u_m = u + \sum_{i=0}^{k} (R_m^i)^{\frac{p-n}{n}} f(x_o^j)^{\frac{p-n}{p^2}} \eta_\delta(\exp_{x_m^i}^{-1}(x)) v_i((R_m^i)^{-1}\exp_{x_m^i}^{-1}(x)) + \mathcal{W}_m,$$

$$\textit{with } \mathcal{W}_m \to 0 \textit{ in } H_1^p(M),$$

and

$$J_{f,h,s}(u_m) = J_{f,h,s}(u) + \sum_{i=1}^{k} f(x_o^i)^{\frac{p-n}{p}} E(v_i) + o(1).$$

Where

$$J_{f,h,s}(u) = \frac{1}{p}\left(\int_M \left(|\nabla_g u|^p - \frac{h}{(\rho_{x_o})^s}|u|^p\right) dv_g\right) - \frac{1}{p^*}\int_M f\,|u|^{p^*} dv_g$$

is the energy functional of (E_s), $s < p$, and

$$E(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p dx - \frac{1}{p^*}\int_{\mathbb{R}^n} |u|^{p^*} dx$$

is the energy functional of (0.0.3).

Theorem 0.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Let f and h be two smooth functions on M. Let x_o be a point of M as defined in* (0.0.1). *Assuming f and h satisfy the following conditions*

1. $f(x) > 0, x \in M$,
2. $h(x_o) = \sup_M h(x)$ *and* $0 < h(x_o) < (\frac{n-p}{p})^p$.

Let u_m be a Palais-Smale sequence of the $J_{f,h,s}$ functional at level β. Then, there exist $k \in \mathbb{N}$ sequences $\mathcal{T}_m^i \geq 0$, $\mathcal{T}_m^i \underset{m\to\infty}{\to} 0$, $l \in \mathbb{N}$ sequences $\tau_m^j \geq 0$, $\tau_m^j \underset{m\to\infty}{\to} 0$, $l \in \mathbb{N}$, sequences of converging points in M, $y_m^j \underset{m\to\infty}{\to} y_o^j \neq x_o$, a weak solution $u \in H_1^p(M)$ of (E_s), $s = p$, non-trivial weak solutions $\nu_j \in D^{1,p}(\mathbb{R}^n)$ of (0.0.3) *and weak solutions* $v_i \in D^{1,p}(\mathbb{R}^n)$ *of* (0.0.4) *such that, up to a subsequence, we have*

$$\begin{aligned} u_m = u \quad &+ \quad \sum_{i=1}^{k} (\mathcal{T}_m^i)^{\frac{p-n}{n}} \eta_\delta(\exp_{x_o}^{-1}(x)) v_i((\mathcal{T}_m^i)^{-1} \exp_{x_o}^{-1}(x)) \\ &+ \quad \sum_{j=1}^{l} (\tau_m^j)^{\frac{p-n}{n}} f(y_o^j)^{\frac{p-n}{p^2}} \eta_\delta(\exp_{y_m^j}^{-1}(x)) \nu_j((\tau_m^j)^{-1} \exp_{y_m^j}^{-1}(x)) + \mathcal{W}_m \end{aligned}$$

$$\text{with } \mathcal{W}_m \to 0 \text{ in } H_1^p(M)$$

and

$$J_{f,h,p}(u_m) = J_{f,h,p}(u) + \sum_{i=0}^{k} E_{f,h}(v_i) + \sum_{j=1}^{l} f(y_o^j)^{\frac{p-n}{p}} E(\nu_j) + o(1).$$

Where

$$E_{f,h}(u) = \frac{1}{p} \int_{I\!R^n} |\nabla u|^p dx - \frac{h(x_o)}{p} \int_{I\!R^n} \frac{|u|^p}{|x|^p} dx - \frac{f(x_o)}{p^*} \int_{I\!R^n} |u|^{p^*} dx.$$

is the energy functional of (0.0.4), *and* $J_{f,h,p}$ *is the energy functional of* (E_s), $s = p$.

In chapter three, we show some existence results. We exhibit some geometrical conditions that ensure existence of a solution. More precisely we prove the following theorems

Theorem 0.3. *Let* (M, g) *be a compact Riemannian manifold of dimension* $n \geq 3$. *Let* p *and* s *be real numbers such that* $0 < s < p$, $1 < p < n$ *and* $n > p^2 - sp + s$. *Let* f *and* h *be two regular functions on* M. *Let* x_o *be a point of* M *as defined in* (0.0.1). *We assume that* h *is such that the operator*

$$L_{h,s}(u) = \int_M \left(|\nabla_g u|^p - \frac{h}{(\rho_{x_o})^s} |u|^p \right) dv_g$$

is coercive. Assume that f *and* h *satisfy the following conditions*

1. $f(x_o) = \sup_M f(x)$, $f(x) > 0$, $x \in M$,
2. $0 < h(x_o) < (\frac{n-p}{p})^p$.

Suppose we are in one of the following cases:

1. $0 < p < s + 2$ *and* $h(x_o) > 0$

2. $p = s + 2$ *and*
$$\left(\frac{p-1}{n-p}\right)^p \frac{p}{n(p-1)} \frac{\Gamma(n - \frac{n+2}{p} + 3 - p)\,\Gamma(n)}{\Gamma(n-p)} h(x_o)$$
$$> \quad \frac{\Gamma(n - \frac{n}{p} - \frac{2}{p} + 2)}{2n^2} \left((n + 2 - 3p)\frac{\Delta_g f\,(x_o)}{f\,(x_o)} - p\,\mathrm{Scal}(g)\,(x_o)\right)$$

3. $p > s + 2$ *and*
$$\left(\frac{n+2-3p}{p}\right)\frac{\Delta_g f(x_o)}{f(x_o)} < Scal(g)(x_o)$$

Then equation (E_s)*,* $0 < s < p$*, has a positive weak solution* $u \in H_1^p(M)$*.*

Theorem 0.4. *Let* (M, g) *be a compact Riemannian manifold of dimension* n*. Let* p *be a real number such that* $1 < p < n$ *and* $n > p^2$*. Let* f *and* h *be two regular functions on* M *such that* f *is positive everywhere on* M*. Let* x_o *be a point on* M *as defined in* (0.0.1)*. Assume that* h *is such that the operator* $L_{h,s}$ *is coercive. Suppose there exists a point* $x_1 \neq x_o$ *such that* $f(x_1) = \sup_M f(x)$ *and*
$$f(x_1) = \sup_M f(x) \geq \frac{f(x_o)}{\left(1 - h(x_o)(\frac{p}{n-p})^p\right)^{\frac{n}{n-p}}}.$$

Suppose we are in one of the following cases:

1. $1 < p < 2$ *and* $h(x_1) > 0$*,*

2. $p = 2$ *and*
$$\frac{8(n-1)}{(n-2)(n-4)} h(x_1) > dist_g(x_o, x_1)^s \left(\frac{\Delta_g f(x_1)}{f(x_1)} - \frac{2Scal_g(x_1)}{n-4}\right), 0 < s \leq p. \tag{0.0.7}$$

3. $p > 2$ *and*
$$\left(\frac{n+2-3p}{p}\right)\frac{\Delta_g f(x_1)}{f(x_1)} < Scal(g)(x_1), \tag{0.0.8}$$

Then, the equation (E_s)*,* $0 < s \leq p$*, has a positive weak solution* $u \in H_1^p(M)$*.*

Introduction (French version)

Le sujet de cette thèse s'inscrit dans le domaine de l'analyse non linéaire sur les variétés. Nous nous intéressons principalement à l'étude d'une équation elliptique quasi-linéaire contenant un terme de Hardy et un exposant critique de Sobolev.
Soit (M, g) une variété riemannienne de dimension $n \geq 3$, avec $Scal_g$ sa courbure scalaire. Parmi les équations bien connues dans le domaine des équations aux dérivées partielles, on trouve l'équation de Yamabe, qui est donnée par :

$$\Delta_g u + \frac{(n-2)Scal_g}{4(n-1)} u = \lambda u^{2^*-1}$$

avec $u \in C^\infty(M), u > 0, \Delta_g u$ est le Laplacien sur (M, g), $2^* = \frac{2n}{n-2}$ et $\lambda \in \mathbb{R}^*$.
L'origine de cette équation remonte au célèbre problème de Yamabe posé en 1960 par Yamabe [51] et qui s'énonce comme suit : trouver une métrique conforme g' à g (i.e $g' = fg$, f dans $C^\infty(M), f > 0$), telle que la courbure scalaire de g' soit constante. En 1968, Trudinger [49] montrait qu'il était très difficile de prouver cette affirmation, et donnait ainsi lieu à l'un des problèmes majeurs de l'analyse non linéaire sur les variétés. Il a été démontré que la recherche de cette métrique est équivalente à la recherche d'une solution positive et régulière u de l'équation ci-dessus. La métrique conforme recherchée est alors donnée par $g' = u^{\frac{4}{n-2}} g$ avec la courbure scalaire est la constante λ. Ce problème est maintenant complètement résolu par les travaux de [51], [49], [4], [41] and [31]. Pour un recueil sur ce problème et les sujets connexes, nous suggérons de lire les livres [4], [25].
Le problème de Yamabe se généralise naturellement au problème de la courbure

scalaire prescrite, énoncé comme suit : pour une fonction positive et régulière donnée f, existe-t-il une métrique conforme g' à g dont la courbure scalaire est la fonction f ? Ceci, à son tour, est équivalent à la recherche d'une solution positive et régulière de l'équation

$$\Delta_g u + \frac{(n-2)Scal_g}{4(n-1)} u = f u^{2^*-1}.$$

Cette équation a été largement étudiée, nous citons à titre d'exemple [3], [6], [12], [16], [19], [24], [27], [29], [32] and [41].
L'équation de la courbure scalaire prescrite a été généralisée à ce qu'on appelle la courbure scalaire prescrite généralisée où on trouve l'opérateur p-Laplacien, par O.Druet [17]. La même équation a été étudiée sur une variété Riemannienne non compacte complète par M.Benalili et Y.Maliki dans [7], [8], [9].
Maintenant, notons par Inj_g le rayon d'injectivité de (M, g). Soit x_o un point de M. On définit une fonction sur M par

$$\rho_{x_o}(x) = \begin{cases} dist_g(x_o, x), & x \in B(x_o, Inj_g), \\ Inj_g, & x \in M \setminus B(x_o, inj_g). \end{cases} \tag{0.0.9}$$

Soit $s \in (0,1)$ et considérons des métriques sur M de la forme

$$g_s = (1 + (\rho_{x_o}(x))^{2-s})^m g, \; m \in \mathbb{N}^*.$$

Il n'est pas difficile de voir que la courbure scalaire des métriques g_s est de la forme $\frac{h(x)}{(\rho_{x_o}(x))^s}$, où h est une fonction régulière. Comme mentionné plus haut, trouver une métrique conforme à g_s avec une courbure scalaire une fonction f revient à trouver une solution positive et régulière de l'équation suivante

$$\Delta_g u - \frac{h(x)}{(\rho_{x_o(x)})^s} u = f u^{2^*-1}, \quad u \in H_1^2(M). \tag{0.0.10}$$

Ce problème, qui est un problème de Yamabe singulier, a été étudié par F.Madani [33].
En traitant le problème de l'existence de solutions de cette équation, puisque

$s \in (0,2)$, par compacité de l'inclusion $H_1^p(M) \subset L_p(M, (\rho_{x_o})^s)$, on constate que le terme singulier n'ajoute pas d'autres difficultés autre que celles rencontrées dans l'étude de la courbure scalaire prescrite plus haut. Par contre, lorsque $s = 2$, des difficultés techniques plus sérieuses apparaissent. Ce cas a déjà été traité dans F.Z. Terki and Y. Maliki [46].

Maintenant, pour un reél p, $1 < p < n$, on considère l'opérateur $p-$Laplacien $\Delta_{g,p}$ défini par :

$$\Delta_{g,p}u = -\mathrm{div}(|\nabla_g u|^{p-2}\nabla_g u), \quad u \in H_1^p(M).$$

Soit h et f deux fonctions régulières sur M. Pour $0 < s \leq p$, on considère l'équation elliptique quasi-linéaire suivante.

$$\Delta_{g,p}u - \frac{h(x)}{(\rho_{x_o}(x))^s} |u|^{p-2} u = f(x) |u|^{p^*-2} u, \tag{E_s}$$

avec $p^* = \frac{np}{n-p}$. Nous constatons immédiatement que lorsqu'on fait varier $s \in (0,p]$ et $p \in (1,n)$, l'équation (E_s) couvre toutes les équations que nous avons mentionné jusqu'à présent. En fait, lorsque $s = 0$ et $p = 2$, nous rencontrons l'équation de la courbure scalaire prescrite. Lorsque, $s \in (0.2]$ et $p = 2$, on rencontre l'équation considérée dans F.Madani [34] et F.Z.Terki et Y.Maliki [46]. Enfin, lorsque $s = 0$ et $p \in (1,n)$, on rencontre l'équation de la courbure scalaire prescrite généralisée étudiée dans O.Druet [17]. Dans cette thèse, nous considérons les équations (E_s) avec $s \in (0,p]$. Nous établissons d'abord un résultat de décomposition de type Struwe. Puis, dans une seconde partie, nous prouvons quelques résultats d'existence on s'appuyant sur le résultat de décomposition.

Nous concluons cette introduction en donnant un aperçu général du contenu de la thèse.

Le premier chapitre est consacré à quelques rappels de géométrie riemannienne et à certains résultats de l'analyse non linéaire et de la théorie du point critique, que nous utiliserons tout au long de la thèse. Dans le deuxième chapitre, nous établissons une décomposition des suites de Palais-Smale selon que $s \in (0,p)$ ou $s = p$. En fait, dans le cas sous-critique, nous prouvons que les séquences de

Palais-Smale de notre équation peuvent être décomposées (a une sous- suite près) en une somme d'une solution faible de (E_s), un terme qui tend vers zéro, et une somme de solutions faibles non triviales ré-échelonnées de l'équation Euclidienne

$$\Delta_{\xi,p} u = |u|^{p^*-2} u,\ u \in D^{1,p}(\mathbb{R}^n) \tag{0.0.11}$$

Où $D^{1,p}(\mathbb{R}^n)$ est l'espace de Sobolev défini par la fermeture de l'espace $C_0^\infty(\mathbb{R}^n)$, et ξ est la métrique Euclidienne de $\mathbb{R}^n$. Noté que l'existence et la classification des solutions positive de (0.0.11) sont étudiés dans [11], [50] and [42].
Dans le cas critique $s = p$, un autre terme entre dans la décomposition qui est la somme de solutions faibles non triviales de l'équation Euclidienne

$$\Delta_{\xi,p} u - \frac{h(x_o)}{|x|^p} |u|^{p-2} u = f(x_o) |u|^{p^*-2} u,\ u \in D^{1,p}(\mathbb{R}^n), \tag{0.0.12}$$

dont l'existence de solution est étudie dans [1].
Plus explicitement, dans ce chapitre, nous prouvons les deux théorèmes suivants.

Théorème 0.1. *Soit (M, g) une variété Riemannienne compacte de dimension $n \geq 3$. Soient f et h deux fonctions régulières sur M. Soit x_o le point de M ainsi défini dans* (0.0.1)*. On suppose que $f(x) > 0$, $x \in M$.*
Soit u_m une suite Palais-Smale de la fonctionnelle $J_{f,h,s}$ à niveau β_s, $0 < s < p$. Alors, il existe $k \in \mathbb{N}$, suites $R_m^i \geq 0$, $R_m^i \underset{m\to\infty}{\to} 0$, suites de points convergentes en M, $x_m^i \underset{m\to\infty}{\to} x_o^i$, une solution faibe $u \in H_1^p(M)$ de (E_s), $0 < s < p$, des solutions faibles non triviales $v_i \in D^{1,p}(\mathbb{R}^n)$ de (0.0.11) *telles que, à une sous-suite près, pour $0 < s < p$, on ait*

$$u_m = u + \sum_{i=0}^{k} (R_m^i)^{\frac{p-n}{n}} f(x_o^j)^{\frac{p-n}{p^2}} \eta_\delta(\exp_{x_m^i}^{-1}(x)) v_i((R_m^i)^{-1} \exp_{x_m^i}^{-1}(x)) + \mathcal{W}_m,$$

$$\textit{avec } \mathcal{W}_m \to 0 \textit{ in } H_1^p(M),$$

et

$$J_{f,h,s}(u_m) = J_{f,h,s}(u) + \sum_{i=1}^{k} f(x_o^i)^{\frac{p-n}{p}} E(v_i) + o(1).$$

Avec

$$J_{f,h,s}(u) = \frac{1}{p}\left(\int_M \left(|\nabla_g u|^p - \frac{h}{(\rho_{x_o})^s}|u|^p\right) dv_g\right) - \frac{1}{p^*}\int_M f\,|u|^{p^*} dv_g$$

est la fonctionnelle d'énergie de (E_s), $s < p$, *et*

$$E(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p dx - \frac{1}{p^*}\int_{\mathbb{R}^n} |u|^{p^*} dx$$

est la fonctionnelle d'énergie de (0.0.11). *et* η_δ *est la fonction défini dans* (0.0.6).

Théorème 0.2. *Soit* (M, g) *une variété Riemannienne compacte de dimension* $n \geq 3$. *Soient* f *et* h *deux fonctions régulières sur* M. *Soit* x_o *le point de* M *ainsi défini dans* (0.0.1).*On suppose que* f *et* h *satisfont les conditions suivantes*

1. $f(x) > 0$, $x \in M$,
2. $h(x_o) = \sup_M h(x)$ *and* $0 < h(x_o) < (\frac{n-p}{p})^p$.

Soit u_m *une suite Palais-Smale de la fonctionnelle* $J_{f,h,s}$ *à niveau* β. *Alors, il existe* $k \in \mathbb{N}$ *suites* $\mathcal{T}_m^i \geq 0$, $\mathcal{T}_m^i \underset{m\to\infty}{\to} 0$, $l \in \mathbb{N}$ *suites* $\tau_m^j \geq 0$, $\tau_m^j \underset{m\to\infty}{\to} 0$, $l \in \mathbb{N}$, *suites de point convergentes dans* M, $y_m^j \underset{m\to\infty}{\to} y_o^j \neq x_o$, *une solution faible* $u \in H_1^p(M)$ *of* (E_s), $s = p$, *des solutions faibles non triviale* $\nu_j \in D^{1,p}(\mathbb{R}^n)$ *de* (0.0.11) *et des solutions faibles* $v_i \in D^{1,p}(\mathbb{R}^n)$ *de* (0.0.12) *telles que, à une sous-suite près, on ait*

$$\begin{aligned} u_m &= u + \sum_{i=1}^{k} (\mathcal{T}_m^i)^{\frac{p-n}{n}} \eta_\delta(\exp_{x_o}^{-1}(x)) v_i((\mathcal{T}_m^i)^{-1}\exp_{x_o}^{-1}(x)) \\ &+ \sum_{j=1}^{l} (\tau_m^j)^{\frac{p-n}{n}} f(y_o^j)^{\frac{p-n}{p^2}} \eta_\delta(\exp_{y_m^j}^{-1}(x)) \nu_j((\tau_m^j)^{-1}\exp_{y_m^j}^{-1}(x)) + \mathcal{W}_m \end{aligned}$$

$$\textit{avec } \mathcal{W}_m \to 0 \textit{ in } H_1^p(M)$$

et

$$J_{f,h,p}(u_m) = J_{f,h,p}(u) + \sum_{i=0}^{k} E_{f,h}(v_i) + \sum_{j=1}^{l} f(y_o^j)^{\frac{p-n}{p}} E(\nu_j) + o(1).$$

Avec

$$E_{f,h}(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p dx - \frac{h(x_o)}{p}\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx - \frac{f(x_o)}{p^*}\int_{\mathbb{R}^n} |u|^{p^*} dx$$

est la fonctionnelle d'énergie de (0.0.12), *et* $J_{f,h,p}(u)$ *est la fonctionnelle d'énergie de* (E_s), $s = p$.

Dans le troisième chapitre, nous montrons quelques résultats d'existence. Nous donnons des conditions géométriques qui assurent l'existence d'une solution. Plus précisément, nous prouvons les théorèmes suivants

Théorème 0.3. *Soit* (M, g) *une variété Riemannienne compacte de dimension* n. *Soient* p *et* s *des nombres réels tels que* $0 < s < p$, $1 < p < n$ *et* $n > p^2 - sp + s$. *Soient* f *et* h *deux fonctions régulières sur* M. *Soit* x_o *un point de* M *tel que défini dans* (E_s). *Nous supposons que* h *est tel que la fonctionnelle*

$$L_{h,s}(u) = \int_M \left(|\nabla_g u|^p - \frac{h}{(\rho_{x_o})^s} |u|^p \right) dv_g$$

est coercive. Supposons que f *et* h *vérifient les conditions suivantes*

1. $f(x_o) = \sup_M f(x), f(x) > 0, x \in M$,
2. $0 < h(x_o) < (\frac{n-p}{p})^p$.

Supposons que l'une des conditions suivantes soit satisfaite :

1. $0 < p < s + 2$ *et* $h(x_o) > 0$
2. $p = s + 2$ *et*

$$\begin{aligned} &\left(\frac{p-1}{n-p}\right)^p \frac{p}{n(p-1)} \frac{\Gamma(n - \frac{n+2}{p} + 3 - p)\,\Gamma(n)}{\Gamma(n-p)} h(x_o) \\ > \quad &\frac{\Gamma(n - \frac{n}{p} - \frac{2}{p} + 2)}{2n^2} \left((n + 2 - 3p) \frac{\Delta_g f\,(x_o)}{f\,(x_o)} - p\,\mathrm{Scal}(g)\,(x_o) \right) \end{aligned}$$

3. $p > s + 2$ *et*

$$\left(\frac{n + 2 - 3p}{p}\right) \frac{\Delta_g f(x_o)}{f(x_o)} < Scal(g)(x_o)$$

Alors, l'équation (E_s), $0 < s < p$, *possède une solution faible positive* $u \in H_1^p(M)$.

Théorème 0.4. *Soit* (M, g) *une variété Riemannienne compacte de dimension* n*. Soit* p *un nombre réel tel que* $1 < p < n$ *et* $n > p^2$*. Soient* f *et* h *deux fonctions régulières sur* M *telles que* f *est positive partout sur* M*. Soit* x_o *un point de* M *tel que défini dans* (E_s)*. On suppose que* h *est tel que l'opérateur* $L_{h,s}$ *est coercif. Supposons qu'il existe un point* $x_1 \neq x_o$ *tel que* $f(x_1) = \sup_M f(x)$ *et*

$$f(x_1) = \sup_M f(x) \geq \frac{f(x_o)}{\left(1 - h(x_o)(\frac{p}{n-p})^p\right)^{\frac{n}{n-p}}}.$$

Supposons que l'une des conditions suivantes soit satisfaite :

1. $1 < p < 2$ *et* $h(x_1) > 0$,

2. $p = 2$ *et*
$$\frac{8(n-1)}{(n-2)(n-4)} h(x_1) > dist_g(x_o, x_1)^s \left(\frac{\Delta_g f(x_1)}{f(x_1)} - \frac{2Scal_g(x_1)}{n-4} \right), 0 < s \leq p. \tag{0.0.13}$$

3. $p > 2$ *et*
$$\left(\frac{n+2-3p}{p} \right) \frac{\Delta_g f(x_1)}{f(x_1)} < Scal(g)(x_1), \tag{0.0.14}$$

Alors, l'équation (E_s), $0 < s \leq p$, *possède une solution faible positive* $u \in H_1^p(M)$.

Chapter 1

Background materials

In this chapter, for reader convenience, we introduce some basics of Riemannian geometry and nonlinear analysis. Certainly, we will be brief and partial in that we select only those that we need throughout the thesis. For more details, we suggest to consult the books [25] and [30]

1.1 Basics in Riemannian geometry

1.1.1 Manifolds

Topological manifolds

Definition 1.1. *Let M be a separated topological space , we say that M is an $n-$dimensional topological Manifold if every point x of M, possesses an open neighborhood Ω, homoeomorphic to an open set V of $\mathbb{R}^n$.*

The pair (Ω, ϕ) is called a local chart of M around the point x.
For every point y in Ω, the coordinates of $\phi(y)$ in $\mathbb{R}^n$, are known as coordinate of y in the chart (Ω, ϕ).

Differentiable Manifold

An atlas of M, denoted by $\mathcal{A}$, is a collection of locale charts $(\Omega_i, \phi_i)_{i\in I}$, such that $M = \bigcup_{i\in I} \Omega_i$.
Let $\mathcal{A}$ be an atlas on M such that $\Omega_i \cup \Omega_j \neq \emptyset$. The functions Φ_{ij}defined by

$$\Phi_{ij} = \phi_j \circ \phi_i^{-1} : \phi_i\left(\Omega_i \cap \Omega_j\right) \to \phi_j\left(\Omega_i \cap \Omega_j\right),$$

are known as transition functions.
If the transition functions define a diffeomorphism of class C^k from $\phi_i\left(\Omega_i \cap \Omega_j\right)$ to $\phi_j\left(\Omega_i \cap \Omega_j\right)$, we say that the atlas is of class C^k.
In a topological manifold M, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two atlases of class C^k, such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is also an atlas of class C^k, then we say that $\mathcal{A}_1$ et $\mathcal{A}_2$ are C^k-compatibles. The relation of C^k-compatibility defines an equivalence relation in the set of class of C^k atlases. A complete C^k-atlas, is the union of all atlases that are in the same class of equivalence, note that every atlas of class C^k is contained in a complete C^k-atlas.

Definition 1.2. *A topological manifold equipped with a complete C^k-atlas is said to be of class C^k.*

Differentiable maps

The following definition introduces the concept of differentiability for mapping between manifolds.

Definition 1.3. *Let M et N be two manifold of class C^k, and $f : M \mapsto N$ a continuous map. If for every charts (Ω, ϕ) and $(\bar{\Omega}, \bar{\phi})$ such that $f(\Omega) \subset \bar{\Omega}$ the map*

$$\bar{\phi} \circ f \circ \phi^{-1} : \phi(\Omega) \to \bar{\phi}(\bar{\Omega})$$

is of class C^k, then f is said to be of class C^k.

Tangent space , Tangent bundle, Cotangent space, Cotangent bundle

Let M be a manifold of class C^k and x a point of M. Let $\mathbb{F}_x$ be the vector space of all functions defined on M with value in $\mathbb{R}$ which are differentiable at x, an $\mathbb{F}_x$-function is said to be flat at x, if for any local map (Ω,ϕ) at the point x, we have $D\left(f \circ \phi^{-1}\right)_{\phi(x)} = 0$. Let $\mathcal{L}_x$ be the sub-vector space of $\mathbb{F}_x$ formed of flat functions at x.

Definition 1.4. *A tangent vector of M at point x is any linear form $X : \mathbb{F}_x \to \mathbb{R}$ whose value on $\mathcal{L}_x$ equals to zero .*
The tangent space of M at point x, denoted by $T_x(M)$, is the vector space of all tangent vectors.

Let (Ω,ϕ) be a local chart at the point x with coordinates $(x_1, ..., x_n)$, Let the n linear forms defined on $\mathbb{F}_x$ by

$$\left(\frac{\partial}{\partial x_i}\right)_x \cdot (f) = D_i\left(f \circ \varphi^{-1}\right)_{\varphi(x)},$$

where D_i is the i-th partial derivative.
The linear forms $\left(\frac{\partial}{\partial x_i}\right)_x$ are tangent vectors that form a basis of $T_x(M)$. The dual of $T_x(M)$, denoted by $T_x^*(M)$, is said to be the cotangent space to M at x. The basis of this space is the family $\{dx_x^i\}_{i=1,...,n}$, with

$$dx_x^i \cdot \left(\frac{\partial}{\partial x_j}\right)_x = \varsigma_i^j,$$

where $\varsigma_i^j = 1$ if $i = j$, et 0 if $i \neq j$.

Definition 1.5. *The tangent bundle $T(M)$, is the disjoint union of all tangent spaces $(T_x(M))_{x\in M}$.*

It is proved (see [25], Theorem (1.5.2)), that if M is of dimension n, then $T(M)$ has a natural structure of a manifold of dimension $2n$. If (Ω, ϕ) is a local chart of M then $(\bigcup_{x\in\Omega} T_xM, \Phi)$ is a local chart of $T(M)$, for $X \in T_x(M)$ and $x \in M$, the homeomorphism Φ is defined as follows

$$\Phi(X) = (\phi^1(x), ..., \phi^n(x), X(\phi^1), ..., X(\phi^n)).$$

The space defined by the disjoint union of $T_x^*(M)$, $x \in M$, is said to be the cotangent bundle of M, which we denote $T^*(M)$. It has the natural structure of a $2n$-dimensional variety (see [25], theorem (1.6.1)).

Linear tangent application

Definition 1.6. *Let M, N be two manifolds of class C^k and $f : M \to N$ a differentiable application. Define the linear tangent application of f at a point x, denoted $f_\star$, as the application from $T_x(M)$ to $T_{f(x)}(N)$, which assigns to every $X \in T_x(M)$, the vector $f_\star(x).X \in T_{f(x)}(N)$ which is defined for a differentiable map $g : N \to \mathbb{R}$ by:*

$$(f_\star(x).X)(g) = X(g \circ f)$$

the linear application takes on the role previously played by the differential in euclidean spaces.

Vector field, differentiable form.

Definition 1.7.
1. *A vector field on M is any application $X : M \to T(M)$ of class C^k, such that for any $x \in M$, $X(x) \in T_x(M)$.*
2. *A differentiable form on M, is any application $\omega : M \to T^*(M)$ of class C^k, such that for every $x \in M$, $w(x) \in T_x^*(M)$.*

Tensors, Tensors fields, the pullback

Now we introduce the concept of tensor on a differentiable manifold.

Definition 1.8. *Let F be a finite-dimensional real vector space and F^* its dual. A p-times covariant and q-times contravariant tensor on F, denoted by (p, q)-tensor, is a $(p+q)$-linear form on $\underbrace{F^* \times F^* \times ... \times F^*}_{p\ times} \times \underbrace{F \times F \times ... \times F}_{q\ times}$.*

Let $T_p^q(T_x(M))$ be the space of (p,q)-tensors on $T_x(M)$. Let let $T_p^q(M)$ be the space defined by the disjoint union of $T_p^q(T_x(M))$, $x \in M$, this space has the natural structure of a C^k- manifold of dimension $n(1+n^{p+q-1})$.

Definition 1.9. *A (p,q)-tensor field on M is an application $T : M \to T_p^q(M)$, which to any $x \in M$, assigns $T(x) \in T_p^q(T_x(M))$. It is said to be of class C^k, if it is of class C^k from M to $T_p^q(M)$.*

Definition 1.10. *Let M and N be two manifolds, f an application of class C^{k+1} from M to N and a $(p,0)$-tensor field T on N. The pull-back of T by f, denoted $f^\star T$, is a $(p,0)$-tensor field of class C^k on M, defined for $x \in M$, and $X_1, X_2, ..., X_p \in T_x(M)$ by*

$$f^\star T(x).(X_1, X_2, ..., X_p) = T(f(x)).(f_\star(x)X_1, f_\star(x).X_2, ..., f_\star(x).X_p)$$

Linear connection

Let us denote the space of differentiable vector fields on M by $\Gamma(M)$.

Definition 1.11. *A linear connection D is an application from $T(M) \times \Gamma(M)$ to $T(M)$, which satisfies the following properties:*

1. *For all $X \in T_x(M)$, $D(X,Y) \in T_x(M)$.*
2. *D is bilinear on $T_x(M) \times \Gamma(M)$*
3. *For a differentiable function f, $X \in T_x(M)$, and $Y \in \Gamma(M)$ we have :*

$$D(X, fY) = X(f)Y(x) + f(x)D(X,Y)$$

4. *if $X, Y \in \Gamma(M)$, are respectively of class C^k and C^{k+1}, the $D(X,Y)$ is of class C^k.*

Given a linear connection D, a chart (Ω, ϕ). There are n^3 functions, denoted by Γ_{ij}^k, from Ω into $\mathbb{R}$, called the Christoffel symbols of the connection D in (Ω, ϕ), such that for any $x \in \Omega$, $Y \in \Gamma(M)$ and $X \in T_x(M)$

$$D(X,Y) = D_X(Y) = X^i(\nabla_i Y)(x) = X^i \left(\left(\frac{\partial Y^j}{\partial x_i} \right)_x + \Gamma_{i\alpha}^j(x) Y^\alpha \right) \left(\frac{\partial}{\partial x_j} \right)_x$$

where X^i,Y^i are the components of X and Y in (Ω, ϕ). Besides, for $f : M \to \mathbb{R}$ differentiable in x,

$$\left(\frac{\partial f}{\partial x_i} \right)_x = D_i \left(f \circ \varphi^{-1} \right)_{\varphi(x)},$$

and

$$\nabla_i(Y) = D_{\left(\frac{\partial}{\partial x_i}\right)}(Y).$$

It should be noted that the Christofell Symbols are not the components of a tensor.

The covariant derivative

Definition 1.12. *The covariant derivative is applied to the tensors as follows:*

1. *If f is a function, then $D_X(f) = X(f)$.*
2. *D_X doesn't change the type of tensor.*
3. *$D_X(T \otimes \tilde{T}) = (D_X(T)) \otimes \tilde{T} + T \otimes (D_X(\tilde{T}))$*
4. *if T is a (p,q)-tensor, then for all $1 \leq k_1 \leq p$ and $1 \leq k_1 \leq q$,*

   $$D_X(C_{k_1}^{k_2} T) = C_{k_1}^{k_2} D_X(T),$$

   where $C_{k_1}^{k_2} T$ is the (k_1, k_2) contraction of T, which is a $(p-1, q-1)$-tensor.

Torsion and curvature

Definition 1.13 (Torsion)**.** *The application T of $\Gamma(M) \times \Gamma(M)$ in $\Gamma(M)$ defined by*

$$(X,Y) \longrightarrow T(X,Y) = D(X,Y) - D(Y,X) - [X,Y]$$

is called a torsion of D, where $[X,Y]$ is the vector field defined by:

$$[X,Y](f) = X(Y(f)) - Y(X(f)),\ f \in C^2(M)$$

If $T \equiv 0$, we say that the connection is torsion-free or without torsion.

Definition 1.14. *The application R from $\Gamma(M) \times \Gamma(M)$ to the homomorphism group of $\Gamma(M)$ defined by*

$$R(X,Y) = D_X(D_Y) - D_Y(D_X) - D_{[X,Y]},\ \forall X,Y \in \Gamma(M)$$

is called the curvature of D.

1.1.2 Riemannian manifold

Riemannian Metric

Definition 1.15. *A Riemannian metric g on a smooth manifold M, is a $(2,0)$-C^∞ tensor field, such that for all $x \in M$, g_x define a scalar product on $T_x(M)$, namely we have:*

$$g_x(X,Y) = g_x(Y,X),\ \textit{for all} X,Y \in T_x(M)$$
$$g_x(X,X) > 0,\ \textit{for } X \in T_x(M) \setminus \{0\}.$$

Definition 1.16. *A Reimannian manifold of dimension n of class C^∞, is the pair (M,g), with M is a differentiable manifold of class C^∞, and g a Riemannian metric.*

Levi-Civita connection, associated curvature

Definition 1.17. *Let (M,g) be a Riemannian manifold. The Levis-Civita connection is the only torsion-free connection for which the covariant derivative of g is zero.*

Given a chart (Ω,ϕ), with x^i its associated coordinates. Then, for all x in Ω, the Christoffel symbols of the Levi-Civita connection are given by :

$$\Gamma_{ij}^k = \frac{1}{2}\left(\frac{\partial g_{mj}}{\partial x_i} + \frac{\partial g_{mi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\right) g^{mk},$$

where g_{ij} are the components of g in (Ω, ϕ) and g^{ij} are the components of the inverse matrix of g. In the next definition we introduce four type of curvature associated to this connection.

Definition 1.18. *Let R be the curvature associated with the Levis-Civita connection. There are several types of curvature associated with this connection, and we will introduce four types:*

1. ***Riemann curvature**: The Riemann curvature is the $(4,0)$-C^∞ tensor field, denoted by Rm_g, defined for all $X, Y, Z, T \in \Gamma(T(M))$ by:*

$$Rm_g(X,Y,Z,T) = g(D_X(D(Y,Z)) - D_Y(D(X,Z)), T),$$

in a local chart (Ω, ϕ), the components of Rm_g are given by:

$$R_{ijkl} = g_{im} R^m_{jkl}$$

2. ***Ricci curvature**: The Ricci curvature is the $(2,0)$-C^∞ tensor field, denoted by $Ricc_g$, defined as :*

$$Ricc_g(X,Y) = \sum_{i=1}^{n} Rm_g(x)(e_i, X, ei, Y),$$

where $\{e_i\}_{1\le i\le n}$ is a basis of $T_x(M)$, the components of this curvature are given as:

$$(Ricc_g)_{ij} = R_{mikj} g^{mk}$$

3. ***Scalar curvature** : The scalar curvature is the function of class C^∞ from M to $\mathbb{R}$, denoted by $Scal_g$ defined as:*

$$Scal_g(x) = \sum_{i,j=1}^{n} Rm_g(x)(e_i, e_j, e_i, e_j).$$

In a local chart we have:

$$Scal_g(x) = R_{ij} g^{ij}$$

4. ***Sectional curvature**: For any $x \in M$ and for any linearly independent $X, Y \in T_x(M)$, the sectional curvature $K(X, Y)$, is defined as:*

$$K(X,Y) = \frac{Rm_g(x)(X,Y,X,Y)}{g(x)(X,X)g(x)(Y,Y) - g(x)(X,Y)^2}.$$

This curvature does not depend on the choice of (X, Y), and defines the Riemann curvature.

Riemannian distance

Let $\zeta : [a, b] \to M$ be a curve of class C^1. Denote by $\frac{d\zeta}{dt}$ the tangent vector of $T_{\zeta(t)}(M)$, defined for any function $f : M \to \mathbb{R}$ differentiable at $\zeta(t)$, by $(\frac{d\zeta}{dt})_t.f = (f \circ \zeta)'(t)$. The length of ζ is defined by:

$$L(\zeta) = \int_a^b \sqrt{g(\zeta(t)).\left(\left(\frac{d\zeta}{dt}\right)_t, \left(\frac{d\zeta}{dt}\right)_t\right)}\, dt.$$

For $x, y \in M$, we define a set L_x^y, by:

$$L_x^y = \left\{\zeta \in C^1([a,b]\,; \zeta(\{a\}), \zeta(\{b\}) \subset \{x,y\}\right\}$$

Definition 1.19. *The distance on M between two points x, y, is defined by:*

$$d_g(x,y) = \inf_{\zeta \in L_x^y} L(\zeta)$$

.

The exponential application

Definition 1.20. *Let (M, g) be a Riemannian manifold, ∇ the connection of $Levis - Civita$. A geodesic is a curve $\zeta : [a, b] \to M$, such that for every $t \in [a, b]$*

$$\nabla_{\frac{d\zeta}{dt}}\left(\frac{d\zeta}{dt}\right) = 0.$$

This translates into the fact that in any chart (Ω, ϕ), and for any integer k, and for any $t \in [a, b]$ such that $\zeta(t) \in \Omega$

$$(\zeta_k)"(t) + \Gamma_{ij}^k(\zeta(t))(\zeta_i)'(t)(\zeta_j)'(t) = 0$$

For any $X \in T_x(M)$, there is a single geodesic, $\zeta : [0, b] \mapsto M$ such that,

$$\zeta(0) = x \text{ et } \frac{d\zeta(t)}{dt}\Big|_{t=0} = X$$

Let $\zeta_{x,X}$ be this geodesic, for all real $\lambda > 0$, this curve satisfies $\zeta_{x,\lambda X}(t) = \zeta_{x,X}(\lambda t)$, then, for $X \in T_x(M)$ such that $g(X, X)$ is small, $\zeta_{x,X}$ is defined on every point of the interval $[0, 1]$. The exponential application is defined as follows:

Definition 1.21. *Let x be a point of M and V a neighborhood of 0 in $T_x(M)$. The exponential application in x, denoted by exp_x, is the application from V to M that assigns to any X of V, $exp_x(X) = \zeta_{x,X}(1)$.*

Since $T_x(X)$ is a vector space of dimension n, it can be identified with $\mathbb{R}^n$, then the pair (Ω, exp_x^{-1}) is a local chart of M, called the normal chart. For every $y \in \Omega$, the local coordinates of y in this chart ($exp_x^{-1}(y) \in \mathbb{R}^n$), are called normal geodesic coordinates. The components of g in this chart satisfy $g_{ij} = \delta_i^j$, and the Christofell symbols of the Levi-Civita connection, is this chart are null.

The injectivity radius

Let (M, g) be a Riemannian manifold, the injectivity radius $i_g(x)$ at a point x of M, is the largest real r, for which any geodesic ζ starting from x of length r is minimizing. This means that for any $y \in M$, we have $d_g(x, y) = L(\zeta)$, where ζ is a geodesic joining x and y. The radius of injectivity of M is defined by

$$Inj_g = inf_{x \in M} i_g(x).$$

If (M, g) is compact, Inj_g is strictly positive, but if it is just complete, i_g can be zero.

Cut-locus

The set C_x of M of measure zero, such that

$$i_g(x) = d_g(x, C_x)$$

is said to be the cut-locus of M in x.

1.1.3 Integration on Riemannian manifolds

Partition of the unity

Let (M, g) be a Riemannian manifold and $\mathcal{A} = (\Omega_i, \phi_i)_{i\in I}$ an atlas of M. The family $(\Omega_j, \phi_j, \varsigma_j)_{j\in J}$, is said to be a partition of unity subordinate to $\mathcal{A}$ if :

1. $(\varsigma_j)_j$ is a smooth unit partition of the unity subordinate to the covering $(\Omega_i)_i$,

2. $(\Omega_j, \varphi_j)_J$ is also an atlas of M,

3. For all $j \in J$, supp$\varsigma_j \subset \Omega_j$.

For every atlas $\mathcal{A}$ of M, there exists a partition of the unity l subordinate to $\mathcal{A}$. Let f be a function on M, continuous and of a compact support, and $(\Omega_i, \phi_i)_{i\in I}$ an atlas of M. Let $(\Omega_j, \varphi_j, \eta_j)_{j\in J}$ be a partition of the unity subordinate to $(\Omega_i, \phi_i)_{i\in I}$. We set

$$\int_M f dv_g = \sum_{j\in J} \int_{\varphi_j(\Omega_j)} (\varsigma_j \sqrt{|detg|} f) o \varphi_j^{-1} dx$$

The application that for f assigns $\int_M f dv_g$, defines a Radon measure that does not depend on the atlas and the partition of the unity chosen. This integral is called a Riemannian integral or Riemannian measure.

Laplacian

Let f be a function of class C^2 on M, the Laplacian denoted Δ_g is defined as follows.

$$\Delta_g f = -g^{ij}(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial f}{\partial x_k}).$$

1.2 Basics of nonlinear analysis

In this section, we introduce some basic notions of nonlinear analysis that will be used throughout the thesis.

1.2.1 Functional Spaces

- We define the Lebesgue space $L_p(M)$, $p \geq 1$ as the space of function $u : M \mapsto \mathbb{R}$, such that $|u|^p$ is intergrable on M. This space equipped with the norm
$$||u||^p_{L_p(M)} = \int_M |u|^p \, dv_g,$$
is a Banach space.

- We define the weighted Lebesgue space $L_p(M, (\rho_{x_o})^s)$, $p > 1, 0 < s \leq p$, as the space of function $u : M \mapsto \mathbb{R}$, such that $\frac{|u|^p}{(\rho_{x_o})^s}$ is integrable on M. This space equipped with the norm
$$||u||^p_{L_p(M,(\rho_{x_o})^s)} = \int_M \frac{|u|^p}{(\rho_{x_o})^s} \, dv_g,$$
is a Banach space.

- We define the Sobolev space $H_1^p(M)$, $p > 1$ as the completion of the space $C^\infty(M)$ with respect to the norm
$$||u||^p_{H_1^p(M)} = \int_M (|\nabla_g u|^p + |u|^p) dv_g$$

- The sobolev space $D^{1,p}(\mathbb{R}^n)$, $p > 1$ is defined as the completion of the space $C_0^\infty(\mathbb{R}^n)$, with respect to the norm
$$||u||^p_{D^{1,p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (|\nabla u|^p) dx$$

The following theorem is proved in ([26], page 215):

Theorem 1.22 (Rellich-Kondrakov)**.** *Let (M, g) be a compact Riemannian manifold of dimension n. Then*

1. *the inclusion $H_1^p(M) \subset L_q(M)$ is compact for $1 \leq q < p^* = \frac{pn}{n-p}$.*
2. *the inclusion $H_1^p(M) \subset L_{p^*}(M)$ is continuous.*
3. *the inclusion $H_1^p(M) \subset C^\alpha(M)$ is compact for $\alpha < 1 - \frac{n}{p}$ and $0 \leq \alpha < 1$.*

1.2.2 Sobolev inequality

In $\mathbb{R}^n$, for $1 \leq p < n$ and $p^* = \frac{np}{n-p}$, the Sobolev inequality asserts that for all $u \in D^{1,p}(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} |u|^{p^*} dx \leq K(n,p)^{p^*} \left(\int_{\mathbb{R}^n} |\nabla u|^p dx \right)^{\frac{p^*}{p}},$$

with $K(n,p)$ is the best constant in the Sobolev's inequality. The value of $K(n,p)$ is calculated by Aubin [2] and Talenti [45] and is given by:

$$\frac{p-1}{n-p} \left(\frac{n-p}{n(p-1)} \right)^{\frac{1}{p}} \left(\frac{\Gamma(n+1)}{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p}) w_{n-1}} \right)^{\frac{1}{n}} \tag{1.2.1}$$

On a compact Riemannian manifold (M,g), in [2], the following Sobolev inequality is proved: for all $\varepsilon > 0$, there exists a positive constant $A_\varepsilon > 0$ such that for all $u \in H_1^p(M)$,

$$\int_M |u|^{p^*} dv_g \leq (K(n,p)^{p^*} + \varepsilon) \left(\int_M |\nabla_g u|^p dv_g \right)^{\frac{p^*}{p}} + A_\varepsilon \left(\int_M |u|^p dv_g \right)^{\frac{p^*}{p}}. \tag{1.2.2}$$

1.2.3 Hardy inequality

For $u \in D^{1,p}(\mathbb{R}^n)$, the Hardy inequality writes

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx \leq \left(\frac{p}{n-p} \right)^p \int_{\mathbb{R}^n} |\nabla u|^p dx. \tag{1.2.3}$$

This inequality has been extended to compact Riemannian manifold in [33] as follows: For all $\varepsilon > 0$ there exists a constant $B_\varepsilon > 0$ such that for all $u \in H_1^p(M)$,

$$\int_M \frac{|u|^p}{(\rho_{x_o})^p} dv_g \leq \left(\left(\frac{p}{n-p} \right)^p + \varepsilon \right) \int_M |\nabla_g u|^p dv_g + B_\varepsilon \int_M |u|^p dv_g. \tag{1.2.4}$$

For a $u \in H_1^p(M)$ with support included in $B(x_o, \delta)$, with $\delta < Inj_g$, we have

$$\int_M \frac{|u|^p}{(\rho_{x_o})^p} dv_g \leq (K_\delta(n,p,-p))^p \int_M |\nabla_g u|^p dv_g, \tag{1.2.5}$$

with $K_\delta(n,p,-p) \to \frac{p}{n-p}$ as $\delta \to 0$.

Note that in [33], its proven that the inclusion $H_1^p(M) \subset L_p(M, (\rho_{x_o})^p)$ is continuous and the inclusion $H_1^p(M) \subset L_p(M, (\rho_{x_o})^s)$, with $0 < s < p$, is compact.

1.2.4 Convergence theorems

Now we introduce two convergence theorems.

Egorov theorem

Egorov's theorem, which we now recall, establishes a relationship between almost everywhere and uniform convergence.

Theorem 1.23 (Egorov)**.** *Let M be a compact Riemannian manifold, if $(w_m)_m$ converges to w almost everywhere, then*

$$\forall d > 0, \quad \exists E_d \subset M, \quad \int_{M \setminus E_d} dv_g < d$$

and $(w_m)_m$ converges uniformly to w in E_d.

Brezis-Lieb lemma

Lemma 1.24. *Let $(w_m)_n$ be a sequence of functions bounded in $L^p(M)_{1 \leq p < +\infty}$, converging almost everywhere to w, then $w \in L^p(M)$ and:*

$$\|w\|_p^p = \lim_{m \to \infty} \left(\|w_m\|_p^p - \|w_m - w\|_p^p \right)$$

We can find the proof of this lemma in ([28], page 10, lemma 4.6).

Other results

The next lemma is mentioned in ([28], page 11, lemma 4.8).

Lemma 1.25. *If $(w_m)_n$ is a sequence of bounded functions in $L^p(M)_{1<p<+\infty}$, converging almost everywhere to w, then w_n converges weakly to w in $L^p(M)$.*

The following lemma is due to N. Saintier ([40], page 20)

Lemma 1.26. *Let $(X_m)_m \subset \mathbb{R}^n$, and $X \in \mathbb{R}^n$, such that*

$$\left(|X_m|^{p-2} X_m - |X|^{p-2} X\right)(X_m - X) \to 0.$$

then $X_m \to X$

1.2.5 Inequalities

Now we introduce some inequalities that we will use further. Recall the following inequality (see [40, page 56]): for all x and y in a normed vector space and $p > 1$

$$|||x+y\|^{p-2}(x+y) - \|x\|^{p-2}x - \|y\|^{p-2}y\| \leq C(||x||^{p-1-\theta}||y||^{\theta} + ||y||^{p-1-\theta}||x||^{\theta}), \tag{1.2.6}$$

where θ is a small constant that depends on p.
The next inequalities are due to lemma A.4 in [5]

1. If $1 < p < 2$, for a given $\gamma \in (1, p)$, there exists a constant such that

$$(1 + t^2 + 2t\cos\alpha)^{\frac{p}{2}} \leq 1 + t^p + pt\cos\alpha + Ct^{\gamma}, \tag{1.2.7}$$

for $t \geq 0$ uniformly in α.

2. If $2 \leq p \leq 3$, for a given $\gamma \in [p-1, 2]$, there exists a constant such that

$$(1 + t^2 + 2t\cos\alpha)^{\frac{p}{2}} \leq 1 + t^p + pt\cos\alpha + Ct^{\gamma}, \tag{1.2.8}$$

for $t \geq 0$ uniformly in α.

3. If $p \geq 3$, there exists a constant such that

$$(1 + t^2 + 2t\cos\alpha)^{\frac{p}{2}} \leq 1 + t^p + pt\cos\alpha + C(t^2 + t^{p-1}), \tag{1.2.9}$$

for $t \geq 0$ uniformly in α.

1.2.6 Ekeland variational principle

One of the powerful tools of variational methods is the Ekeland lemma and its applications. It is used to minimize lower semicontinous and bounded from below functionals.

Definition 1.27. *Let B be a Banach space, and F be a C^1 functional. We say that w_m is a Palais-Smale sequence of the functional F at level c, if we have*

$$F(w_m) \to c \text{ in } \mathbb{R}, \quad \text{and} \quad F'(u_m) \to 0 \text{ in } B'.$$

Where B' is the dual of B.

We say that the functional F satisfies the Palais-Smale condition, if every Palais-Smale sequence of F posses a strongly convergent subsequence.
Now, we introduce the variational principle of Ekeland in the following lemma, see The proof of the previous lemma is in ([28], page 162, lemme 6.8).

Lemma 1.28. *Given a complete metric space (B, d), and a lower semi-continuous functional F. We assume that F is bounded from bellow on B. Then for all $\varepsilon > 0$, there exist a $\kappa_\varepsilon \in B$ such that*

$$\begin{cases} \inf_B F \leq F(\kappa_\varepsilon) \leq \inf_B F + \varepsilon \\ \forall x \in B,\, x \neq \kappa_\varepsilon,\, J(x) + \varepsilon d(x, \kappa_\varepsilon) > J(\kappa_\varepsilon) \end{cases}$$

An important application of this lemma is the following corollary.

Corollary 1.29. *Given a Banach space B, and a C^1-functional F, we assume that F, is bounded from below on B, then there exist a Palais-Smale sequence u_m, at level $\inf_B F$, that is a minimizing Palais-Smale sequence. Furthermore, if F satisfies the Palais-smale condition, then F reaches its minimum.*

This corollary is obtained from the proof of corollary 6.9 in ([28], page 163).

Chapter 2

Decomposition of Palais-Smale sequences

In this chapter, we show a Sruwe-type decomposition [44] (or, as known in the literature, H_1^p decomposition), for Palais-Smale sequences associated to the energy functional defined below. In the paper [44], M. Struwe proved a decomposition result for the well-known boundary value problem studied by Brezis and Nirenberg. Namely, let O be a bounded domain of $\mathbb{R}^n$, and $\varsigma \in \mathbb{R}$. We introduce the following boundary value problem

$$\begin{cases} \Delta u - \varsigma u = |u|^{2^*-2}u, & \text{dans } O \\ u = 0 & \text{sur } \partial O. \end{cases} \tag{2.0.1}$$

The energy functional associated to this problem is defined on $D^{1,2}(O)$, by

$$E_\varsigma(u) = \frac{1}{2}\int_O |\nabla u|^2 dx - \frac{\varsigma}{2}\int_O |u|^2 dx - \frac{1}{2^*}\int_O |u|^{2^*} dx.$$

Consider also the functional defined on $D^{1,2}(I\!R^n)$ by

$$E_0(u) = \frac{1}{2}\int_{I\!R^n} |\nabla u|^2 dx - \frac{1}{2^*}\int_{I\!R^n} |u|^{2^*} dx.$$

M. Struwe [44] has shown that any Palais-Smale sequence ω_m of E_ς can be written as follows: there exists a weak solution ω^o of (2.0.1), there exists $k \in \mathbb{N}$ solutions

$\omega^1, ..., \omega^k$ of the equation

$$\Delta u = |u|^{2^*-2}u, \ \text{In } \mathbb{R}^n,$$

sequences of point $z_m^1, ..., z_m^k$ in $\mathbb{R}^n$, real sequences $\varrho_m^1, ..., \varrho_m^k$, such that, up to a subsequence we have

$$\begin{aligned}
\omega_m^o &\equiv \omega_m \to \omega^o \text{ weakly in } D^{1,2}(O),\\
\omega_m^j &\equiv (\varrho_m^j)^{\frac{n-2}{2}}(\omega_m^{j-1} - \omega^{j-1})(\varrho_m^j(x - z_m^j)) \to \omega^j \text{ weakly in } D^{1,2}(\mathbb{R}^n),\\
E_\varsigma(\omega_m) &= E_\varsigma(\omega^o) + \sum_{j=1}^{k} E_0(\omega^j) + o(1).
\end{aligned}$$

This type of decomposition is extended to the case of compact Riemannian manifolds by several authors. We begin with the interesting work done by O. Druet, et al.[18], where a decomposition result has been proven for the equation :

$$\Delta_g u - h(x)u = f(x)u^{2^\star-1}, \quad u \in H_1^2(M).$$

Afterwards, N. Saintier [39] generalized this decomposition result to the following equation:

$$\Delta_{g,p} u - h(x)u^{p-1} = f(x)u^{p^\star-1}, \quad u \in H_1^p(M).$$

Subsequently, the authors in [35] have proved a similar decomposition result for the equation:

$$\Delta_g u - \frac{h(x)}{(\rho_{x_o(x)})^2}u = fu^{2^\star-1}, \quad u \in H_1^2(M). \tag{2.0.2}$$

where ρ_{x_o} is defined by (0.0.1).

It is worth mentioning that the singular term adds a new contribution as it leads to a new term to be added in the decomposition formulas. For reasons of relevance, it seems useful to cite this decomposition result. Let δ be a positive real, and consider a smooth cut-off function ι_δ on $\mathbb{R}^n$ defined by

$$\iota_\delta(x) = \begin{cases} 1, & x \in B(\delta), \\ 0, & x \in \mathbb{R}^n \setminus B(2\delta). \end{cases} \tag{2.0.3}$$

Where $B(\delta)$ is the ball of center zero and radius δ. Let us also consider the equation

$$\Delta_\xi u - \frac{h(x_o)}{|x|^2}u = f(x_o)|u|^{2^*-2}u, \quad u \in D^{1,2}(\mathbb{R}^n). \tag{2.0.4}$$

By I_0, we denote the energy functional of (2.0.4)

$$I_0 = \frac{1}{2}\int_{I\!R^n} |\nabla u|^2 dx - \frac{h(x_o)}{2}\int_{I\!R^n} \frac{|u|^2}{|x|^2}dx - \frac{f(x_o)}{2^*}\int_{I\!R^n} |u|^{2^*}dx$$

For $x \in M$, and $\delta < \frac{Inj_g}{2}$, we consider the function $\eta_{\delta,x}$ on M defined by:

$$\iota_{\delta,x}(y) = \iota_\delta(\exp_x^{-1}(y)).$$

In [35], the authors show that any Palais-Smale sequence of the energy functional

$$I_{h,f} = \frac{1}{2}\left(\int_M \left(|\nabla_g u|^2 - \frac{h}{(\rho_{x_o})^2}|u|^2\right)dv_g\right) - \frac{1}{2^*}\int_M f\,|u|^{2^*}dv_g,$$

can be splitted (up to a sub sequence) to a sum of a weak solution of the equation (2.0.2), non-trivial solutions of (2.0.4), and non-trivial solutions of the equation

$$\Delta_\xi u = |u|^{2^*-2}u, \quad u \in D^{1,p}(\mathbb{R}^n). \tag{2.0.5}$$

Precisely, they show that a sequence of Palais-Smale u_m, can be written in the following form:

$$\begin{aligned} u_m &= u + \sum_{i=1}^{k}(R_m^i)^{\frac{2-n}{n}}\iota_\delta(\exp_{x_o}^{-1}(x))v_i((R_m^i)^{-1}\exp_{x_o}^{-1}(x)) \\ &+ \sum_{j=1}^{l}(\tau_m^j)^{\frac{2-n}{n}}\left(f(x_o^j)\right)^{\frac{2-n}{4}}\iota_\delta(\exp_{x_m^j}^{-1}(x))\nu_j((\tau_m^j)^{-1}\exp_{x_m^j}^{-1}(x)) + \mathcal{W}_m, \\ &\quad \text{with } \mathcal{W}_m \to 0 \text{ in } H_1^2(M), \end{aligned}$$

and the energy functional satisfies the following

$$I_{h,f}(u_m) = I_{h,f}(u) + \sum_{i=1}^{k} I_0(v_i) + \sum_{j=1}^{l} E_0(\nu_j) + o(1).$$

With $(R_m^i)_{i=\overline{1,k}}$ and $(\tau^j)_{j=\overline{1,l}}$ are positive real sequences that converge to 0 when m tends to infinity, $(x_m^j)_{j=\overline{1,l}}$, are point sequences in M, such that $x_m^j \to x_o^j \neq x_o$,

u is a weak solution of (2.0.2), v_i and ν_j are weak solutions of (2.0.4) and (2.0.5) respectively.

In this chapter we will generalize the results obtained in [35] and [39] to the equations (E_s) by establishing two decomposition results, one for the subcritical case $s < p$, and the second for the Hardy critical case $s = p$.

We begin with introducing some definitions that will be used hereafter. We consider the energy functional associated to (E_s) defined on $H_1^p(M)$ by:

$$J_{f,h,s}(u) = \frac{1}{p}\left(\int_M \left(|\nabla_g u|^p - \frac{h}{(\rho_{x_o})^s}|u|^p\right) dv_g\right) - \frac{1}{p^*}\int_M f\,|u|^{p^*} dv_g. \tag{2.0.6}$$

This functional is of class C^2 in $H_1^p(M)$, and its Fréchet derivative at a point $v \in H_1^p(M)$ is given by:

$$\begin{aligned}(DJ_{f,h,s}u).v &= \int_M \left(|\nabla_g u|^{p-2} g(\nabla_g u, \nabla_g v) - \frac{h}{(\rho_{x_o})^s}|u|^{p-2} u.v\right) dv_g \\ &- \int_M f\,|u|^{p^*-2} u.v dv_g\end{aligned}$$

A Palais-Smale (P.S. for short) sequence of the functional $J_{f,h,s}$ at a level $\beta_s \in I\!R$, $0 < s \leq p$, is defined as the sequence $u_m \in H_1^p(M)$ that satisfies

$$J_{f,h,s}(u_m) \to \beta_s \quad \text{and} \qquad (DJ_{f,h,s}u_m).v \to 0, \forall v \in H_1^p(M) \text{ as } m \to \infty.$$

To abbreviate, β_p is denoted by β. We say that $J_{f,h,s}$, satisfies the Palais-smale condition, if every P.S. sequence, admits a convergent subsequence in $H_1^p(M)$. A weak solution of (E_s), $0 < s \leq p$, is a function $u \in H_1^p(M)$ such that

$$(DJ_{f,h,s}u).v = 0, \quad \forall v \in H_1^p(M).$$

We recall the following equations and their associated energy functionals :

$$\Delta_{\xi,p} u = |u|^{p^*-2}u, \tag{2.0.7}$$

$$\Delta_{\xi,p} u - \frac{h(x_o)}{|x|^p}|u|^{p-2}u = f(x_o)|u|^{p^*-2}u, \tag{2.0.8}$$

where ξ is the Euclidean metric on $I\!R^n$.

$$
\begin{aligned}
E(u) &= \frac{1}{p}\int_{I\!R^n} |\nabla u|^p dx - \frac{1}{p^*}\int_{I\!R^n} |u|^{p^*} dx, \\
E_{f,h}(u) &= \frac{1}{p}\int_{I\!R^n} |\nabla u|^p dx - \frac{h(x_o)}{p}\int_{I\!R^n} \frac{|u|^p}{|x|^p} dx - \frac{f(x_o)}{p^*}\int_{I\!R^n} |u|^{p^*} dx.
\end{aligned}
$$

Now we state the theorems that we are going to prove.

Theorem 2.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Let f and h be two regular functions on M. Let x_o be a point of M as defined in (0.0.1). Assume that $f(x) > 0, x \in M$.*
Let u_m be a Palais-Smale sequence of the functional $J_{f,h,s}$ at level β_s, $0 < s < p$. Then, there exist $k \in \mathbb{N}$, sequences $R^i_m \geq 0$, $R^i_m \underset{m\to\infty}{\to} 0$, convergent sequences of points in M, $x^i_m \underset{m\to\infty}{\to} x^i_o$, a weak solution $u \in H^p_1(M)$ of (E_s), $0 < s < p$, non-trivial weak solutions $v_i \in D^{1,p}(\mathbb{R}^n)$ of (2.0.7) such that, up to a subsequence, for $0 < s < p$, we have

$$
u_m = u + \sum_{i=1}^{k} (R^i_m)^{\frac{p-n}{n}} f(x^j_o)^{\frac{p-n}{p^2}} \eta_\delta(\exp^{-1}_{x^i_m}(x)) v_i((R^i_m)^{-1} \exp^{-1}_{x^i_m}(x)) + \mathcal{W}_m, \quad (2.0.9)
$$

$$
\textit{with } \mathcal{W}_m \to 0 \textit{ in } H^p_1(M),
$$

and

$$
J_{f,h,s}(u_m) = J_{f,h,s}(u) + \sum_{i=1}^{k} f(x^i_o)^{\frac{p-n}{p}} E(v_i) + o(1). \quad (2.0.10)
$$

Theorem 2.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Let f and h be two smooth functions on M. Let x_o be a point of M as defined in (0.0.1). Assume that f and h satisfy the following conditions*

1. *$f(x) > 0, x \in M$,*

2. *$h(x_o) = \sup_M h(x)$ and $0 < h(x_o) < (\frac{n-p}{p})^p$.*

Let u_m be a Palais-Smale sequence of the $J_{f,h,s}$ functional at level β. Then, there exist $k \in \mathbb{N}$ sequences $\mathcal{T}^i_m \geq 0$, $\mathcal{T}^i_m \underset{m\to\infty}{\to} 0$, $l \in \mathbb{N}$ sequences $\tau^j_m \geq 0$, $\tau^j_m \underset{m\to\infty}{\to} 0$,

$l \in \mathbb{N}$, sequences of converging points in M, $y_m^j \underset{m\to\infty}{\to} y_o^j \neq x_o$, a weak solution $u \in H_1^p(M)$ of (E_s), $s = p$, non-trivial weak solutions $\nu_j \in D^{1,p}(\mathbb{R}^n)$ of (2.0.7) and weak solutions $v_i \in D^{1,p}(\mathbb{R}^n)$ of (2.0.8) such that, up to a subsequence, we have

$$\begin{aligned} u_m &= u + \sum_{i=1}^{k} (\mathcal{T}_m^i)^{\frac{p-n}{n}} \eta_\delta(\exp_{x_o}^{-1}(x)) v_i((\mathcal{T}_m^i)^{-1} \exp_{x_o}^{-1}(x)) \qquad (2.0.11) \\ &+ \sum_{j=1}^{l} (\tau_m^j)^{\frac{p-n}{n}} f(y_o^j)^{\frac{p-n}{p^2}} \eta_\delta(\exp_{y_m^j}^{-1}(x)) \nu_j((\tau_m^j)^{-1} \exp_{y_m^j}^{-1}(x)) + \mathcal{W}_m \end{aligned}$$

$$\text{with } \mathcal{W}_m \to 0 \text{ in } H_1^p(M)$$

and

$$J_{f,h,p}(u_m) = J_{f,h,p}(u) + \sum_{i=0}^{k} E_{f,h}(v_i) + \sum_{j=1}^{l} f(y_o^j)^{\frac{p-n}{p}} E(\nu_j) + o(1). \qquad (2.0.12)$$

The proofs of these two theorems go through several steps that we formulate in lemmas.

Lemma 2.3. *Let u_m be a Palais-Smale sequence of $J_{f,h,s}$, $0 < s \leq p$, at level β_s. We assume that f is positive and $1-h(x_o)(\frac{p}{n-p})^p > 0$. If the sequence u_m converges weakly to a function u in $H_1^p(M)$ and $L_p(M, \rho_{x_o}^p)$, strongly in $L_q(M), 1 \leq q < p^*$ and almost everywhere in M, then, the function u is a weak solution of (E_s) and $v_m = u_m - u$ is a Palais-Smale sequence of $J_{f,h,s}$ such that $J_{f,h,s}(v_m) = \beta_s - J_{f,h,s}(u) + o(1)$.*

Proof. Let u_m be a P.S. sequence of $J_{f,h,s}$ at level β_s. As a first step in proving this lemma, we show that the sequence u_m is bounded in $H_1^p(M)$.
Firstly, on the one hand, since u_m is P.S. sequence of $J_{f,h,s}$, we have

$$J_{f,h,s}(u_m) - \frac{1}{p^*} DJ_{s,f,h}(u_m)u_m = \beta_s + o(1) + o(||u_m||_{H_1^p(M)}).$$

On the other hand we have

$$\begin{aligned} J_{f,h,s}(u_m) - \frac{1}{p^*} DJ_{f,h,s}(u_m)u_m &= \frac{1}{n} \int_M (|\nabla_g u_m|^p - \frac{h}{(\rho_{x_o})^s} |u_m|^p) dv_g \\ &= \frac{p}{n} \left(J_{f,h,s}(u_m) + \frac{1}{p^*} \int_M f\, |u_m|^{p^*} dv_g \right), \end{aligned}$$

then

$$\frac{p}{np^*}\int_M f\,|u_m|^{p^*}\,dv_g = \Big(1-\frac{p}{n}\Big)\,\beta_s + o(1) + o(||u_m||_{H_1^p(M)}).$$

since f is strictly positive on (M,g), we deduce that u_m is bounded in $L_{p^*}(M)$ and therefore in $L_p(M)$. In addition we have

$$\begin{aligned}\int_M |\nabla_g u_m|^p\,dv_g &= nJ_{f,h,s}(u_m) + \int_M \frac{h}{(\rho_{x_o})^s}\,|u_m|^p\,dv_g + o(\|u_m\|_{H_1^p(M)})\\ &= n\beta_s + \int_M \frac{h}{(\rho_{x_o})^s}\,|u_m|^p\,dv_g + o(1) + o(\|u_m\|_{H_1^p(M)}).\end{aligned}$$

Let $\delta > 0$ be a constant close to zero. then we have,

$$\begin{aligned}\int_M |\nabla_g u_m|^p\,dv_g &= n\beta_s + \int_{B(x_o,\delta)} (\rho_{x_o})^{p-s}\frac{h}{(\rho_{x_o})^p}\,|u_m|^p\,dv_g\\ &+ \int_{M\setminus B(x_o,\delta)} \frac{h(x)}{(\rho_{x_o})^s}\,|u_m|^p\,dv_g + o(1) + o(\|u_m\|_{H_1^p(M)}),\end{aligned}$$

since $p \geq s$, we have

$$\begin{aligned}\int_M |\nabla_g u_m|^p\,dv_g &\leq n\beta_s + \delta^{p-s}\max_{x\in B(x_o,\delta)}|h(x)|\int_{B(x_o,\delta)}\frac{|u_m|^p}{(\rho_{x_o})^p}dv_g\\ &+ \delta^{-s}\max_{x\in M}|h(x)|\int_{M\setminus B(x_o,\delta)}|u_m|^p\,dv_g + o(1) + o(\|u_m\|_{H_1^p(M)}).\end{aligned}$$

By Hardy inequality (1.2.5), since u_m is bounded in $L_p(M)$ we obtain that there exists a positive constant C such that

$$\left(1-\delta^{p-s}\max_{x\in B(x_o,\delta)}|h(x)|K_\delta(n,p,-p)^p\right)\int_M |\nabla_g u_m|^p\,dv_g \leq n\beta_s + C + o(1) + o(\|u_m\|_{H_1^p(M)}).$$

Now, for $p > s$, we can choose δ small enough so that we have

$$1-\delta^{p-s}\max_{x\in B(x_o,\delta)}|h(x)|K_\delta(n,p,-p)^p > 0,$$

we obtain that $\int_M |\nabla_g u_m|^p\,dv_g$ is bounded.

For $p = s$, since $\max_{B(x_o,\delta)}|h(x)|K_\delta(n,p,-p)$ goes to $h(x_o)(\frac{p}{n-p})^p$ when $\delta \to 0$ and since by assumption $1-h(x_o)(\frac{p}{n-p})^p > 0$, there exists $\delta_o > 0$ such that for any $\delta < \delta_o$ we have

$$1-\max_{x\in B(x_o,\delta)}|h(x)|K_\delta(n,p,-p)^p > 0,$$

then $\int_M |\nabla_g u_m|^p \, dv_g$ is bounded, which completes the proof that u_m is bounded in $H_1^p(M)$.

Now, we assume that the sequence u_m converges weakly to a function u in $H_1^p(M)$. We show that for $\varphi \in H_1^p(M)$, $(DJ_{f,h,s}(u_m)).\varphi$ converges to $(DJ_{f,h,s}(u).\varphi)$. In other words, u is a weak solution of (E_s). First, since the sequence u_m converges almost everywhere to u in M, by lemma 1.25 , we can conclude that the sequence $f|u_m|^{p^*-2}u_m$ converges to $f|u|^{p^*-2}u$ weakly in $L_{\frac{p^*}{p^*-1}}(M)$ and the sequence $h|u_m|^{p-2}u_m$ converges weakly to $h|u_m|^{p-2}u_m$ in $L_{\frac{p}{p-1}}(M, \rho_{x_o}^s)$. It remains to show that

$$\int_M |\nabla_g u_m|^{p-2} g(\nabla_g u_m, \nabla_g \varphi) dv_g = \int_M |\nabla_g u|^{p-2} g(\nabla_g u, \nabla_g \varphi) dv_g + o(1) \quad (2.0.13)$$

We proceed by the same method used in [39]. To lighten the notation, we denote $|\nabla_g u_m|^{p-2}\nabla_g u_m$ by $\mathcal{D}_m$, and $|\nabla_g u|^{p-2}\nabla_g u$ by $\bar{\mathcal{D}}$. The fact that $\nabla_g u_m$ is bounded in $L_p(M)$, gives that $\mathcal{D}_m$ is bounded in $L_{\frac{p}{p-1}}(M)$, which implies that $\mathcal{D}_m$ converges weakly in $L_{\frac{p}{p-1}}(M)$ to a vector field $\mathcal{D}$ in $L_{\frac{p}{p-1}}(M)$.
Now, fixing $d > 0$ and using Egorov's theorem, we get that there exists $E_d \subset M$ such that

$$\int_{M \setminus E_d} dv_g < d$$

$(u_m)_m$ converges uniformly to u in E_d. For positive ε, we can take an m large enough so that $|u_m - u| < \frac{\varepsilon}{2}$ on E_d. Let β_ε be the truncation function defined by

$$\beta_\varepsilon(x) = \begin{cases} x & \text{if } |x| < \varepsilon \\ \frac{\varepsilon x}{|x|} & \text{if } |x| \geq \varepsilon \end{cases} \quad (2.0.14)$$

It is easy to verify that

$$g((\mathcal{D}_m - \bar{\mathcal{D}}), \nabla_g (\beta_\varepsilon \circ (u_m - u))) \geq 0$$

almost everywhere in M. Now for m large enough,

$$\begin{aligned}\int_{E_d} g((\mathcal{D}_m-\bar{\mathcal{D}}),\nabla_g(u_m-u))dv_g &= \int_{E_d} g((\mathcal{D}_m-\bar{\mathcal{D}}),\nabla_g(\beta_\varepsilon\circ(u_m-u)))dv_g \\ &\le \int_M g((\mathcal{D}_m-\bar{\mathcal{D}}),\nabla_g(\beta_\varepsilon\circ(u_m-u)))dv_g.\end{aligned}$$

Observing that $\beta_\varepsilon\circ(u_m-u)$ converges weakly to 0 in $H_1^p(M)$, we deduce that,

$$\int_M g(\bar{\mathcal{D}},\nabla(\beta_\varepsilon\circ(u_m-u)))dv_g\to 0$$

On the other hand, for m sufficiently large, since $\beta_\varepsilon\circ(u_m-u)$ is bounded in $H_1^p(M)$, and u_m a Palais-Smale sequence of $J_{f,h,s}$,

$$DJ_{f,h,s}(u_m)(\beta_\varepsilon\circ(u_m-u))=o(1).$$

Then

$$\int_M g(\mathcal{D}_m,\nabla_g(\beta_\varepsilon\circ(u_m-u)))dv_g=o(1)+l_1+l_2$$

with

$$\begin{aligned}|l_1|=\left|\int_M\frac{h}{(\rho_{x_o})^s}|u_m|^{p-2}u_m.\beta_\varepsilon\circ(u_m-u)dv_g\right| &\le \frac{\varepsilon}{2}\int_M\frac{h}{(\rho_{x_o})^s}|u_m|^{p-1}dv_g,\\ |l_2|=\left|\int_M f|u|^{p^*-2}u.\beta_\varepsilon\circ(u_m-u)dv_g\right| &\le \frac{\varepsilon}{2}\int_M f|u|^{p^*-1}dv_g,\end{aligned}$$

then we find that

$$\int_M g(\mathcal{D}_m,\nabla_g(\beta_\varepsilon\circ(u_m-u)))dv_g\le C\varepsilon.$$

Finally, we get that

$$\limsup_{m\to+\infty}\int_{E_d} g((\mathcal{D}_m-\bar{\mathcal{D}}),\nabla_g(u_m-u))dv_g\le C\varepsilon.$$

Since ε is arbitrary, we deduce that $g((\mathcal{D}_m-\bar{\mathcal{D}}),\nabla_g(u_m-u))$ converges to 0 in $L_1(E_d)$, it follows that $g((\mathcal{D}_m-\bar{\mathcal{D}}),\nabla_g(u_m-u))$ has a subsequence that converges almost everywhere to 0 in E_d. By lemma 1.26, we find that $\nabla_g u_m$ converges almost everywhere to $\nabla_g u$ in E_d. Since d is arbitrary, we have convergence almost everywhere of $\nabla_g u_m$ to $\nabla_g u$ in M. Consequently, $|\nabla_g u_m|^{p-2}\nabla_g u_m$ converges almost everywhere to $\bar{\mathcal{D}}$, since $|\nabla_g u_m|^{p-2}\nabla_g u_m$ is bounded in $L_{\frac{p}{p-1}}(M)$, and converges almost everywhere to $\bar{\mathcal{D}}$, so by lemma 1.25, we conclude that $|\nabla_g u_m|^{p-2}\nabla_g u_m$ converges weakly to $\bar{\mathcal{D}}$ in $L_{\frac{p}{p-1}}(M)$, therefore, $\mathcal{D} = \bar{\mathcal{D}}$, which proves (2.0.13), and consequently that u is a weak solution of (E_s).

Now we show that the sequence $v_m = u_m - u$ is a P.S. sequence for $J_{f,h,s}$ at level $\beta_s - J_{f,h,s}(u)$. For $\varphi \in H_1^p(M)$, we write

$$\begin{aligned}
& D(J_{s,f,h}(v_m)).\varphi = D(J_{s,f,h}(u_m)).\varphi - D(J_{s,f,h}(u)).\varphi \qquad (2.0.15)\\
+ & \int_M g(|\nabla_g v_m|^{p-2}\nabla_g v_m - |\nabla_g v_m + \nabla_g u|^{p-2}(\nabla_g v_m + \nabla_g u) + \nabla_g u|^{p-2}\nabla_g u, \nabla_g \varphi) dv_g \\
- & \int_M \frac{h}{\rho_o^s}(|v_m|^{p-2}v_m - |v_m + u|^{p-2}(v_m + u) + |u|^{p-2}u)\varphi dv_g \\
- & \int_M f(|v_m|^{p^*-2}v_m - |v_m + u|^{p^*-2}(v_m + u) + |u|^{p^*-2}u)\varphi dv_g
\end{aligned}$$

The inequality (1.2.6) implies that

$$\begin{aligned}
& \int_M g(|\nabla_g v_m|^{p-2}\nabla_g v_m - |\nabla_g v_m + \nabla_g u|^{p-2}(\nabla_g v_m + \nabla_g u) + \nabla u|^{p-2}\nabla_g u, \nabla_g \varphi) dv_g \\
\leq \; & C\int_M \left(|\nabla_g v_m|^{p-1-\theta}|\nabla_g u|^{\theta} + |\nabla_g v_m|^{\theta}|\nabla_g u|^{p-1-\theta}\right)|\nabla_g \varphi| dv_g \\
\leq \; & C\|\nabla_g \varphi\|_{L_p(M)}\left[\left(\int_M |\nabla_g v_m|^{\frac{p(p-1-\theta)}{p-1}}|\nabla_g u|^{\frac{p\theta}{p-1}} dv_g\right)^{\frac{p-1}{p}}\right. \\
+ & \left.\left(\int_M |\nabla_g v_m|^{\frac{p\theta}{p-1}}|\nabla_g u|^{\frac{p(p-1-\theta)}{p-1}} dv_g\right)^{\frac{p-1}{p}}\right].
\end{aligned}$$

Now, the sequence $|\nabla_g v_m|^{p\frac{p-1-\theta}{p-1}}$ is bounded in $L_{\frac{p-1}{p-1-\theta}}(M)$ and converge almost everywhere to 0 in M. Then, it converges weakly to 0 in $L_{\frac{p-1}{p-1-\theta}}(M)$, which implies that $\int_M |\nabla_g v_m|^{p\frac{p-1-\theta}{p-1}}\varphi dv_g \to 0, \forall \varphi \in L_{\frac{p-1}{\theta}}(M)$. Since $|\nabla_g u|^{p\frac{\theta}{p-1}} \in L_{\frac{p-1}{\theta}}(M)$, we have

$$\int_M |\nabla_g v_m|^{\frac{p(p-1-\theta)}{p-1}}|\nabla_g u|^{\frac{p\theta}{p-1}} dv_g \to 0.$$

By the same method, we also obtain

$$\int_M |\nabla_g v_m|^{\frac{p\theta}{p-1}}|\nabla_g u|^{\frac{p(p-1-\theta)}{p-1}} dv_g \to 0.$$

Similarly, the second and third integrals of (2.0.15) tend to zero. Then $(DJ_{f,h,s}(v_m)).\varphi \to 0, \forall \varphi \in H_1^p(M)$.
Finally, to show that $J_{f,h,s}(v_m)$ goes to $\beta_s - J_{f,h,s}(u)$, we apply the Brezis-Lieb lemma (1.24) for the sequences u_m and $\nabla_g u_m$. Since u_m and $\nabla_g u_m$ converge almost everywhere to u and $\nabla_g u$ in M , and since $\nabla_g u_m$ is bounded in $L_p(M)$, u_m is bounded in $L_{p^*}(M)$, by the Brezis-Lieb lemma we have

$$\int_M |\nabla_g u|^p dv_g = \lim_{m\to\infty} \left(\int_M |\nabla_g u_m|^p dv_g - \int_M |_g(u_m - u)|^p dv_g \right),$$

and

$$\int_M f|u|^{p^*} dv_g = \lim_{m\to\infty} \left(\int_M f|u_m|^{p^*} dv_g - \int_M f|u_m - u|^{p^*} dv_g \right)$$

In addition, by Hardy's inequality (1.2.4) we have

$$\int_M \frac{|u_m|^p}{(\rho_{x_o})^s} dv_g \leq Diam(M)^{p-s} \int_M \frac{|u_m|^p}{(\rho_{x_o})^p} dv_g \leq C\|u_m\|_{H_1^p(M)},$$

which means that the sequence u_m is also bounded in $L_p(M, \rho_{x_o}^s)$ and then we obtain by Brezis-Lieb's lemma that

$$\int_M \frac{h}{(\rho_{x_o})^s}|u|^p dv_g = \lim_{m\to\infty} \left(\int_M \frac{h}{(\rho_{x_o})^s}|u_m|^p dv_g - \int_M \frac{h}{(\rho_{x_o})^s}|u_m - u|^p dv_g \right)$$

which implies that

$$J_{f,h,s}(v_m) = \beta_s - J_{f,h,s}(u) + o(1),$$

and by this, the proof is done. □

The next lemma gives us a level, under which every Palais-Smale sequence converging to 0, converges strongly.

Lemma 2.4. *We assume that* $\sup_M f > 0$ *and* $1 - h(x_o)(\frac{p}{n-p})^p > 0$. *Let* v_m *be a P.S. sequence of* $J_{f,h,s}$ *at level* β_s, $0 < s \leq p$, *that converges weakly to* 0 *in* $H_1^p(M)$. *if*

$$\beta_s < \beta^* = \begin{cases} \dfrac{1}{n(\sup_M f)^{\frac{n-p}{p}} K(n,p)^n}, & \text{if } s < p \\ \dfrac{(1 - h(x_o)(\frac{p}{n-p})^p)^{\frac{n}{p}}}{n(\sup_M f)^{\frac{n-p}{p}} K(n,p)^n}, & \text{if } s = p, \end{cases}$$

then $\beta_s = 0$, *and* v_m *converge strongly to* 0 *in* $H_1^p(M)$.

Proof. First, we have

$$\begin{aligned} DJ_{f,h,s}(v_m).v_m &= o(\|v_m\|_{H_1^p(M)}) \\ &= \int_M (|\nabla_g v_m|^p - \frac{h}{(\rho_{x_o})^s}|v_m|^p)dv_g - \int_M f\,|v_m|^{p^*}\,dv_g, \end{aligned}$$

then

$$\beta_s = \frac{1}{n}\int_M (|\nabla_g v_m|^p - \frac{h}{(\rho_{x_o})^s}|v_m|^p)dv_g + o(1) = \frac{1}{n}\int_M f\,|v_m|^{p^*}\,dv_g + o(1) \quad (2.0.16)$$

which implies that $\beta_s \geq 0$. In addition, for $\delta > 0$ a small constant, we have

$$\begin{aligned} \int_M (|\nabla_g v_m|^p - \frac{h}{(\rho_{x_o})^s}|v_m|^p)dv_g &= \int_M |\nabla_g v_m|^p\,dv_g - \int_{B(x_o,\delta)} \frac{h}{(\rho_{x_o})^s}|v_m|^p\,dv_g \\ &- \int_{M\setminus B(x_o,\delta)} \frac{h}{(\rho_{x_o})^s}|v_m|^p\,dv_g \\ &\geq \int_M |\nabla_g v_m|^p\,dv_g - \max_{x\in B(x_o,\delta)}|h(x_o)|\delta^{p-s}\int_{B(x_o,\delta)} \frac{|v_m|^p}{(\rho_{x_o})^p}dv_g \\ &- \delta^{-s}|\max_{x\in M}|h(x_o)|\int_{M\setminus B(x_o,\delta)} |v_m|^p\,dv_g \end{aligned}$$

Now, since the sequence v_m is bounded in $L_p(M)$ and $L_p(M,(\rho_{x_o})^p)$, we have:
For $0 < s < p$, by letting δ go to 0, we obtain from (2.0.16)

$$\int_M |\nabla_g v_m|^p\,dv_g \leq n\beta_s + o(1). \quad (2.0.17)$$

For $s = p$, by letting δ go to 0, we obtain from (2.0.16) and Hardy's inequality (1.2.5)

$$\int_M |\nabla_g v_m|^p\,dv_g \leq \frac{n\beta_s}{1 - h(x_o)(\frac{p}{n-p})^p} + o(1), \quad (2.0.18)$$

On the other hand, by Sobolev's inequality and (2.0.16), we also obtain by that for $0 < s \leq p$,

$$\int_M |\nabla_g v_m|^p\,dv_g \geq \Big(\frac{n\beta_s}{(\sup_M f)\,(K(n,p)+\varepsilon)^{p^*}}\Big)^{\frac{p}{p^*}} + o(1) \quad (2.0.19)$$

Now, suppose by contradiction that $\beta_s > 0$. Then, after passing m to ∞, the inequalities (2.0.17), (2.0.18) and (2.0.19) gives

$$\beta_s \geq \frac{1}{n\,(\sup_M f)^{\frac{n-p}{p}}\,(K(n,p)+\varepsilon)^n}, \text{ for } 0 < s < p,$$

and

$$\beta_s \geq \frac{(1-(h(x_o)K^p(n,p,-p))^{\frac{n}{p}}}{n(\sup_M f)^{\frac{n-p}{p}}(K(n,p))^n}, \text{ for } s=p.$$

Both cases present a clear contradiction with the hypothesis of the lemma. Therefore, under the assumption of the lemma, $\beta_s = 0$ and hence $v_m \to 0$ in $H_1^p(M)$. □

Now we divide the proof of the main theorems into two parts depending on whether $0 < s < p$ or $s = p$, starting with the case $0 < s < p$.

2.1 The subcritical Hardy potential

Lemma 2.5. *Let v_m be a P.S. of $J_{f,h,s}$, with $0 < s < p$, at level β_s that converges weakly and not strongly to 0 in $H_1^p(M)$. Then, there exists a convergent sequence of points $x_m \to x^o$ dans M, a sequence of positive reals $R_m \to 0$ as $m \to \infty$ and a non trivial weak solution $v \in D^{1,p}(I\!R^n)$ of*

$$\Delta_{\xi,p} v = f(x^o)|v|^{p^*-2}v, \tag{2.1.1}$$

such that the sequence,

$$w_m(x) = v_m(x) - R_m^{\frac{p-n}{p}}\eta_\delta(\exp_{x_m}^{-1}(x))v(R_m^{-1}\exp_{x_m}^{-1}(x)),$$

where $0 < \delta < \frac{Inj_g}{2}$, possesses a subsequence w_m, that is a P.S. sequence of $J_{f,h,s}$, with $0 < s < p$, at level $J_{f,h,s}(w_m) = \beta_s - (f(x^o))^{\frac{p-n}{p}}E(u) + o(1)$, with u is a non trivial weak solution of (2.0.7), and converges weakly to 0 in $H_1^p(M)$.

Proof. Let v_m be a P.S. sequence of $J_{f,h,s}$ at level β_s that converges to 0 weakly and not strongly in $H_1^p(M)$. Then, up to a subsequence, we can assume that v_m converges strongly to 0 in $L_p(M)$. For $t > 0$, we assume that

$$F_m(t) = \max_{x\in M}\int_{B(x,t)} |\nabla_g v_m| dv_g.$$

For t_o small, by the (2.0.19), there exists z_o in M and $\gamma_o > 0$ such that

$$\int_{B(z_o,t_o)} |\nabla_g v_m| dv_g \geq \gamma_o.$$

Since F_m is continuous in t, we obtain that for each $\gamma \in (0, \gamma_o)$ and for each $m > 0$, we can find a point x_m and a constant $r_m \in (0, t_o)$ such that

$$\int_{B(x_m, r_m)} |\nabla_g v_m|^p \, dv_g = \gamma \tag{2.1.2}$$

Let $0 < r_o < \frac{Inj_g}{2}$ such that there exists a positive constant $C_o \in [1, 2]$, so that for all $x \in M$ and $y, z \in B(r_o) \subset I\!R^n$, the following inequality holds true

$$dist_g(\exp_x(y), \exp_x(z)) \leq C_0 |y - z|. \tag{2.1.3}$$

Let $0 < R_m < 1$ and $x \in B(R_m^{-1} Inj_g)$. We define

$$\begin{aligned} \hat{v}_m(x) &= R_m^{\frac{n-p}{p}} v_m(\exp_{x_m}(R_m x)), \; x \in I\!R^n \\ \hat{g}_m(x) &= \exp_{x_m}^* g(R_m x) \end{aligned}$$

Then we have

$$|\nabla_{\hat{g}_m} \hat{v}_m|^p_{\hat{g}_m}(x) = R_m^n |\nabla_g v_m|^p_g(\exp_{x_m}(R_m x)) \tag{2.1.4}$$

Thus, it follows that if $z \in I\!R^n$ is such that $|z| + r < Inj_g R_m^{-1}$, then we have

$$\int_{B(z,r)} |\nabla_{\hat{g}_m} \hat{v}_m|^p_{\hat{g}_m} dv_{\hat{g}_m} = \int_{\exp_{x_m}(R_m B(z,r))} |\nabla_g v_m|^p_g dv_g. \tag{2.1.5}$$

Furthermore, for $|z| + r < r_o R_m^{-1}$, using (2.1.3) we have

$$\exp_{x_m}(R_m B(z,r)) \subset B_{\exp_{x_m}(R_m z)}(r C_o R_m) \tag{2.1.6}$$

Since for $y \in B(r C_o R_m) \subset B(Inj_g)$, we have $dist_g(x_m, \exp_{x_m}(R_m y)) = R_m |y|$, which gives us

$$\exp_{x_m}(B(r C_o R_m)) = B(x_m, r C_o R_m). \tag{2.1.7}$$

Now, for $r \in (0, r_o)$, we consider that $R_m = \frac{r_m}{r C_o}$, where r_m is as defined above. By (2.1.4), (2.1.5) and (2.1.6), we obtain

$$\int_{B(z,r)} |\nabla_{\hat{g}_m} \hat{v}_m|^p_{\hat{g}_m} dv_{\hat{g}_m} \leq \gamma, \tag{2.1.8}$$

and

$$\int_{B(rC_o)} |\nabla_{\hat{g}_m} \hat{v}|^p_{\hat{g}_m} dv_{\hat{g}_m} = \gamma, \tag{2.1.9}$$

Let $\delta \in (0, Inj_g)$ and $u \in D^{1,2}(I\!R^n)$ with support included in $B(\delta R^{-1})$, where $0 < R \leq 1$ is a constant. There exists a constant C_1 such that if $\hat{g}(x) = \exp_p^*(g(Rx))$, then

$$\frac{1}{C_1} \int_{\mathbb{R}^n} |\nabla u|^p \, dx \leq \int_{\mathbb{R}^n} |\nabla_{\hat{g}} u|^p \, dv_{\hat{g}} \leq C_1 \int_{\mathbb{R}^n} |\nabla u|^p \, dx. \tag{2.1.10}$$

Without loss of generality, we can also assume that for any $u \in L_1(I\!R^n)$ with support in $B(\delta R^{-1})$, we have

$$\frac{1}{C_1} \int_{\mathbb{R}^n} |u|^p \, dx \leq \int_{\mathbb{R}^n} |u|^p \, dv_{\hat{g}} \leq C_1 \int_{\mathbb{R}^n} |u|^p \, dx. \tag{2.1.11}$$

Now, we consider a cut-off function $\eta \in C_o(I\!R^n)$ such that

$$0 \leq \eta \leq 1, \eta(x) = 1, x \in B(\frac{1}{4}) \text{ and } \eta(x) = 0, x \in B(\frac{3}{4}). \tag{2.1.12}$$

We put $\hat{\eta}_m(x) = \eta(\delta^{-1} R_m x)$, where $\delta \in (0, Inj_g)$. We obtain that there exists a positive constant C such that

$$\begin{aligned}
&\int_{I\!R^n} |\nabla_{\hat{g}_m} (\hat{\eta}_m \hat{v}_m)|^p \, dv_{\hat{g}_m} = \int_{B(\frac{3\delta R_m^{-1}}{4})} |\nabla_{\hat{g}_m} (\hat{\eta}_m \hat{v}_m)|^p \, dv_{\hat{g}_m} \\
\leq \; & 2^{p-1} \int_{B(\frac{3\delta R_m^{-1}}{4}))} \left(|\eta(\delta^{-1} R_m x)|^p \, |\nabla_{\hat{g}_m} \hat{v}_m|^p + \delta^{-p} R_m^p |(\nabla_{\hat{g}_m} \eta)(\delta^{-1} R_m x)|^p \, |\hat{v}_m|^p \right) dv_{\hat{g}_m} \\
= \; & 2^{p-1} \int_{B(x_m, \frac{3\delta}{4})} \left(|\eta(\delta^{-1} \exp_{x_m}^{-1}(x))|^p \, |\nabla_g v_m|^p + |(\nabla_g \eta)(\delta^{-1} \exp_{x_m}^{-1}(x))|^p \, |v_m|^p \right) dv_g \\
\leq \; & C \int_{B(x_m, \frac{3\delta}{4})} (|\nabla_g v_m|^p + |v_m|^p) \, dv_g.
\end{aligned}$$

Since the sequence is bounded in $H_1^p(M)$, this implies by (2.1.10) that the sequence $\hat{\eta}_m \hat{v}_m$ is bounded in $D^{1,p}(I\!R^n)$ and therefore converges weakly in $D^{1,p}(I\!R^n)$ and almost everywhere in $\mathbb{R}^n$ to some function $v \in D^{1,p}(I\!R^n)$.

Now, we divide the rest of the proof of the lemma into steps.

step 1

For γ small and $s \in (0,p)$, the sequence $\hat{\eta}_m\hat{v}_m$ converges strongly to v in $H_1^p(B(C_o r))$.

Proof of step 1. Let $a \in I\!R^n$ and $\mu \in [r, 2r]$. We define $A = B(a,3r) \setminus B(a,\mu)$. In [39](see also [18]), it was proved that there exists a sequence $z_m \in H_1^p(\mathcal{A})$ that converges strongly to 0 in $H_1^p(\mathcal{A})$ and that z_m is a solution of

$$\begin{cases} \Delta_{\xi,p} z_m = 0 \text{ in } \mathcal{A}, \\ z_m - \varphi_m - \varphi_m^o \in D^{1,p}(\mathcal{A}), \end{cases} \tag{2.1.13}$$

where $\varphi_m = \hat{\eta}_m\hat{v}_m - v$ in $B(a,\mu+\varepsilon)$, $\varphi_m = 0$ in $I\!R^n \setminus B(a, 3\mu - \varepsilon)$ and φ_m^o is such that $\|\varphi_m + \varphi_m^o\|_{H_1^p(\mathcal{A})} \le C\|\varphi_m\|_{H^p_{\frac{p-1}{p}}(\partial\mathcal{A})}$. We let $\hat{\psi}_m \in D^{1,p}(I\!R^n)$ be the sequence

$$\hat{\psi}_m = \begin{cases} \hat{\eta}_m\hat{v}_m - v & \text{in } \overline{B}(a,\mu), \\ z_m & \text{in } \overline{B}(a,3r) \setminus B(a,\mu), \\ 0 & \text{in } I\!R^n \setminus B(a,3r). \end{cases}$$

For $r < \frac{\delta}{24}$, we consider the rescaling sequence ψ_m de $\hat{\psi}_m$

$$\begin{cases} \psi_m(x) = R_m^{\frac{p-n}{p}}\hat{\psi}_m(R_m^{-1}\exp_{x_m}^{-1}(x)), & \text{if } x < d_g(x_m, 6r), \\ \psi_m(x) = 0, & \text{otherwise.} \end{cases}$$

Let η be the cut-off function considered above. Then , $\eta(\delta^{-1}\exp_{x_m}^{-1}(x)) = 1$ For x such that $d_g(x_m, x) < 6r$. We put $\hat{\eta}_m(x) = \eta(\delta^{-1}\exp_{x_m}^{-1}(x))$, in addition if we have $|a| < 3r$, then we get

$$\begin{aligned} DJ_{f,h,s}(v_m).\psi_m &= DJ_{f,h,s}(\eta(\delta^{-1}\exp_{x_m}^{-1}(x))v_m).\psi_m \\ &= \int_{B(a,3r)} |\nabla_{\hat{g}_m}(\hat{\eta}_m\hat{v}_m)|^{p-2}\hat{g}\left(\nabla_{\hat{g}_m}(\hat{\eta}_m\hat{v}_m), \nabla_{\hat{g}_m}\hat{\psi}_m\right) dv_{\hat{g}_m} \\ &- R_m^{p-s}\int_{B(a,3r)} \frac{h(\exp_{x_m}(R_m(x))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m} \\ &- \int_{B(a,3r)} f(\exp_{x_m}(R_m(x))|\hat{\eta}_m\hat{v}_m|^{p^*-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}. \end{aligned}$$

It is clear that the sequence $\hat{\psi}_m$ is bounded in $D^{1,p}(I\!R^n)$ and we have that $||\psi_m||_{H_1^p(M)} \leq C||\hat{\psi}_m||_{D^{1,p}(I\!R^n)}$. Then the sequence ψ_m is bounded in $H_1^p(M)$ and since v_m is a P.S. sequence of $J_{f,h,s}$, we have

$$\begin{aligned} o(1) = & \int_{B(a,3r)} |\nabla_{\hat{g}_m}(\hat{\eta}_m\hat{v}_m)|^{p-2}\hat{g}\left(\nabla_{\hat{g}_m}(\hat{\eta}_m\hat{v}_m), \nabla_{\hat{g}_m}\hat{\psi}_m\right) dv_{\hat{g}_m} \\ - & R_m^{p-s}\int_{B(a,3r)} \frac{h(\exp_{x_m}(R_m(x))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m} \\ - & \int_{B(a,3r)} f(\exp_{x_m}(R_m(x))|\hat{\eta}_m\hat{v}_m|^{p^*-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}. \end{aligned} \tag{2.1.14}$$

With the same arguments as those used in [39], we can have

$$\int_{B(a,3r)} |\nabla_{\hat{g}_m}(\hat{\eta}_m\hat{v}_m)|^{p-2}\hat{g}\left(\nabla_{\hat{g}_m}(\hat{\eta}_m\hat{v}_m), \nabla_{\hat{g}_m}\hat{\psi}_m\right) dv_{\hat{g}_m} = \int_{I\!R^n} |\nabla_{\hat{g}_m}\hat{\psi}_m|^p dv_{\hat{g}_m} + o(1),$$

and

$$\begin{aligned} & \int_{B(a,3r)} f(\exp_{x_m}(R_m x))|\hat{\eta}_m\hat{v}_m|^{p^*-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m} \\ = & \int_{I\!R^n} f(\exp_{x_m}(R_m x))|\hat{\psi}_m|^{p^*} dv_{\hat{g}_m} + o(1). \end{aligned}$$

Instead, we prove that

$$\begin{aligned} & \int_{B(a,3r)} \frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m} \\ = & \int_{I\!R^n} \frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}|\hat{\psi}_m|^p dv_{\hat{g}_m} + o(1) \end{aligned} \tag{2.1.15}$$

We distinguish between two cases : $0 \in B(a,\mu)$ and $0 \notin B(a,\mu)$.
Si $0 \notin B(a,\mu)$, then there exists $\varrho >$ such that, $B(\varrho) \cap B(a,\mu) = \emptyset$. Then, by

using convexity, the Hölder inequality (1.2.6) inequality, we find

$$
\begin{aligned}
&\left|\int_{B(a,\mu)} \frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}\left[|\hat{\psi}_m+v|^{p-2}(\hat{\psi}_m+v)-|\hat{\psi}_m|^{p-2}\hat{\psi}_m-|v|^{p-2}v\right]\hat{\psi}_m dv_{\hat{g}_m}\right| \\
\leq\ & C\varrho^{-s}\sup|h|\|\hat{\psi}_m\|_{L_p(\mathbb{R}^n)}\left(\int_{B(a,\mu)}\left[|\,|\hat{\psi}_m+v|^{p-2}(\hat{\psi}_m+v)-|\hat{\psi}_m|^{p-2}\hat{\psi}_m-|v|^{p-2}v\,|\right]^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}} \\
\leq\ & C'\|\hat{\psi}_m\|_{L_p(\mathbb{R}^n)}\left(\int_{B(a,\mu)}\left[|\,|\hat{\psi}_m|^{p-1-\theta}|v|^{\theta}-\hat{\psi}_m|^{\theta}|v|^{p-1-\theta}\,|\right]^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}} \\
\leq\ & C''\|\hat{\psi}_m\|_{L_p(\mathbb{R}^n)}\left[\left(\int_{B(a,\mu)}|\hat{\psi}_m|^{\frac{p(p-1-\theta)}{p-1}}|v|^{\frac{p\theta}{p-1}}dx\right)^{\frac{p-1}{p}}+\left(\int_{B(a,\mu)}|\hat{\psi}_m|^{\frac{p\theta}{p-1}}|v|^{\frac{p(p-1-\theta)}{p-1}}dx\right)^{\frac{p-1}{p}}\right].
\end{aligned}
$$

Since $\hat{\psi}_m$ converges to 0 almost everywhere and is bounded in $L_p(\mathbb{R}^n)$, we get that $|\hat{\psi}_m|^{\frac{p(p-\theta-1)}{p-1}}$ and $|\hat{\psi}_m|^{\frac{p\theta}{p-1}}$ converge almost everywhere to 0 and are bounded respectively in $L_{\frac{p-1}{p-1-\theta}}(\mathbb{R}^n)$ and $L_{\frac{p-1}{\theta}}(\mathbb{R}^n)$. We get then

$$
\left(\int_{B(a,\mu)}|\hat{\psi}_m|^{\frac{p(p-1-\theta)}{p-1}}|v|^{\frac{p\theta}{p-1}}dx\right)^{\frac{p-1}{p}}+\left(\int_{B(a,\mu)}|\hat{\psi}_m|^{\frac{p\theta}{p-1}}|v|^{\frac{p(p-1-\theta)}{p-1}}dx\right)^{\frac{p-1}{p}}=o(1).
$$

Hence, we get

$$
\begin{aligned}
&\int_{B(a,\mu)}\frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m} \\
=\ &\int_{B(a,\mu)}\frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}\left[|\hat{\psi}_m|^p+|v|^{p-2}v\hat{\psi}_m\right]dv_{\hat{g}_m}+o(1).
\end{aligned}
$$

Now, if $0\in B(a,\mu)$, let $\varrho'>0$ be such that $B(\varrho')\subset B(a,\mu)$. Then, as above we have

$$
\begin{aligned}
&\int_{B(a,\mu)\backslash B(\varrho')}\frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m} \\
=\ &\int_{B(a,\mu)\backslash B(\varrho')}\frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}\left[|\hat{\psi}_m|^p+|v|^{p-2}v\hat{\psi}_m\right]dv_{\hat{g}_m}+o(1).
\end{aligned}
$$

Moreover, by Hölder inequality we have

$$\begin{aligned}
&\int_{B(\varrho')} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\eta}_m \hat{v}_m|^{p-2} (\hat{\eta}_m \hat{v}_m) \hat{\psi}_m dv_{\hat{g}_m} \\
\leq\ & C \sup |h| \left(\int_{B(\varrho')} \frac{|\hat{\eta}_m \hat{v}_m|^p}{|x|_\xi^s} dx \right)^{\frac{1}{p}} \left(\int_{B(\varrho')} \frac{|\hat{\psi}_m|^p}{|x|_\xi^s} dx \right)^{1-\frac{1}{p}} \\
\leq\ & C \sup |h| \varrho'^{\frac{p-s}{p}} \left(\int_{B(\varrho')} \frac{|\hat{\eta}_m \hat{v}_m|^p}{|x|_\xi^p} dx \right)^{\frac{1}{p}} \left(\int_{B(\varrho')} \frac{|\hat{\psi}_m|^p}{|x|_\xi^s} dx \right)^{1-\frac{1}{p}}.
\end{aligned}$$

Now, by Hardy inequality (1.2.3), $\left(\int_{B(\varrho')} \frac{|\hat{\eta}_m \hat{v}_m|^p}{|x|_\xi^p} dx \right)^{\frac{1}{p}}$ is bounded. Since $\hat{\psi}_m$ converges to 0 strongly in $L_p(B(\varrho'), |x|^s)$, $0 < s < p$, then

$$\int_{B(\varrho')} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\eta}_m \hat{v}_m|^{p-2} (\hat{\eta}_m \hat{v}_m) \hat{\psi}_m dv_{\hat{g}_m} = o(1).$$

Thus, in both cases we have

$$\begin{aligned}
&\int_{B(a,\mu)} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\eta}_m \hat{v}_m|^{p-2} (\hat{\eta}_m \hat{v}_m) \hat{\psi}_m dv_{\hat{g}_m} \\
=\ & \int_{B(a,\mu)} \frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s} \left[|\hat{\psi}_m|^p + |v|^{p-2} v \hat{\psi}_m \right] dv_{\hat{g}_m} + o(1).
\end{aligned}$$

Now, using the fact that $\hat{\psi}_m$ converges to 0 strongly in $D^{1,p}(\mathcal{A})$ and weakly to 0 in $D^{1,p}(\mathbb{R}^n)$, we get

$$\begin{aligned}
&\int_{B(a,3r)} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\eta}_m \hat{v}_m|^{p-2} (\hat{\eta}_m \hat{v}_m) \hat{\psi}_m dv_{\hat{g}_m} \\
=\ & \int_{B(a,\mu)} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\eta}_m \hat{v}_m|^{p-2} (\hat{\eta}_m \hat{v}_m) \hat{\psi}_m dv_{\hat{g}_m} \\
=\ & \int_{B(a,\mu)} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} \left[|\hat{\psi}_m|^p + |v|^{p-2} v \hat{\psi}_m \right] dv_{\hat{g}_m} + o(1) \\
=\ & \int_{\mathbb{R}^n} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\psi}_m|^p dv_{\hat{g}_m} + o(1).
\end{aligned}$$

We deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla_{\hat{g}_m} \hat{\psi}_m|^p dv_{\hat{g}_m} - R_m^{p-s} \int_{\mathbb{R}^n} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\psi}_m|^p dv_{\hat{g}_m} \\
=\ & \int_{\mathbb{R}^n} f(\exp_{x_m}(R_m x)) |\hat{\psi}_m|^{p^*} dv_{\hat{g}_m} + o(1).
\end{aligned}$$

Since the sequence $\hat{\psi}_m$ converges strongly to 0 in $L_p(B(a,3\mu),|x|^s)$, $s<p$ and since $R_m \le 1$, we get that

$$R_m^{p-s}\left|\int_{I\!R^n} \frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}|\hat{\psi}_m|^p dv_{\hat{g}_m}\right| \le \sup hC\int_{I\!R^n}\frac{|\hat{\psi}_m|^p}{|x|_\xi^s}dx = o(1).$$

We get then

$$\int_{I\!R^n}|\nabla_{\hat{g}_m}\hat{\psi}_m|^p dv_{\hat{g}_m} = \int_{I\!R^n} f(\exp_{x_m}(R_m x))|\hat{\psi}_m|^{p^*}dv_{\hat{g}_m} + o(1). \qquad (2.1.16)$$

By the same way as in [39], we can prove that for $|a|+3r<r_o$

$$\int_{I\!R^n}|\nabla_{\hat{g}_m}\hat{\psi}_m|^p dv_{\hat{g}_m} \le N\gamma + o(1), \qquad (2.1.17)$$

where $N\in\mathbb{N}$ is such that $B(a,\mu)\subset B(a,2r)\subset \bigcup_{1\le i\le N} B(x_i,r)$, with $x_i\in B(a,2r)$. We get then by the Sobolev inequality, that

$$\begin{aligned}\int_{I\!R^n} f(\exp_{x_m}(R_m x))|\hat{\psi}_m|^{p^*}dv_{\hat{g}_m} &\le \sup_M fC_1\int_{I\!R^n}|\hat{\psi}_m|^{p^*}dx\\ &\le \sup_M fC_1^{\frac{p^*}{p}+1}K(n,p)^{p^*}\left(\int_{I\!R^n}|\nabla_{\hat{g}_m}\hat{\psi}_m|^p dv_{\hat{g}_m}\right)^{\frac{p^*}{p}}.\end{aligned}$$

Then, by (2.1.16) and (2.1.17), we get

$$\begin{aligned}\int_{I\!R^n}|\nabla_{\hat{g}_m}\hat{\psi}_m|^p dv_{\hat{g}_m} &\le \sup_M fC_1\int_{I\!R^n}|\hat{\psi}_m|^{p^*}dx\\ &\le \sup_M fC_1^{\frac{p^*}{p}+1}K(n,p)^{p^*}(N\gamma+o(1))^{\frac{p^*}{p}-1}\int_{I\!R^n}|\nabla_{\hat{g}_m}\hat{\psi}_m|^p dv_{\hat{g}_m}.\end{aligned}$$

By taking γ such that

$$\sup_M fC_1^{\frac{p^*}{p}+1}K(n,p)^{p^*}(N\gamma)^{\frac{p^*}{p}-1}<1, \qquad (2.1.18)$$

we get

$$\int_{I\!R^n}|\nabla_{\hat{g}_m}\hat{\psi}_m|^p dv_{\hat{g}_m} = o(1),$$

which means that $\hat{\psi}_m$ converges strongly in $D^{1,p}(I\!R^n)$. Thus, since $r\le\mu$, we get that $\hat{\eta}_m\hat{v}_m$ converges strongly to v in $H_1^p(B(a,r))$. This strong convergence holds as soon as μ and r are small enough, $|a|<3r$ and $|a|+3r<\min(r_o,\delta)$. Then, let μ be small enough such that condition (2.1.18), then $\hat{\eta}_m\hat{v}_m$ converges strongly to v in $H_1^p(B(a,r))$ for all $|a|<2r$. Since $C_o\le 2$, $B(C_o r)$ can be covered by N balls $B(a,r)$, with $a\in B(2r)$ and thus $\hat{\eta}_m\hat{v}_m$ converges strongly to v in $H_1^p(B(C_o r))$. $\square$

Step 2

For any $R > 0$ and $s \in (0, p)$ the sequence $\hat{v}_m$ converges strongly to v in $H_1^p(B(R))$ and v is a nontrivial solution of (2.1.1).

Proof. First, to prove that $v \neq 0$, we use step 1 above. Take r small enough so that $\hat{\eta}_m = 1$ on $B(C_o r)$, we then obtain

$$\begin{aligned}\gamma &= \int_{B(C_o r)} |\nabla_{\hat{g}_m}(\hat{\eta}_m \hat{v}_m)|^p dv_{\hat{g}_m} \\ &\leq \int_{B(C_o r)} |\nabla v|^p dx + o(1).\end{aligned}$$

Hence $v \neq 0$. As consequence, we get that $R_m \to 0$.
In fact, if $R_m \to R > 0$. Since v_m converges weakly to 0, we get that $\hat{v}_m$ converges weakly to 0 in $H_1^p(B(C_o r))$. Since $v \neq 0$ and $(\hat{\eta}_m \hat{v}_m)$ converges strongly to v in $H_1^p(B(C_o r))$, we get a contradiction. Thus $R_m \to 0$.
Now, let $R > 1$. For m large, $R < R_m^{-1}$ and (2.1.8) and (2.1.9) are satisfied for $z + r < R r_o$. Thus, as one can easily check from the proof of Step 1, $\hat{\eta}_m \hat{v}_m$ converges strongly to v in $H_1^p(B(a, r))$ for $|a| + 3r < rR$ and $|a| \leq 3r(2R - 1)$. In particular, $\hat{\eta}_m \hat{v}_m$ converges strongly to v in $H_1^p(B(a, r))$ for $|a| < 2rR$. Hence $\hat{\eta}_m \hat{v}_m$ converges strongly to v in $H_1^p(B(2rR))$. Since for m large, $\hat{\eta}_m = 1$ and R is arbitrary chosen, we get that $\hat{v}_m$ converges strongly to v in $H_1^p(B(R))$.
Now, let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with compact support included in a ball $B(R), R > 0$. For m large, define on M the sequence φ_m as

$$\varphi_m(x) = R_m^{\frac{p-n}{p}} \varphi(R_m^{-1}(\exp_{x_m}^{-1}(x))).$$

Then, we have

$$\int_M |\nabla_g v_m|^{p-2} g(\nabla_g v_m, \nabla_g \varphi_m) dv_g = \int_{\mathbb{R}^n} |\nabla_{\hat{g}_m}(\hat{\eta}_m \hat{v}_m)|^{p-2} \hat{g}(\nabla_{\hat{g}_m}(\hat{\eta}_m \hat{v}_m), \nabla_{\hat{g}_m} \varphi) dv_{\hat{g}_m}. \tag{2.1.19}$$

Knowing that $d_g(y, \exp_y(R_m x)) = R_m |x|$, we have

$$d_g(x_o, x_m) - R_m|x| \leq d_g(x_o, \exp_{x_m}(R_m x)) \leq d_g(x_o, x_m) + R_m|x|. \tag{2.1.20}$$

Suppose that $x_m \to x_o$ as $m \to \infty$. Then, either $\frac{R_m}{d_g(x_o,x_m)} \to 0$ as $m \to \infty$, then $\frac{d_g(x_o,\exp_{x_m}(R_m x))}{d_g(x_o,x_m)} \to 1$ as $m \to \infty$ and consequently

$$\frac{R_m}{d_g\left(x_o, \exp_{x_m}(R_m x)\right)} \to 0 \text{ as } m \to \infty,$$

or $\frac{R_m}{d_g(x_o,x_m)} \to A > 0$ as $m \to \infty$. Then, always by (2.1.20), we get

$$\frac{1}{\frac{1}{A}+|x|} \leq \lim_{m\to\infty} \frac{R_m}{d_g\left(x_o, \exp_{x_m}(R_m x)\right)} \leq \frac{1}{\frac{1}{A}-|x|}.$$

Hence, by writing

$$\begin{aligned}&\int_M \frac{h}{\rho_{x_o}^s}|v_m|^{p-2}v_m\varphi_m dv_g \\ &\qquad = R_m^{p-s}\int_{\mathbb{R}^n}\frac{R_m^s}{d_g(x_o,\exp_{x_m}(R_m x))^s}h(\exp_{x_o}(R_m x))|(\hat\eta_m\hat v_m)|^{p-2}(\hat\eta_m\hat v_m)\varphi dv_{\hat g_m},\end{aligned}$$

and

$$\int_M f|v_m|^{p^*-2}v_m\varphi_m dv_g = \int_{\mathbb{R}^n} f(\exp_{x_m}(R_m x))|(\hat\eta_m\hat v_m)|^{p^*-2}(\hat\eta_m\hat v_m)\varphi dv_{\hat g_m}. \quad (2.1.21)$$

Since $\hat g_m \to \xi$ in $C^1(B(R))$ for any $R > 0$, the sequence φ_m is bounded in $H_1^p(M)$, the sequence v_m is a P-S sequence of $J_{f,h,s}$ and the sequence $\hat\eta_m\hat v_m$ converges strongly to $v \neq 0$ in $D^{1,p}(\mathbb{R}^n)$, by passing to the limit we get that v is a weak solution of

$$\Delta_{\xi,p}v = f(x^o)|v|^{p^*-2}v.$$

□

Step 3

Let $w_m = v_m - \mathcal{B}_m$, with

$$\mathcal{B}_m(x) = R_m^{\frac{p-n}{p}}\eta_{\delta,x_m}(x)v(R_m^{-1}\exp_{x_m}^{-1}(x)), \quad (2.1.22)$$

where $\eta_{\delta,x_m}(x) = \eta_\delta(\exp_{x_m}^{-1}(x))$. Then, the following statements hold

$$\mathcal{B}_m \text{ converges weakly to } 0 \text{ in } H_1^p(M), \quad (2.1.23)$$

$$DJ_{f,h,s}(\mathcal{B}_m) \to 0, DJ_{f,h,s}(w_m) \to 0 \text{ strongly,} \tag{2.1.24}$$

and

$$J_{f,h,s}(w_m) = J_{f,h,s}(v_m) - (f(x^o))^{\frac{p-n}{p}} E(u), \tag{2.1.25}$$

with $u = (f(x^o))^{\frac{n-p}{p^2}} v$ is a nontrivial weak solution of (2.0.7).

Proof. The proof of (2.1.23) is identical to that of statement (14) of Step 2.4 in [39] and thus we omit it. We prove (2.1.24). Let $\varphi \in H_1^p(M)$. For $x \in B(\delta R_m^{-1})$ put $\varphi_m(x) = R_m^{\frac{n-p}{p}} \varphi(\exp_{x_m}(R_m x))$ and $\bar{\varphi}_m = \eta_\delta(R_m x)\varphi_m(x)$. Let $R > 0$ be a constant, we have

$$\int_M |\nabla_g \mathcal{B}_m|^{p-2} g(\nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g = \int_{B(x_m, R_m R)} |\nabla_g \mathcal{B}_m|^{p-2} g(\nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g$$
$$+ \int_{B(x_m, 2\delta)\setminus B(x_m, R_m R)} |\nabla_g \mathcal{B}_m|^{p-2} g(\nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g.$$

Direct computations give

$$\int_{B(x_m, 2\delta)\setminus B(x_m, R_m R)} |\nabla_g \mathcal{B}_m|^{p-2} g(\nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g = O(||\varphi||_{H_1^p(M)})\varepsilon(R),$$

where $\varepsilon(R) \to 0$ as $R \to \infty$.
For m large, we have

$$\int_{B(x_m, R_m R)} |\nabla_g \mathcal{B}_m|^{p-2} g(\nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g = \int_{B(R)} |\nabla_{\hat{g}_m} v|^{p-2} \hat{g}(\nabla_{\hat{g}_m} v, \nabla_{\hat{g}_m} \overline{\varphi}_m) dv_{\hat{g}_m}$$

knowing that

$$\int_{B(x_m, R_m R)} |\nabla_g \varphi|^p dv_g = \int_{B(R)} |\nabla_{\hat{g}_m} \varphi_m|^p dv_{\hat{g}_m},$$

and that the sequence of metrics $\hat{g}_m$ converges in $C^1(B(R')), R' > R$, we get that

$$\int_{B(x_m, R_m R)} |\nabla_g \mathcal{B}_m|^{p-2} g(\nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g$$
$$= \int_{B(R)} |\nabla_{\hat{g}_m} v|^{p-2} \hat{g}(\nabla_{\hat{g}_m} v, \nabla_{\hat{g}_m} \overline{\varphi}_m) dx + o(||\varphi||_{H_1^p(M)}).$$
$$= \int_{\mathbb{R}^n} |\nabla v|_\xi^{p-2} \nabla v . \nabla \overline{\varphi}_m dx + o(||\varphi||_{H_1^p(M)}) + O(||\varphi||_{H_1^p(M)})\varepsilon(R),$$

where $\varepsilon(R) \to 0$ as $R \to \infty$.Thus

$$
\begin{aligned}
&\int_M |\nabla_g \mathcal{B}_m|^{p-2} g(\nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g \qquad (2.1.26)\\
=\ &\int_{I\!R^n} |\nabla v|_\xi^{p-2} \nabla v . \nabla \overline{\varphi}_m dx + o(||\varphi||_{H_1^p(M)}) + O(||\varphi||_{H_1^p(M)})\varepsilon(R),
\end{aligned}
$$

By the same way, we get that

$$
\begin{aligned}
&\int_M f(x) |\mathcal{B}_m|^{p^*-2} \mathcal{B}_m \varphi dv_g \qquad (2.1.27)\\
=\ &f(x^o) \int_{I\!R^n} |v|^{p^*-2} v \overline{\varphi}_m dx + o(||\varphi||_{H_1^p(M)}) + O(||\varphi||_{H_1^p(M)})\varepsilon(R).
\end{aligned}
$$

Since the sequence $\mathcal{B}_m$ converges to 0 weakly in $H_1^p(M)$ and the inclusion $H_1^p(M) \subset L_p(M, (\rho_{x_o})^s)$ is compact for $s \in (0, p)$, we can assume that $\mathcal{B}_m \to 0$ in $L_p(M, (\rho_{x_o})^s)$. Then, using the fact that v is a weak solution of $\Delta_{\xi,p} v = f(x^o)|v|^{p^*-2} v$, we get

$$
DJ_{f,h,s}(\mathcal{B}_m).\varphi = o(||\varphi||_{H_1^p(M)}) + O(||\varphi||_{H_1^p(M)})\varepsilon(R).
$$

Since R arbitrary, we get that $DJ_{f,h,s}(\mathcal{B}_m) \to 0$. This proves the first part of (2.1.24). For the proof of the second part of (2.1.24), we write

$$
DJ_{f,h,s}(w_m) = DJ_{f,h,s}(v_m) - DJ_{f,h,s}(\mathcal{B}_m) + \mathcal{A}_m.\varphi + \mathcal{C}_m \varphi + \mathcal{D}_m \varphi,
$$

where

$$
\begin{aligned}
\mathcal{A}_m.\varphi &= \int_M g(|\nabla_g w_m|^{p-2} \nabla_g w_m - |\nabla_g v_m|^{p-2} \nabla_g v_m + |\nabla_g \mathcal{B}_m|^{p-2} \nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g,\\
\mathcal{C}_m \varphi &= \int_M \frac{h}{(\rho_{x_o})^s} \left(|w_m|^{p-2} w_m + |v_m|^{p-2} v_m - |\mathcal{B}_m|^{p-2} \mathcal{B}_m \right) .\varphi dv_g,
\end{aligned}
$$

and

$$
\mathcal{D}_m \varphi = \int_M f \left(|w_m|^{p^*-2} w_m + |v_m|^{p^*-2} v_m - |\mathcal{B}_m|^{p^*-2} \mathcal{B}_m \right) .\varphi dv_g.
$$

We repeat the same arguments as in (2.0.15), we get that $\mathcal{A}_m.\varphi \to 0, \mathcal{C}_m.\varphi \to 0$ and $\mathcal{D}_m.\varphi \to 0$ which ends the proof of (2.1.24). Now, we prove (2.1.25). First, we repeat the same calculation in [39], we get

$$
\int_M |\nabla_g w_m|_g^p dv_g = \int_M |\nabla_g v_m|^p dv_g - \int_{\mathbb{R}^n} |\nabla v|^p dx + B_m(R) + o(1), \qquad (2.1.28)
$$

and

$$\int_M f|w_m|^{p^*}dv_g = \int_M f|v_m|^{p^*}dv_g - f(x^o)\int_{\mathbb{R}^n}|v|^{p^*}dx + B_m(R) + o(1), \quad (2.1.29)$$

with $\lim_{R\to\infty}\limsup_{m\to\infty} B_m(R) = 0$.
Since $w_m \to 0$ weakly in $H_1^p(M)$ which is compactly embedded in $L_p(M,(\rho_{x_o})^s)$ for $s\in(0,p)$, we may assume that $w_m \to 0$ strongly in $L_p(M,(\rho_{x_o})^s)$. Therefore, since R is arbitrarily chosen, by combining (2.1.28), (2.1.29), we get

$$J_{f,h,s}(w_m) = J_{f,h,s}(v_m) - (f(x^o))^{\frac{p-n}{p}}E(u) + o(1),$$

with u is a weak solution of (2.0.7). □

□

2.2 The critical Hardy potential.

Lemma 2.6. *Let v_m be a P.S sequence of $J_{f,h,p}$ at a level β that converges weakly and not strongly to 0 in $H_1^p(M)$. Then, there exists a sequence of positive reals $\mathcal{T}_m \to 0$ as $m\to\infty$ such that the sequence $\tilde{\eta}_m\tilde{v}_m$ with*

$$\tilde{v}_m(x) = \mathcal{T}_m^{\frac{n-p}{p}} v_m(\exp_{x_o}(\mathcal{T}_m x)),$$

and $\tilde{\eta}_m(x) = \eta(\delta^{-1}\mathcal{T}_m x)$, $0<\delta\le\frac{Inj_g}{2}$ and η is defined by (2.1.12), converges up to subsequence to a weak solution $v\in D^{1,p}(I\!R^n)$ of

$$\Delta_{\xi,p}v + \frac{h(x_o)}{|x|^p}|v|^{p-2}v = f(x_o)|v|^{p^*-2}v,$$

Moreover, the sequence

$$w_m(x) = v_m(x) - \mathcal{T}_m^{\frac{p-n}{p}}\eta_\delta(\exp_{x_o}^{-1}(x))v(\mathcal{T}_m^{-1}\exp_{x_o}^{-1}(x)),$$

where $0<\delta<\frac{I_g}{2}$, admits a subsequence w_m that is a P-S sequence of $J_{f,h,p}$, at level $\beta - E_{f,h}(v)$ that converges to 0 weakly in $H_1^p(M)$.

Proof. Let v_m be a P.S sequence of $J_{f,h,p}$ at level β that converges to 0 weakly and not strongly in $H_1^p(M)$. Then,up to a subsequence, we can assume that v_m converges strongly to 0 in $L_p(M)$ and that, by (2.0.19) there exists a small positive constant $\tilde{\gamma}$, such that

$$\limsup_{m\to\infty}\int_M |\nabla_g v_m|^p\,dv_g > \tilde{\gamma} > 0.$$

Up to a subsequence, for each $m > 0$, there exists a constant $\tilde{r}_m > 0$ such that

$$\int_{B(x_o,\tilde{r}_m)} |\nabla_g v_m|^p\,dv_g = \tilde{\gamma} \tag{2.2.1}$$

For $0 < r_o < \frac{\mathrm{Inj_g}}{2}$ and C_o as in (2.1.3). For $0 < r < r_o$, put $\mathcal{T}_m = \frac{\tilde{r}_m}{rC_o}$ and for $x \in B(\mathcal{T}_m^{-1}\delta_g)$ and define

$$\begin{aligned}\tilde{v}_m(x) &= \mathcal{T}_m^{\frac{n-p}{p}}\,v_m(\exp_{x_o}(\mathcal{T}_m x)),\ x \in I\!R^n\\ \tilde{g}_m(x) &= \exp_{x_o}^* g(\mathcal{T}_m x)\end{aligned}$$

We let the sequence $\tilde{\eta}_m\tilde{v}_m$ such that $\tilde{\eta}_m = \eta(\delta^{-1}\mathcal{T}_m x), \delta \in (0, \frac{\mathrm{Inj_g}}{2})$ and $\eta \in C_o(I\!R^n)$ is the cut-off function such that $0 \le \eta \le 1$, $\eta(x) = 1, x \in B(\frac{1}{4})$ and $\eta(x) = 0, x \in \mathbb{R}^n \setminus\ B(\frac{3}{4})$. Going through the same way in the proof of Lemma 3.3, we get then that the sequence $\tilde{\eta}_m\tilde{v}_m$ is bounded in $D^{1,p}(I\!R^n)$ and then it converges weakly in $D^{1,p}(I\!R^n)$ to a function $v \in D^{1,p}(I\!R^n)$.

Suppose that $v \ne 0$, we get then that $\mathcal{T}_m \to 0$. To prove that v solves (2.0.8), we let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with compact support included in a ball $B(R), R > 0$. For m large, define on M the sequence φ_m as

$$\varphi_m(x) = \mathcal{T}_m^{\frac{p-n}{p}}\,\varphi(\mathcal{T}_m^{-1}(\exp_{x_o}^{-1}(x)))$$

Identities (2.1.19) and (2.1.21) still hold and we have

$$\int_M \frac{h}{\rho_{x_o}^p}|v_m|^{p-2}v_m\varphi_m dv_g \;=\; \int_{\mathbb{R}^n}\frac{h(\exp_{x_o}(\mathcal{T}_m x))}{|x|^p}|(\tilde{\eta}_m\tilde{v}_m)|^{p-2}(\tilde{\eta}_m\tilde{v}_m)\varphi dv_{\tilde{g}_m}.$$

Since $\mathcal{T}_m \to 0$, $\tilde{g}_m \to \xi$ in $C^1(B(R))$ and thus we can write $dv_{\tilde{g}_m} = \varepsilon_m dx$, with $\varepsilon \to 1$ uniformly in $B(R)$. In addition, we can prove, as in [39] (proof of step 2.1), that $\nabla(\tilde{\eta}_m\tilde{v}_m) \to \nabla v$ a.e. Since we have also $\tilde{\eta}_m\tilde{v}_m \to v$ a.e, and the sequence $\tilde{\eta}_m\tilde{v}_m$ is bounded in $L_p(\mathbb{R}^n, |x|^p)$ we get by basic integration theory together with the fact that the sequence φ_m is bounded in $H_1^p(M)$ and the sequence v_m is a P-S sequence of $J_{f,h,p}$, that v is a weak solution of

$$\Delta_{\xi,p}v - \frac{h(x_o)}{|x|^p}|v|^{p-2}v = f(x_o)|v|^{p^*-2}v,$$

Now, that the sequence w_n converges weakly to 0 in $H_1^p(M)$ follows in the same manner as in the proof of Step 3 above. To prove that $DJ_{f,h,p}(w_m) \to 0$, we consider the sequence $\mathcal{B}_m$ defined by (2.1.22). Let $\varphi \in H_1^p(M)$. For $x \in B(\delta\mathcal{T}_m^{-1})$ put $\varphi_m(x) = \mathcal{T}_m^{\frac{n-p}{p}}\varphi(\exp_{x_o}(\mathcal{T}_m x))$ and $\bar{\varphi}_m = \eta_\delta(\mathcal{T}_m x)\varphi_m(x)$. Then, identities (2.1.26) and (2.1.27) still hold. Let $R > 0$ be a constant, we have

$$\begin{aligned}\int_M \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g &= \int_{B(x_o,\mathcal{T}_mR)} \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g \\ &+ \int_{B(x_o,\delta)\backslash B(x_o,\mathcal{T}_mR)} \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g.\end{aligned}$$

By Hölder and Hardy inequalities we have

$$\begin{aligned}\int_{B(x_o,\delta)\backslash B(x_o,\mathcal{T}_mR)} \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g &\le \sup_M |h|\|\varphi\|_{H_1^p(M)} \int_{B(x_o,\delta)\backslash B(x_o,\mathcal{T}_mR)} |\nabla_g\mathcal{B}_m|^p dv_g + o(1) \\ &= \sup_M |h|\|\varphi\|_{H_1^p(M)} \int_{B(\delta\mathcal{T}_m^{-1}))\backslash B(R)} |\nabla v|^p dx + o(1) \\ &= O(\|\varphi\|_{H_1^p(M)})\varepsilon(R) + o(1),\end{aligned}$$

with $\varepsilon \to 0$ as $R \to \infty$.

Put

$$\overline{\varphi}(x) = \mathcal{T}_m^{\frac{n-p}{p}}\varphi(\exp_{x_o}(\mathcal{T}_m x)).$$

Then, for m large

$$\int_{B(x_o,\mathcal{T}_mR)} \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g = \int_{B(R)} \frac{h(\exp_{x_o}(\mathcal{T}_m x))}{|x|^p}|v|^{p-2}v\overline{\varphi}_m dv_{\tilde{g}_m}$$

Since $\tilde{g} \to \xi$ in $C^1(B(R'), R' > R$, we get

$$\begin{aligned}
\int_{B(R)} \frac{h(\exp_{x_o}(\mathcal{T}_m x))}{|x|^p} |v|^{p-2} v \overline{\varphi}_m dv_{\tilde{g}_m} &= h(x_o) \int_{B(R)} \frac{1}{|x|^p} |v|^{p-2} v \overline{\varphi}_m dx + o(\|\varphi\|_{H_1^p(M)}) \\
&= h(x_o) \int_{\mathbb{R}^n} \frac{1}{|x|^p} |v|^{p-2} v \overline{\varphi}_m dx + o(\|\varphi\|_{H_1^p(M)}) \\
&+ O(\|\varphi\|_{H_1^p(M)}) \varepsilon(R).
\end{aligned}$$

Therefore

$$\int_M \frac{h}{(\rho_{x_o})^p} |\mathcal{B}_m|^{p-2} \mathcal{B}_m \varphi dv_g = h(x_o) \int_{\mathbb{R}^n} \frac{1}{|x|^p} |v|^{p-2} v \overline{\varphi}_m dx + o(\|\varphi\|_{H_1^p(M)}) + O(\|\varphi\|_{H_1^p(M)}) \varepsilon(R) + o(1). \tag{2.2.2}$$

Since v is a weak solution of (2.0.8), we get by (2.1.26), (2.1.27) and (2.2.2) that $DJ_{f,h,p}(\mathcal{B}_m) \to 0$. This implies, as in the proof of (2.1.24) of Step 3, that $DJ_{f,h,p}(w_m) \to 0$.

Now, we prove the last statement of the lemma. Put

$$\hat{w}_m(x) = \mathcal{T}_m^{\frac{n-p}{p}} w_m(\exp_{x_o}(\mathcal{T}_m x)) = \tilde{v}_m - \eta_\delta(\mathcal{T}_m x) v(x)$$

By convexity, we have

$$\begin{aligned}
& \int_{\mathbb{R}^n} |\nabla(v(\eta_\delta(\mathcal{T}_m x) - 1))|^p dx \\
= & \int_{\mathbb{R}^n \setminus B(\delta \mathcal{T}_m^{-1})} |\nabla v|^p dx + \int_{B(2\delta \mathcal{T}_m^{-1}) \setminus B(\delta \mathcal{T}_m^{-1})} |\nabla(v(\eta_\delta(\mathcal{T}_m x) - 1))|^p dx \\
\leq & 2^{p-1} \left(\int_{B(2\delta \mathcal{T}_m^{-1}) \setminus B(\delta \mathcal{T}_m^{-1})} |\eta_\delta(\mathcal{T}_m x) - 1)|^p |\nabla v|^p dx + \mathcal{T}_m^p \int_{B(2\delta \mathcal{T}_m^{-1}) \setminus B(\delta \mathcal{T}_m^{-1})} |v|^p |(\nabla \eta_\delta)(\mathcal{T}_m x)|^p dx \right) \\
+ & \int_{\mathbb{R}^n \setminus B(\delta \mathcal{T}_m^{-1})} |\nabla v|^p dx \\
\leq & 2^{p-1} \left(\int_{B(2\delta \mathcal{T}_m^{-1}) \setminus B(\delta \mathcal{T}_m^{-1})} |\nabla v|^p dx + C \mathcal{T}_m^p \int_{B(2\delta \mathcal{T}_m^{-1}) \setminus B(\delta \mathcal{T}_m^{-1})} |v|^p dx \right) + \int_{\mathbb{R}^n \setminus B(\delta \mathcal{T}_m^{-1})} |\nabla v|^p dx \\
= & o(1).
\end{aligned}$$

Similarly, we get that $\tilde{\eta}_m v = v + o(1)$. Thus, we obtain

$$\tilde{\eta}_m \hat{w}_m = \tilde{\eta}_m \tilde{v}_m - v + o(1).$$

Since $\tilde{\eta}_m\tilde{v}_m \to v$ a.e in $\mathbb{R}^n$ and $\nabla(\tilde{\eta}_m\tilde{v}_m) \to \nabla v$ a.e in $\mathbb{R}^n$, we get, as in the proof of lemma 2.1, that

$$E_{f,h}(\tilde{\eta}_m\hat{w}_m) = E_{f,h}(\tilde{\eta}_m\tilde{v}_m) - E_{h,f}(v) + o(1).$$

By using re-scaling invariance and the fact that $\tilde{g}_m \to \xi$ in $C^1(B(R))$ for any $R > 0$, we get that

$$J_{f,h,p}(w_m) = J_{f,h,p}(v_m) - E_{h,f}(v) + o(1).$$

□

Lemma 2.7. *Suppose that the weak limit v in $D^{1,p}(\mathbb{R}^n)$ of the sequence $\tilde{\eta}_m\tilde{v}_m$ of the above lemma is null. Then, there exists a sequence of positive numbers $\tau_m \to 0$ and a sequence of points $y_i \in M\backslash\{x_o\}$, $y_i \to y_o \neq x_o$ such that up to a subsequence, the sequence $\check{\eta}_m\check{\nu}_m$ with*

$$\check{\nu}_m = \tau_m^{\frac{n-p}{p}} v_m(\exp_{y_i}(\tau_m x)),$$

and $\check{\eta}_m(x) = \eta(\delta^{-1}\tau_m x)$, converges weakly to a nontrivial weak solution ν of the Euclidean equation

$$\Delta_{\xi,p}\nu = f(y_o)|\nu|^{p^\star-2}\nu$$

and the sequence

$$\mathcal{W}_m = v_m - \tau_m^{\frac{p-n}{p}}\eta_\delta(exp_{y_i}^{-1}(x))\nu(\tau_m^{-1}exp_{y_i}^{-1}(x))$$

is a Palais-Smale sequence for $J_{f,h,p}$ that converges weakly to 0 in $H_1^p(M)$ and

$$J_{f,h,p}(\mathcal{W}_m) = J_{f,h,p}(v_m) - f(y_o)^{\frac{p-n}{p}}E(u),$$

with u is a solution of (2.0.7).

Proof. Take a function $\varphi \in \mathcal{C}_0^\infty(B(C_o r))$ and put $\varphi_m(x) = \varphi(\mathcal{T}_m^{-1}\exp_{x_o}^{-1}(x))$. we have

$$\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}\tilde{v}_m|^{p-2}\tilde{g}(\nabla_{\tilde{g}_m}\tilde{v}_m, \nabla_{\tilde{g}_m}(\tilde{v}_m|\varphi|^p))dv_{\tilde{g}_m} = \int_{\mathbb{R}^n} |\varphi|^p|\nabla_{\tilde{g}_m}\tilde{v}_m|^p dv_{\tilde{g}_m}$$
$$+ \int_{\mathbb{R}^n} p|\varphi|^{p-1}|\tilde{v}_m||\nabla_{\tilde{g}_m}\tilde{v}_m|^{p-2}\tilde{g}(\nabla_{\tilde{g}_m}\tilde{v}_m, \nabla_{\tilde{g}_m}|\varphi|)dv_{\tilde{g}_m}.$$

Since the sequence $\tilde{\eta}_m \tilde{v}_m$ is bounded in $D^{1,p}(I\!R^n)$ and it converges strongly to 0 in $L_{p,loc}(I\!R^n)$, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} p|\varphi|^{p-1}|\tilde{v}_m||\nabla_{\tilde{g}_m}\tilde{v}_m|^{p-2}\tilde{g}(\nabla_{\tilde{g}_m}\tilde{v}_m, \nabla_{\tilde{g}_m}|\varphi|)dv_{\tilde{g}_m} \right| \\
\leq\ & C\int_{B(C_o r)} |\tilde{v}_m||\nabla_{\tilde{g}_m}\tilde{v}_m|^{p-1}dv_{\tilde{g}_m} \\
\leq\ & C\left(\int_{B(C_o r)} |\tilde{v}_m|^p dv_{\tilde{g}_m}\right)^{\frac{1}{p}} \left(\int_{B(C_o r)} |\nabla_{\tilde{g}_m}\tilde{v}_m|^p_{\tilde{g}_m} dv_{\tilde{g}_m}\right)^{1-\frac{1}{p}} = o(1).
\end{aligned}$$

Then

$$\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}\tilde{v}_m|^{p-2}\tilde{g}(\nabla_{\tilde{g}_m}\tilde{v}_m, \nabla_{\tilde{g}_m}(\tilde{v}_m|\varphi|^p))dv_{\tilde{g}_m} = \int_{\mathbb{R}^n} |\varphi|^p|\nabla_{\tilde{g}_m}\tilde{v}_m|^p dv_{\tilde{g}_m} + o(1).$$

Using the inequalities (1.2.7), (1.2.8), and (1.2.8), together with Hölder inequality and the strong convergence of $\tilde{\eta}_m\tilde{v}_m$ in $L_{p,loc}(I\!R^n)$, we get

$$\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}(\tilde{v}_m\varphi)|^p dv_{\tilde{g}_m} \leq \int_{\mathbb{R}^n} |\varphi|^p|\nabla_{\tilde{g}_m}\tilde{v}_m|^p dv_{\tilde{g}_m} + o(1),$$

in such way that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}(\tilde{v}_m\varphi)|^p dv_{\tilde{g}_m} \\
\leq\ & \int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}\tilde{v}_m|^{p-2}\tilde{g}(\nabla_{\tilde{g}_m}\tilde{v}_m, \nabla_{\tilde{g}_m}(\tilde{v}_m|\varphi|^p))dv_{\tilde{g}_m} + o(1), \\
=\ & \int_M |\nabla v_m|^{p-2}\, g(\nabla_g v_m, \nabla_g(v_m|\varphi_m|^p))dv_g + o(1)
\end{aligned}$$

Moving to and from re-scaling, using Hölder, Hardy and Sobolev inequalities and the fact that v_m is P-S sequence and that $v_m|\varphi_m|^p$ is bounded in $H_1^p(M)$, we get

$$\begin{aligned}
& \int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}(\tilde{v}_m\varphi)|^p dv_{\tilde{g}_m} \\
\leq \; & \int_M |\nabla_g v_m|^{p-2}\, g(\nabla_g v_m, \nabla_g(v_m|\varphi_m|^p))dv_g + o(1) \\
= \; & (DJ_{f,h,p}(v_m)).(v_m|\varphi_m|^p) + \int_M \frac{h}{\rho_{x_o}^p}\, |v_m\varphi_m|^p\, |dv_g \\
+ \; & \int_M f\, |v_m|^{p^*-p}\, |v_m\varphi_m|^p\, dv_g + o(1) \\
\leq \; & (h(x_o)+\varepsilon)\left(\left(\frac{p}{n-p}\right)^p + \varepsilon\right)\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}(\tilde{v}_m\varphi)|^p\, dv_{\tilde{g}_m} \\
+ \; & (K^{p^\star}(n,p)+\varepsilon)\sup f\Big(\int_{B(C_or)} |\nabla_{\tilde{g}_m}(\tilde{v}_m)|^p\, dv_{\tilde{g}_m}\Big)^{\frac{p}{n-p}} \int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}(\tilde{v}_m\varphi)|^p\, dv_{\tilde{g}_m} \\
+ \; & o(1).
\end{aligned}$$

Thus, since $1-h(x_o)(\frac{p}{n-p})^p > 0$, for $\tilde{\gamma}$ in (2.2.1) chosen small enough, we get that for each t, $0 < t < C_or$

$$\int_{B(x_o,t\mathcal{T}_m)} |\nabla_g v_m|^p\, dv_g = \int_{B(t)} |\nabla_{\tilde{g}_m}\tilde{v}_m|^p\, dv_{\tilde{g}_m} \to 0, m \to \infty \tag{2.2.3}$$

Now, the sequence v_m is a P.S sequence that converges to 0 weakly and not strongly in $H_1^p(M)$, we get as in lemma 2.2 that

$$\int_M |\nabla_g v_m|^p\, dv_g \geq \left(\frac{n\beta^*}{\sup_M f(K(n,p)+\varepsilon)^{p^*}}\right)^{\frac{p}{p^*}} + o(1). \tag{2.2.4}$$

Consider for $t > 0$ the function

$$t \mapsto \mathcal{F}_m(t) = \max_{y\in M}\int_{B(y,t)} |\nabla_g v_m|^p\, dv_g$$

Given t_o small, it follows from (2.2.4) that there exists $y \in M$ and $\lambda_o > 0$ such that up to a subsequence

$$\int_{B(y,t_o)} |\nabla_g v_m|^p\, dv_g \geq \lambda_o \tag{2.2.5}$$

Since $\mathcal{F}_m$ is continuous, it follows that for any $\lambda \in (0,\lambda_o)$, there exist $t_m \in (0,t_o)$ and $y_m \in M$ such that

$$\mathcal{F}_m(t_m) = \int_{B(y_m,t_m)} |\nabla_g v_m|^p\, dv_g = \lambda. \tag{2.2.6}$$

Since M is compact, up to a subsequence, we may assume that y_m converges to some point $y_o \in M$.

Note first that for all $m \geq 0$, $t_m < \tilde{r}_m = C_o r \mathcal{T}_m$, otherwise if there exists $m_o \geq 0$ such that $t_{m_o} \geq \tilde{r}_{m_o}$, we get

$$\lambda = \int_{B(y_{m_o}, t_{m_o})} |\nabla_g v_{m_o}|^p \, dv_g \geq \int_{B(x_o, t_{m_o})} |\nabla_g v_{m_o}|^p \, dv_g \geq \int_{B(x_o, \tilde{r}_{m_o})} |\nabla v_{m_o}|^p \, dv_g = \gamma.$$

Hence, if we choose λ small enough such that $0 < \lambda < \gamma$, we get a contradiction.

Now, suppose that for all $\varepsilon > 0$, there exists $m_\varepsilon > 0$ such that for all $m \geq m_\varepsilon$ $dist_g(y_m, x_o) \leq \varepsilon$. Choose r'_m such that, $t_m < r'_m < \tilde{r}_m$ and take $\varepsilon' = r'_m - t_m$, we get that for some $m_{\varepsilon'} > 0$ and $m \geq m_{\varepsilon'}$

$$B(y_m, t_m) \subset B(x_o, r'_m)$$

which gives, by virtue of (2.2.3) and (2.2.6), a contradiction. We deduce then that $y_o \neq x_o$.

Now, take $0 < \tau_m < 1$ such that $C_o r \tau_m = t_m$, where $r \in (0, r_o)$ and C_o and r_o are as in (2.1.3). Then, for $x \in B(\tau_m^{-1} \delta_g) \subset \mathbb{R}^n$ consider the sequences

$$\begin{aligned} \check{\nu}_m(x) &= \tau_m^{\frac{n-p}{p}} v_m(\exp_{y_m}(\tau_m x)), \\ \check{g}_m(x) &= \exp^*_{y_m} g(\tau_m x) \end{aligned}$$

Put $\check{\eta}_m(x) = \eta(\delta^{-1} \tau_m x)$, where $\delta \in (0, Inj_g)$ and $x \in I\!R^n$. As in the proof of lemma 2.3, we can easily check that there is a subsequence of $\tilde{\eta}_m \tilde{\nu}_m$ that converges weakly in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ to some function ν. We prove that actually the strong convergence holds in $H_1^p(B(R)), R > 0$. In fact, we go through the same proof of Step 1 above by just replacing x_m by y_m and R_m by τ_m. We let then $a \in I\!R^n$ and $\mu \in [r, 2r]$ and consider the sequence

$$\begin{cases} \check{\psi}_m = \check{\eta}_m \check{\nu}_m - \nu & \text{in } \overline{B}(a, \mu), \\ \check{\psi}_m = z_m & \text{in } \overline{B}(a, 3r) \setminus B(a, \mu), \\ \check{\psi}_m = 0 & \text{in } I\!R^n \setminus B(a, 3r). \end{cases}$$

where z_m are solutions of (2.1.13). For $r < \frac{\delta}{24}$, consider the re-scaling sequence ψ_m of $\check{\psi}_m$

$$\begin{cases} \psi_m(x) = \tau_m^{\frac{p-n}{p}} \check{\psi}_m(\tau_m^{-1} \exp_{y_m}^{-1}(x)), & \text{if } x < d_g(y_m, 6r), \\ \psi_m(x) = 0, & \text{otherwise.} \end{cases}$$

As in (2.1.14), we have

$$\begin{aligned} o(1) = & \int_{B(a,3r)} |\nabla_{\check{g}_m}(\check{\eta}_m \check{\nu}_m)|^{p-2} \check{g}\left(\nabla_{\check{g}_m}(\check{\eta}_m \check{\nu}_m), \nabla_{\check{g}_m} \check{\psi}_m\right) dv_{\check{g}_m} \qquad (2.2.7) \\ - & \tau_m^p \int_{B(a,3r)} \frac{h(\exp_{y_m}(\tau_m x))}{\left(\rho_{x_o}(\exp_{y_m}(\tau_m x))\right)^p} |\check{\eta}_m \check{\nu}_m|^{p-2} (\check{\eta}_m \check{\nu}_m) \check{\psi}_m dv_{\check{g}_m} \\ - & \int_{B(a,3r)} f(\exp_{y_m}(\tau_m x)) |\check{\eta}_m \check{\nu}_m|^{p^*-2} (\check{\eta}_m \check{\nu}_m) \check{\psi}_m dv_{\check{g}_m}. \end{aligned}$$

As above, we have

$$\int_{B(a,3r)} |\nabla_{\check{g}_m}(\check{\eta}_m \check{\nu}_m)|^{p-2} \check{g}\left(\nabla_{\check{g}_m}(\check{\eta}_m \check{\nu}_m), \nabla_{\check{g}_m} \check{\psi}_m\right) dv_{\check{g}_m} = \int_{I\!R^n} |\nabla_{\check{g}_m} \check{\psi}_m|^p dv_{\check{g}_m} + o(1),$$

and

$$\begin{aligned} & \int_{B(a,3r)} f(\exp_{y_m}(\tau_m x)) |\check{\eta}_m \check{\nu}_m|^{p^*-2} (\check{\eta}_m \check{\nu}_m) \check{\psi}_m dv_{\check{g}_m} \\ = & \int_{I\!R^n} f(\exp_{y_m}(\tau_m x)) |\check{\psi}_m|^{p^*} dv_{\check{g}_m} + o(1). \end{aligned}$$

Since $\tau_m \to 0$ we get that for all $\varepsilon > 0$ there exists m_o such that for all $m \geq m_o$ have

$$\rho_{x_o}(\exp_{y_m}(\tau_m x)) = dist_g(x_o, \exp_{y_m}(\tau_m x)) \geq dist_g(x_o, y_o) - \varepsilon = \varrho > 0.$$

Then, as in the proof of step 1, we get

$$\begin{aligned} & \int_{B(a,3r)} \frac{h(\exp_{y_m}(\tau_m x))}{\left(\rho_{x_o}(\exp_{y_m}(\tau_m x))\right)^p} |\check{\eta}_m \check{\nu}_m|^{p-2} (\check{\eta}_m \check{\nu}_m) \check{\psi}_m dv_{\check{g}_m} \qquad (2.2.8) \\ = & \int_{I\!R^n} \frac{h(\exp_{y_m}(\tau_m x))}{\left(\rho_{x_o}(\exp_{y_m}(\tau_m x))\right)^p} |\check{\psi}_m|^p dv_{\check{g}_m} + o(1). \end{aligned}$$

Since the sequence $\check{\psi}_m$ converges strongly to 0 in $L_{p,loc}(I\!R^n)$, we get

$$\int_{I\!R^n} \frac{h(\exp_{y_m}(\tau_m x))}{\left(\rho_{x_o}(\exp_{y_m}(\tau_m x))\right)^p} |\check{\psi}_m|^p dv_{\check{g}_m} \leq C \int_{I\!R^n} |\check{\psi}_m|^p dv_{\check{g}_m} = o(1).$$

We deduce that

$$\int_{I\!R^n} |\nabla_{\check{g}_m} \check{\psi}_m|^p dv_{\check{g}_m} = \int_{I\!R^n} f(\exp_{y_m}(\tau_m x))|\check{\psi}_m|^{p^*} dv_{\check{g}_m} + o(1).$$

The remaining of the proof goes in the same way as in the proof of step 1 and step 2. Thus we get that $\nu \neq 0$ and ν is a weak solution of

$$\Delta_{p,\xi}\nu = f(y_o)|\nu|^{p^*-2}\nu.$$

□

Now, we are in position to prove the theorem 2.1 and 2.2

Proof of theorem 2.1. Let us first note that if $u \in D^{1,p}(\mathbb{R}^n)$ is a nontrivial weak solution of (2.0.8), then

$$E_{f,h}(u) \geq \frac{(1 - h(x_o)(\frac{p}{n-p})^p)^{\frac{n}{p}}}{(\sup_M f)^{\frac{n-p}{p}} K^n(n,p))}. \tag{2.2.9}$$

In fact, by Hardy and Sobolev inequalities, we have

$$\begin{aligned}\left(1 - h(x_o)\left(\frac{p}{n-p}\right)^p\right)\int_{\mathbb{R}^n} |\nabla u|^p dx \;&\leq\; \int_{\mathbb{R}^n} |\nabla u|^p dx - h(x_o)\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx = f(x_o)\int_{\mathbb{R}^n} |u|^{p^*} dx \\ &\leq\; f(x_o)K^{p^*}(n,p)\left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{p^*}{p}}\end{aligned}$$

Since u cannot be a constant, we get

$$\int_{\mathbb{R}^n} |\nabla u|^p dx \geq \frac{(1 - h(x_o)(\frac{p}{n-p})^p)^{\frac{n-p}{p}}}{(f(x_o))^{\frac{n-p}{p}} K^n(n,p))}.$$

Hence

$$\begin{aligned} E_{f,h}(u) = \frac{1}{n}\left(\int_{\mathbb{R}^n} |\nabla u|^p dx - h(x_o)\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx\right) \;&\geq\; \frac{(1 - h(x_o)(\frac{p}{n-p})^p)(1 - h(x_o)(\frac{p}{n-p})^p)^{\frac{n-p}{p}}}{n(f(x_o))^{\frac{n-p}{p}} K^n(n,p))} \\ &\geq\; \frac{(1 - h(x_o)(\frac{p}{n-p})^p)^{\frac{n}{p}}}{n(\sup_M f)^{\frac{n-p}{p}} K^n(n,p))}.\end{aligned}$$

By the same way, we can also have that for a nontrivial solution $u \in D^{1,p}(\mathbb{R}^n)$ of (2.0.7),

$$E(u) \geq \frac{1}{nK^n(n,p))}. \tag{2.2.10}$$

Now, let u_m be a P-S sequence for $J_{f,h,s}$ at level β_s^u, $0 < s < p$. Then, u_m is bounded in $H_1^p(M)$ and it converges, up to a subsequence, to a function u weakly in $H_1^p(M)$ and almost everywhere to u in M. Thus, by Lemma 2.3, the function u is a weak solution of (E_s), $0 < s < p$ and the sequence $v_m = u_m - u$ is a Palais-Smale sequence for $J_{f,h,s}$ at level $\beta_s = \beta_s^u - J_{f,h,s}(u)$.

If v_m converges strongly to 0 in $H_1^p(M)$, then the theorem is proved with $k = 0$. If not, by lemma 2.4, $\beta_s \geq \beta^* = \frac{1}{n(\sup_M f)^{\frac{n-p}{p}} K^n(n,p))}$. Then, by Lemma 2.5 and its proof, there exists a nontrivial weak solution $v_1 \in D^{1,p}(\mathbb{R}^n)$ of $\Delta_{p,\xi} v = f(x_1^o)|v|^{p^*-2}v$, a converging sequence of points $x_m^1 \to x_1^o$ and a sequence of reals $R_m^1 \to 0$ such that, the sequence

$$w_m(x) = v_m - (R_m^1)^{\frac{p-n}{p}} \eta_\delta(\exp_{x_m^1}^{-1}(x)) v_1((R_m^1)^{-1} \exp_{x_m^1}^{-1}(x)), x \in M$$

admits a subsequence that is P-S sequence of $J_{f,h,s}$, $0 < s < p$, at level $\beta^1 = \beta_s - (f(x_1^o))^{\frac{p-n}{p}} E(u_1)$, with u_1 is a nontrivial weak solution of (2.0.7). By (2.2.10), $\beta^1 \leq \beta_s - \beta^*$. Then, if $\beta_s < 2\beta^*$, we get $\beta^1 < \beta^*$ and the sequence w_m converges strongly to 0 in $H_1^p(M)$. Hence, the theorem is proved with $k = 1$. If not we repeat the procedure until we obtain a P-S sequence at level $\beta^k \leq \beta_s - k\beta^* < \beta^*$ and theorem 2.1 is proved.

□

Proof of theorem 2.2. In the same way as above, we prove theorem 2.2. We let u_m be a P-S sequence for $J_{f,h,p}$ at a level β^u. Then, u_m is bounded in $H_1^p(M)$ and it converges, up to a subsequence, to a function u weakly in $H_1^p(M)$ and almost everywhere to u in M. Thus, by Lemma 2.3, the function u is a weak solution of (E_s), $s = p$, and the sequence $v_m = u_m - u$ is a Palais-Smale sequence for $J_{f,h,p}$ at level $\beta = \beta^u - J_{f,h,p}(u)$.

If v_m converges strongly to 0 in $H_1^p(M)$, then the theorem is proved with $k = 0$, $l = 0$. If not, by lemma 2.4, $\beta \geq \beta^* = \frac{(1-h(x_o)(\frac{n-p}{p})^p)^{\frac{n}{p}}}{n(\sup_M f)^{\frac{n-p}{p}} K^n(n,p)}$. By lemma 2.6, there exist a sequence of positive reals $\mathcal{T}_m^1 \to 0$ such that the sequence $\tilde{\eta}_m^1 \tilde{v^1}_m$ with

$$\tilde{v}_m^1(x) = \left(\mathcal{T}_m^1\right)^{\frac{n-p}{p}} v_m(\exp_{x_o}(\mathcal{T}_m^1 x)),$$

and $\tilde{\eta}_m^1(x) = \eta(\delta^{-1}\mathcal{T}_m^1 x)$, $0 < \delta \leq \frac{\mathrm{Inj_g}}{2}$ and η is defined by (2.1.12), converges, up to subsequence, weakly to some function $v_1 \in D^{1,p}(I\!R^n)$ such that if $v_1 \neq 0$, then v_1 is solution of

$$\Delta_{\xi,p} v + \frac{h(x_o)}{|x|^p}|v|^{p-2}v = f(x_o)|v|^{p^*-2}v,$$

and the sequence

$$w_m^1(x) = v_m(x) - (\mathcal{T}_m^1)^{\frac{p-n}{p}} \eta_\delta(\exp_{x_o}^{-1}(x)) v_1((\mathcal{T}_m^1)^{-1} \exp_{x_o}^{-1}(x)),$$

where $0 < \delta < \frac{\mathrm{Inj_g}}{2}$, admits a subsequence w_m that is a P-S sequence of $J_{f,h,p}$, at level $\beta^1 = \beta - E_{f,h}(v_1)$ that converges to 0 weakly in $H_1^p(M)$. By (2.2.9), $\beta^1 \leq \beta - \beta^*$. Then, if $\beta < 2\beta^*$, we get $\beta^1 < \beta^*$ and the sequence w_m converges strongly to 0 in $H_1^p(M)$. If not, we repeat the procedure until we obtain a palais-Smale sequence at level $\beta^k \leq \beta - k\beta^* < \beta^*$.

Now, if the weak limit v_1 of the sequence $\tilde{v}_m^1$ is the zero function by lemma 2.7, there exists a nontrivial weak solution ν_1 of $\Delta_{p,\xi}\nu = f(y_o)|\nu|^{p^*-2}\nu$, a sequence of positive reals $\tau_m^1 \to 0$ and a sequence $y_m^1 \to y_o^1 \neq x_o$ such that the sequence

$$\breve{w}_m(x) = v_m - (\tau_m^1)^{\frac{p-n}{p}} \eta_\delta(\exp_{y_m^1}^{-1}(x)) \nu_1((\tau_m^1)^{-1} \exp_{y_m^1}^{-1}(x)),\ x \in M$$

admits a subsequence which is a P-S sequence of $J_{f,h,p}$ at level $\beta - (f(y_o^1))^{\frac{p-n}{p}} E(u_1) \leq \beta^1 = \beta - \beta_s^*$, with u_1 is a nontrivial weak solution of (2.0.7). If $\beta < 2\beta^*$, then $\beta^1 < \beta^*$ and the sequence $\breve{w}_m$ converges strongly to 0 in $H_1^p(M)$. The theorem is then proved with $k = 0$ and $l = 1$. If not, we repeat the procedure until we obtain a P-S sequence at level $\beta^l \leq \beta - l\beta^* < \beta^*$. $\square$

Chapter 3

Existence results for a Hardy-Sobolev equation containing p-Laplacian operator.

In this chapter, we will establish some existence results for the equation (E_s), $0 < s \le p$. Our results generalize (partially) those obtained in [17], [25] and [46].

3.1 Regularity of solutions

As to the regularity of the weak solution of our equation, using the approach used in [1], we can see that the weak solutions of (E_s) are in $\mathcal{C}^{1,\alpha}(M \setminus x_o)$ for $\alpha > 0$. Let $u \in H_1^p(M)$ be a weak solution of (E_s). Let $R > 0, \varepsilon > 0$ be positive constants such that $\varepsilon < R$. Let $N = B(x_o, R) \setminus B(x_o, \varepsilon)$. Consider the problem

$$\begin{cases} \Delta_{g,p} v - \frac{h(x)}{(\rho_{x_o}(x))^s} |v|^{p-2} v = f(x) |v|^{p^*-2} v, 0 < s \le p, \quad x \in N; \\ v|_{\partial B(x_o,R)} = u|_{\partial B(x_o,R)}; \\ v|_{\partial B(x_o,\varepsilon)} = u|_{\partial B(x_o,\varepsilon)}; \\ v \in H_1^p(N). \end{cases}$$

Since u is a weak solution of the problem above, so by the regularity result-$\mathcal{C}^{1,\alpha}$ [17, theorem 2.3], there exists $\alpha > 0$ such that $u \in \mathcal{C}^{1,\alpha}(N)$. Since R and ε are arbitrary, we find that $u \in \mathcal{C}^{1,\alpha}(M \setminus \{x_o\})$.
Now, we define on $H_1^p(M)$ the functional

$$L_{h,s}(u) = \int_M \left(|\nabla_g u|^p - \frac{h}{(\rho_{x_o})^s} |u|^p \right) dv_g, \; 0 < s \leq p. \tag{3.1.1}$$

We say that $L_{h,s}(u)$, $0 < s \leq p$ is coercive if there exists a positive constant λ, such that for any $\in H_1^p(M)$

$$L_{h,s}(u) \geq \lambda \|u\|^2_{H_1^p(M)}.$$

In this chapter we prove the following two existence theorems.

Theorem 3.1. *Let (M,g) be a compact Riemannian manifold of dimension n. Let p be a real number such that $1 < p < n$ and $n > p^2$. Let f and h be two regular functions on M such that f is positive everywhere on M. Let ρ_{x_o} be the function on M as defined in* (0.0.1)*. Assume that h is such that the operator $L_{h,s}$ is coercive. Suppose there exists a point $x_1 \neq x_o$ such that $f(x_1) = \sup_M f(x)$ and*

$$f(x_1) = \sup_M f(x) \geq \frac{f(x_o)}{\left(1 - h(x_o)(\frac{p}{n-p})^p\right)^{\frac{n}{n-p}}}. \tag{3.1.2}$$

Suppose we are in one of the following cases:

1. $1 < p < 2$ *and* $h(x_1) > 0$,

2. $p = 2$ *and*
$$\frac{8(n-1)}{(n-2)(n-4)} h(x_1) > dist_g(x_o, x_1)^s \left(\frac{\Delta_g f(x_1)}{f(x_1)} - \frac{2Scal_g(x_1)}{n-4} \right), 0 < s \leq p. \tag{3.1.3}$$

3. $p > 2$ *and*
$$\left(\frac{n+2-3p}{p} \right) \frac{\Delta_g f(x_1)}{f(x_1)} < Scal(g)(x_1), \tag{3.1.4}$$

Then, the equation (E_s), $0 < s \leq p$, *has a positive weak solution* $u \in H_1^p(M)$.

Theorem 3.2. *Let* (M, g) *be a compact Riemannian manifold of dimension* $n \geq 3$. *Let* p *and* s *be real numbers such that* $0 < s < p$, $1 < p < n$ *and* $n > p^2 - sp + s$. *Let* f *and* h *be two regular functions on* M. *Let* ρ_{x_o} *be the function of* M *as defined in* (0.0.1). *We assume that* h *is such that the operator* $L_{h,s}$ *is coersive. Assume that* f *and* h *satisfy the following conditions*

1. $f(x_o) = \sup_M f(x), f(x) > 0, x \in M$,

2. $0 < h(x_o) < (\frac{n-p}{p})^p$.

Suppose we are in one of the following cases:

1. $0 < p < s + 2$ *and* $h(x_o) > 0$

2. $p = s + 2$ *and*
$$\left(\frac{p-1}{n-p}\right)^p \frac{p}{n(p-1)} \frac{\Gamma(n - \frac{n+2}{p} + 3 - p)\,\Gamma(n)}{\Gamma(n-p)} h(x_o)$$
$$> \frac{\Gamma(n - \frac{n}{p} - \frac{2}{p} + 2)}{2n^2} \left((n + 2 - 3p) \frac{\Delta_g f\,(x_o)}{f\,(x_o)} - p\,\mathrm{Scal}(g)\,(x_o) \right)$$

3. $p > s + 2$ *and*
$$\left(\frac{n + 2 - 3p}{p}\right) \frac{\Delta_g f(x_o)}{f(x_o)} < Scal(g)(x_o)$$

Then, equation (E_s), $0 < s < p$, *has a positive weak solution* $u \in H_1^p(M)$.

In seeking weak solutions to (E_s), we use the variational method in which weak solutions are obtained as limits of P.S. sequences. As our equation contains the critical Sobolev exponent, the P.S. sequences do not converge at all levels. The following proposition gives us the level under which P.S. sequences converge (to a subsequence).

Proposition 3.3. *Suppose that the function* f *is positive,* $h(x_o) > 0$, $1 - h(x_o)(\frac{p}{n-p})^p > 0$ *and* $L_{h,s}(u) = \int_M (|\nabla_g u|^p - h\frac{|u|^p}{(\rho_{x_o}(x))^s}) dv_g \geq 0$, $0 < s \leq p$, $\forall u \in H_1^p(M)$.

Let u_m be a P-S sequence of $J_{f,h,s}, 0 < s \leq p$, at level β. Then, the sequence u_m converges, up to a subsequence, to a non zero function $u \in H_1^p(M) \setminus \{0\}$ in the following cases

1. $0 < s < p$ *and*
$$0 < \beta < \frac{1}{n(\sup_M f)^{\frac{n-p}{p}} K(n,p)^n}, \tag{3.1.5}$$

2. $s = p$ *and*
$$0 < \beta < \min\left((\sup_M f)^{-\frac{n-p}{p}}, (f(x_o))^{-\frac{n-p}{p}} \left[1 - h(x_o)(\frac{p}{n-p})^p\right]^{\frac{n}{p}} \right) \frac{1}{nK(n,p)^n}. \tag{3.1.6}$$

Proof. Let u_m be a Palais-Smale sequence of $J_{f,h,s}, 0 < s \leq p$ at level β. In the case $s < p$ we have by theorem 2.1 that identities (2.0.9) and (2.0.10) hold. Since v_i are solutions of (2.0.7), by (2.2.10) we have that
$$\left(f(x_o^i)\right)^{\frac{p-n}{p}} E(v_i) \geq \frac{1}{n(\sup_M f)^{\frac{n-p}{p}} K(n,p)^n}. \quad i = \overline{1,k}$$
Then (3.1.5), implies that all the functions v_i are equal to zero, and the expression (2.0.9) implies that u_m converge strongly up to a subsequence to a non zero weak solution of (E_s), $s < p$.

Similarly, in the case $s = p$, we have by theorem 2.2 that identities (2.0.11) and (2.0.12) hold, Since ν_j and v_i are non trivial weak solutions of (2.0.7) and (2.0.8) respectively, the inequalities (2.2.9) and (2.2.10) will gives us that
$$\left(f(y_o^j)\right)^{\frac{p-n}{p}} E(\nu_j) \geq \frac{1}{n(\sup_M f)^{\frac{n-p}{p}} K(n,p)^n}, \quad j = \overline{1,l}.$$
and
$$E_{f,h}(v_i) \geq \frac{\left(1 - h(x_o)(\frac{p}{n-p})^p\right)^{\frac{n}{p}}}{n(f(x_o))^{\frac{n-p}{p}} K(n,p)^n}, \quad i = \overline{1,k}.$$
Then (3.1.6), implies that all the functions v_i and the functions ν_j are equal to zero, and the expression (2.0.11) implies that u_m converge strongly up to a subsequence to a non zero weak solution of (E_s), $s = p$.

□

Let us introduce the Nehari manifold for the functional $J_{f,s,h}$.

$$\mathcal{N}_{h,f,s} = \{u \in H_1^p(M)\backslash\{0\}, (DJ_{h,f,s}(u)).u = 0\}$$

We can easily check that for $u \in H_1^p(M)\backslash\{0\}$, the function defined by

$$\Phi(u) = \left(\frac{\displaystyle\int_M \left(|\nabla u|^p - h\frac{|u|^p}{(\rho_{x_o})^s} \right) dv_g}{\displaystyle\int_M f|u|^{p^*} dv_g} \right)^{\frac{n-p}{p^2}} u, \tag{3.1.7}$$

belongs to $\mathcal{N}_{h,f,s}$, and $J_{h,f,s}(\Phi(u)) = \max_{t>0} J_{h,f,s}(tu)$.
Let $G_{h,f,s}(u) = DJ_{h,f,s}(u)u$, $u \in H_1^p(M)\backslash\{0\}$ and

$$\nabla_{\mathcal{N}_{h,f,s}} J_{h,f,s}(u) = \nabla \cdot J_{h,f,s}(u) - \frac{\nabla J_{h,f,s}(u) \cdot \nabla G_{h,f,s}(u)}{\|\nabla G_{h,f,s}(u)\|^p} \nabla G_{h,f,s}(u), u \in \mathcal{N}_{h,f,s}$$

$\nabla_{\mathcal{N}_{h,f,s}} J_{h,f,s}(u)$ is the projection of $\nabla J_{h,f,s}$ on the tangent space $T_u\mathcal{N}_{h,f,s}$.
We define a constraint P.S. sequence of $J_{h,f,s}$ on $\mathcal{N}_{h,f,s}$ at level β as a sequence u_m such that $\nabla_{\mathcal{N}_{h,f,s}} J_{h,f,s}(u_m) \to 0$ and $J_{h,f,s}(u_m) \to \beta$.
The next lemma follows immediately from the expression of $\nabla_{\mathcal{N}_{h,f,s}} J_{h,f,s}(u)$

Lemma 3.4. *If u_m is a constrained Palais-Smale sequence of $J_{h,f,s}$ on $\mathcal{N}_{h,f,s}$, then u_m is a Palais-Smale sequence of $J_{h,f,s}$ on $H_1^p(M)$,.*

Now, we set

$$\mu = \inf_{u \in H_1^p(M)\backslash\{0\}} \frac{\displaystyle\int_M \left(|\nabla u|^p - h\frac{|u|^p}{(\rho_{x_o})^s} \right) dv_g}{\left(\displaystyle\int_M f|u|^{p^*} dv_g \right)^{\frac{p}{p^*}}}. \tag{3.1.8}$$

The following proposition is the first step towards proving theorems 3.1 and 3.2. This proposition gives generic condition for the existence of a weak solution of equation (E_s).

Proposition 3.5. *Assume that the function f is positive on the manifold M, that $h(x_o) > 0, 1 - h(x_o)(\frac{p}{n-p})^p > 0$ and that the operator $L_{h,s}$ is coercive, and*

$$\mu < \frac{1}{(\sup_M f)^{\frac{n-p}{n}} K(n,p)^p},$$

then the equation (E_s) with $0 < s < p$, possesses a positive weak solution. In addition, if

$$\sup_{x \in M}(f(x)) \geq \frac{f(x_o)}{\left(1 - h(x_o)(\frac{p}{n-p})^p\right)^{\frac{n}{n-p}}}, \tag{3.1.9}$$

and

$$\mu < \frac{1}{(\sup_M f)^{\frac{n-p}{n}} K(n,p)^p},$$

equation (E_s) with $s = p$, possesses a positive weak solution.

Proof. First of all, the functional $J_{h,f,s}$ is bounded from bellow on $\mathcal{N}_{h,f,s}$. Then the Ekeland variational principle (see corollary 1.29) gives a Palais-Smale sequence $u_m \in \mathcal{N}_{h,f,s}$ at level $\beta = \inf_{u \in \mathcal{N}_{h,f,s}} J_{h,f,s}(u)$ and by the coercivity of $L_{h,s}(u)$, we get that β is positive. Now, let $u \in H_1^p(M)\backslash\{0\}$, since $\Phi(u) \in \mathcal{N}_{h,f,s}$, then

$$J_{h,f,s}(\Phi(u)) = \frac{1}{n}\int_M f|\Phi(u)|^{p^*} dv_g = \frac{1}{n}\left(\frac{\int_M \left(|\nabla_g u|^p - h\frac{|u|^p}{(\rho_{x_o(x)})^s}\right) dv_g}{\left(\int_M f|u|^{p^*} dv_g\right)^{\frac{p}{p^*}}}\right)^{\frac{n}{p}}. \tag{3.1.10}$$

Since $\beta = \inf_{u \in \mathcal{N}_{h,f,s}} J_{h,f,s}(u)$, then

$$J_{h,f,s}(\Phi(u)) = \frac{1}{n}\left(\frac{\int_M \left(|\nabla_g u|^p - h\frac{|u|^p}{(\rho_{x_o(x)})^s}\right) dv_g}{\left(\int_M f|u|^{p^*} dv_g\right)^{\frac{p}{p^*}}}\right)^{\frac{n}{p}} \geq \beta$$

Given that u is arbitrary in $H_1^p(M)\backslash\{0\}$, we obtain by definition of μ that

$$\mu \geq n\beta^{\frac{p}{n}}.$$

However, under the assumptions on μ, we obtain by (3.1.9), that

$$\beta < \begin{cases} \dfrac{1}{n(\sup_M f)^{\frac{n-p}{p}} K(n,p)^n}, & \text{for } 0 < s < p; \\ \min\left(\dfrac{1}{(\sup_M f)^{\frac{n-p}{p}}}, \dfrac{(1-h(x_o)(\frac{p}{n-p})^p)^{\frac{n}{p}}}{(f(x_o))^{\frac{n-p}{p}}} \right) \dfrac{1}{nK(n,p)^n}, & \text{for } s = p. \end{cases} \tag{3.1.11}$$

Therefore, by Proposition 3.3, we conclude that the sequence u_m converges, to a subsequence, to a weak solution u_0 of (E_s), $0 < s \leq p$, which verifies that $\beta = J_{h,f,s}(u_0) = \inf_{u \in \mathcal{N}_{h,f,s}} J_{h,f,s}(u)$. Finally, to prove that u_0 is positive, from (3.1.10), we see that :

$$J_{h,f,s}(\Phi(|u_0|)) = J_{h,f,s}(\Phi(u_0)) = J_{h,f,s}(u_0),$$

and since, $\Phi(|u_0|) = |u_0| \in \mathcal{N}_{h,f,s}$. Then $u_0 = |u_0|$ which implies that u_0 is positive. $\square$

3.2 Proof of theorem 3.1

Let $x_1 \in M$, such that $x_1 \neq x_o$, and $\delta > 0$ be chosen so that $B(x_1,\delta) \cap B(x_o,\delta) = \emptyset$. Let φ be a regular cut-off function defined on $\mathbb{R}$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ for $|r| < \frac{\delta}{2}$ and $\varphi = 0$ for $|r| \geq \delta$.
The distance from x_1 to x is denoted by $r = d_g(x_1, x)$. For $\varepsilon \in (0,1)$ let

$$u_\varepsilon(x) = \varphi(r)\left(\varepsilon + r^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}}.$$

We assume that $f(x_1) = \sup_{x \in M} f(x)$ and

$$f(x_1) \geq \frac{f(x_o)}{\left(1 - h(x_o)(\frac{p}{n-p})^p\right)^{\frac{n}{n-p}}}.$$

Then by proposition 3.5, theorem 3.1 is proved if

$$E(u_\varepsilon) = \frac{\displaystyle\int_M \left(|\nabla u_\varepsilon|^p - h\frac{|u_\varepsilon|^p}{(\rho_{x_o})^s} \right) dv_g}{\left(\displaystyle\int_M f|u_\varepsilon|^{p^*} dv_g \right)^{\frac{p}{p^*}}} < \frac{1}{(f(x_1))^{\frac{n-p}{n}} K(n,p)^p}, 0 < s \leq p. \quad (3.2.1)$$

Let $b(x) = \frac{h(x)}{\rho^s_{x_o}(x)}, 0 < s \leq p$. Then, for δ small enough, the function $b(x)\varphi$ is regular on M, so by using Theorem 1.5, and formula (5.5) in [17], we have

$$\begin{aligned} E(u_\varepsilon(x)) &\leq \frac{1}{K(n,p)^p f(x_1)^{\frac{p}{p^*}}} [\, 1 + \varepsilon^{\frac{n}{p-1}} \left(A_1 - A_2 \frac{h(x_1)}{\rho^s_{x_o}(x_1)} \varepsilon^{\frac{p^2-n}{p}} + A_3 \varepsilon^{\frac{3p-2-n}{p}} \right) \\ &+ o(\varepsilon^{\frac{p-n^2}{p}}) + o(\varepsilon^{2\frac{p-1}{p}+1-\frac{n}{p}}) \,], \end{aligned}$$

where A_1,A_2 and A_3 are positive constants. Now, we distinguish three cases:

1. $1 < p < 2$ et $h(x_1) > 0$. Then,

   $$\frac{p^2 - n}{p} < \min(0, \frac{3p-2-n}{p}),$$

   and the term $\varepsilon^{\frac{p-n^2}{p}}$ is dominant. Therefore, we can find ε small enough so that the inequality (3.2.1) is satisfied.

2. $p = 2$, then $\frac{p^2-n}{p} = \frac{3p-2-n}{p} < 0$ and by [17, page 787], we have

   $$A_3 - A_2 \frac{h(x_1)}{\rho^s_{x_o}(x_1)} = \frac{1}{2n} \left(\frac{\Delta_g f(x_1)}{f(x_1)} - \frac{2}{n-4} Scal_g(x_1) - \frac{8(n-1)}{(n-2)(n-4)} \frac{h(x_1)}{\rho^s_{x_o}(x_1)} \right)$$

   Thus, the condition (3.1.3) of theorem 3.1 is sufficient to find ε small so that (3.2.1) is satisfied.

3. $p > 2$. Then, $3p - 2 - n < p^2 - n < 0$ and the term $\varepsilon^{\frac{3p-2-n}{p}}$ is dominant. Since we have by [17]

   $$A_3 = C(n,p)((n+2-3p)\frac{\Delta_g f(x_1)}{f(x_1)} - pScal_g(x_1)),$$

   the condition (3.1.4) of theorem 3.1 is sufficient for (3.2.1) hold.

3.3 Proof of theorem 3.2

Let δ be a small positive constant. Let ψ denote a smooth cut-off function on $\mathbb{R}$ such that $0 \leq \psi \leq 1$, $\varphi = 1$ for $|r| < \delta$ and $\psi = 0$ for $|r| \geq 2\delta$, for $\varepsilon \in (0,1)$, consider the functions

$$U_\varepsilon(x) = \psi(\rho_{x_o}(x)) \left(\varepsilon + (\rho_{x_o}(x))^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}} \tag{3.3.1}$$

As in the proof of theorem 3.1, by proposition 3.5, theorem 3.2 will be proved if

$$E(U_\varepsilon) = \frac{\displaystyle\int_M \left(|\nabla U_\varepsilon|^p - h\frac{|U_\varepsilon|^p}{(\rho_{x_o})^s}\right) dv_g}{\left(\displaystyle\int_M f|U_\varepsilon|^{p^*} dv_g\right)^{\frac{p}{p^*}}} < \frac{1}{(f(x_o))^{\frac{n-p}{n}} K(n,p)^p}. \tag{3.3.2}$$

Consider geodesic normal coordinates around the point x_o. In these coordinates we have

$$dv(g) = \left(1 + r^{1+\eta} O(1)\right) dx, 0 < \eta < 1.$$

Besides, we have

$$h(x)\psi(r)^p = h\left(x_o\right) + \partial_i h(x_o) x^i + r^{1+\eta} O(1).$$

Then,

$$\begin{aligned}
\int_M \frac{h(x)}{(\rho_{x_o})^s} U_\varepsilon^p dv_g &= h(x_o) \int_{B(0,\delta)} \frac{1}{|x|^s} \left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{p-n} dx \\
&+ \partial_i h(x_o) \int_{B(0,\delta)} \frac{1}{|x|^s} \left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{p-n} x^i dx \\
&+ \int_{B(0,\delta)} \frac{1}{|x|^s} \left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{p-n} |x|^{1+\eta} O(1) dx.
\end{aligned}$$

We remark that

$$\int_{B(0,\delta)} \frac{1}{|x|^s} \left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{p-n} x^i dx = 0$$

for all i. Which implies

$$\begin{aligned}
\int_M \frac{h(x)}{(\rho_{x_o})^s} U_\varepsilon^p dv_g &= h(x_o)\omega_{n-1} \int_0^\delta \frac{1}{r^s} \left(\varepsilon + r^{\frac{p}{p-1}}\right)^{p-n} r^{n-1} dr \\
&+ \int_{S^{n-1}} \int_0^\delta \frac{1}{r^s} \left(\varepsilon + r^{\frac{p}{p-1}}\right)^{p-n} r^{n+\eta} O(1) dr d\sigma.
\end{aligned}$$

By setting $r = \varepsilon^{\frac{p-1}{p}} t$, we get

$$\int_M \frac{h(x)}{(\rho_{x_o})^s} U_\varepsilon^p dv_g = \varepsilon^{\frac{p^2-ps-n+s}{p}} h(x_o)\omega_{n-1} \int_0^{+\infty} (1+t^{\frac{p}{p-1}})^{p-n} t^{n-s-1} dt + o(\varepsilon^{\frac{p^2-ps-n+s}{p}}) \tag{3.3.3}$$

Note here that the integral

$$I_4 = \int_0^{+\infty} (1+t^{\frac{p}{p-1}})^{p-n} t^{n-s-1} dt$$

exists as soon as $n > p^2 - s(p-1)$.

Now, the following relations are established in [17, Proof of theorem 1.5]

$$\begin{aligned} \int_M f(x) U_\varepsilon^{p^\star} dv(g) = & \left(f(x_o)\omega_{n-1} \int_0^\infty \left(1+t^{\frac{p}{p-1}}\right)^{-n} t^{n-1} dt \right) \varepsilon^{-\frac{n}{p}} \\ & - \left[\frac{\omega_{n-1}}{2n} \left(\Delta_g f(x_o) + \frac{\mathrm{Scal}(g)(x_o) f(x_o)}{3} \right) \right. \\ & \left. \times \int_0^\infty \left(1+t^{\frac{p}{p-1}}\right)^{-n} t^{n+1} dt \right] \varepsilon^{\frac{-n+2p-2}{p}} \\ & + o\left(\varepsilon^{\frac{-n+2(p-1)}{p}} \right), \end{aligned} \tag{3.3.4}$$

and

$$\begin{aligned} \int_M |\nabla_g U_\varepsilon|^p dv(g) \le & C + K(n,p)^{-p} \left(\omega_{n-1} \int_0^\infty \left(1+t^{\frac{p}{p-1}}\right)^{-n} t^{n-1} dt \right)^{\frac{p}{p^*}} \varepsilon^{\frac{p-n}{p}} \\ & - \left[\left(\frac{n-p}{p-1} \right)^p \omega_{n-1} \frac{\mathrm{Scal}(g)(x_o)}{6n} \int_0^\infty \left(1+t^{\frac{p}{p-1}}\right)^{-n} t^{\frac{p}{p-1}+n+1} dt \right] \times \varepsilon^{\frac{3p-2-n}{p}} \\ & + o\left(\varepsilon^{\frac{3p-2-n}{p}} \right), \end{aligned} \tag{3.3.5}$$

with C a positive constant independent of ε. Knowing that the function

$$\psi(x) = \left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}}, x \in I\!R^n$$

achieves the best constant for the inclusion $H_1^p(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$, we have

$$\begin{aligned} & \left(\omega_{n-1} \int_0^\infty \left(1+t^{\frac{p}{p-1}}\right)^{-n} t^{n-1} dt \right)^{\frac{p}{p^*}} = \\ & K(n,p)^p \left(\frac{n-p}{p-1} \right)^p \omega_{n-1} \int_0^\infty \left(1+t^{\frac{p}{p-1}}\right)^{-n} t^{\frac{p}{p-1}+n-1} dt. \end{aligned}$$

So we obtain

$$\begin{aligned} E(U_\varepsilon) &\leq \frac{1}{K(n,p)^p f(x_o)^{\frac{p}{p^*}}} \left[\, 1+\varepsilon^{\frac{n}{p}-1} \left(B_1 - B_2 \varepsilon^{\frac{p^2-ps+s-n}{p}} + B_3 \varepsilon^{\frac{3p-2-n}{p}} \right.\right. \qquad (3.3.6) \\ &+ \left.\left. o(\varepsilon^{\frac{p^2-ps-n+s}{p}}) + o\left(\varepsilon^{\frac{3p-2-n}{p}}\right) \right) \right] \end{aligned}$$

with

$$\begin{aligned} B_1 &= CK(n,p)^p(\omega_{n-1} I_5)^{-\frac{p}{p^*}}, \\ B_2 &= \omega_{n-1}^{1-\frac{p}{p^*}} K(n,p)^p h(x_o) \frac{I_4}{(I_5)^{\frac{p}{p^*}}}, \\ B_3 &= \frac{p}{2np^*}\left(\frac{\Delta_g f(x_o)}{f(x_o)} + \frac{\mathrm{Scal}(g)(x_o)}{3}\right)\frac{I_1}{I_5} - \frac{\mathrm{Scal}(g)(x_o)}{6n}\frac{I_3}{I_2}, \end{aligned}$$

and

$$\begin{aligned} I_1 &= \int_0^\infty \left(1+t^{\frac{p}{p-1}}\right)^{-n} t^{n+1} dt \\ I_2 &= \int_0^\infty \left(1+t^{\frac{p}{p-1}}\right)^{-n} t^{\frac{p}{p-1}+n-1} dt \\ I_3 &= \int_0^\infty \left(1+t^{\frac{p}{p-1}}\right)^{-n} t^{\frac{p}{p-1}+n+1} dt \\ I_4 &= \int_0^{+\infty} (1+t^{\frac{p}{p-1}})^{p-n} t^{n-s-1} dt \\ I_5 &= \int_0^\infty \left(1+t^{\frac{p}{p-1}}\right)^{-n} t^{n-1} dt. \end{aligned}$$

All the above integrals exist if $n > p^2$.

From [17, Proof of theorem 1.5] and [15, Lemma 7], we have

$$\begin{aligned} I_1 &= \frac{p-1}{p}\frac{\Gamma(\frac{(n+2)(p-1)}{p})\,\Gamma(\frac{n-2(p-1)}{p})}{\Gamma(n)}, \\ I_2 &= \frac{p-1}{p}\frac{\Gamma(n-\frac{n}{p}+1)\,\Gamma(\frac{n}{p}-1)}{\Gamma(n)}, \\ I_3 &= \frac{p-1}{p}\frac{\Gamma(n-\frac{n+2}{p}+3)\,\Gamma(\frac{n+2}{p}-3)}{\Gamma(n)}, \\ I_4 &= \frac{p-1}{p}\frac{\Gamma(\frac{(n-s)(p-1)}{p})\,\Gamma(\frac{n-p^2+s(p-1)}{p})}{\Gamma(n-p)}, \\ I_5 &= \frac{p-1}{p}\frac{\Gamma(\frac{n(p-1)}{p})\,\Gamma(\frac{n}{p})}{\Gamma(n)}. \end{aligned}$$

We deduce that

$$B_3 = C(n,p)\left((n+2-3p)\frac{\Delta_g f(x_o)}{f(x_o)} - p\,\mathrm{Scal}(g)(x_o)\right),$$

with

$$C(n,p) = \frac{\Gamma(n-\frac{n}{p}-\frac{2}{p}+2)\Gamma(n-\frac{n}{p}-\frac{2}{p}-3)}{2n^2\Gamma(n-\frac{n}{p})\Gamma(\frac{n}{p}-1)}.$$

For $p>1$ we distinguish three different cases:

1- When $1<p<s+2$, we have

$$\frac{p^2-ps+s-n}{p} < \min\left(0, \frac{3p-2-n}{p}\right).$$

According to (3.3.6), and for ε sufficiently small, since $h(x_o)>0$, we obtain that $E(U_\varepsilon) < \frac{1}{K(n,p)^p f(x_o)^{\frac{p}{p^*}}}$.

2- When $p=s+2$, we will have

$$\frac{p^2-ps+s-n}{p} = \frac{3p-2-n}{p} < 0.$$

By relation (1.2.1), after making the necessary calculations, we find that

$$B_2 = \left(\frac{p-1}{n-p}\right)^p h(x_o)\frac{\Gamma(n-\frac{n+2}{p}+3-p)\,\Gamma(\frac{n+2}{p}-3)\,\Gamma(n)}{\Gamma(n-\frac{n}{p}+1)\,\Gamma(\frac{n}{p}-1)\,\Gamma(n-p)}.$$

By a simple calculation, we obtain that

$$\begin{aligned} B_3 - B_2 &= \frac{\Gamma(\frac{n+2}{p}-3)}{\Gamma(n-\frac{n}{p}+1)\,\Gamma(\frac{n}{p}-1)} \\ &\left[\frac{\Gamma(n-\frac{n}{p}-\frac{2}{p}+2)}{2n^2}\left((n+2-3p)\frac{\Delta_g f(x_o)}{f(x_o)} - p\,\mathrm{Scal}(g)(x_o)\right)\right. \\ &- \left.\left(\frac{p-1}{n-p}\right)^p \frac{p}{n(p-1)}\frac{\Gamma(n-\frac{n+2}{p}+3-p)\,\Gamma(n)}{\Gamma(n-p)}h(x_o)\right] \end{aligned}$$

According to (3.3.6), and for ε sufficiently small, we have that $E(U_\varepsilon) < \frac{1}{K(n,p)^p f(x_o)^{\frac{p}{p^*}}}$ if

$$\left(\frac{p-1}{n-p}\right)^p \frac{p}{n(p-1)}\frac{\Gamma(n-\frac{n+2}{p}+3-p)\,\Gamma(n)}{\Gamma(n-p)}h(x_o) >$$
$$\frac{\Gamma(n-\frac{n}{p}-\frac{2}{p}+2)}{2n^2}\left((n+2-3p)\frac{\Delta_g f(x_o)}{f(x_o)} - pScal(g)(x_o)\right).$$

3- When $p > s + 2$, we will have

$$\frac{3p - 2 - n}{p} < \frac{p^2 - ps - n + s}{p} < 0.$$

Once more, according to (3.3.6), we obtain that for ε small enough, $E(U_\varepsilon) < \frac{1}{K(n,p)^p f(x_o)^{\frac{p}{p^*}}}$ if

$$(n + 2 - 3p)\frac{\Delta_g f\left(x_o\right)}{f\left(x_o\right)} - p\,\mathrm{Scal}(g)\left(x_o\right) < 0.$$

This proves theorem 3.2.

Conclusion

In this thesis we studied a singular quasilinear equation that contains a critical Sobolev exponent. The goal was to prove some results on compactness of Palais-Smale sequences and on the existence of solutions. Some of these goals are successfully reached, but, unfortunately, not all of them. In fact, existence result could not been proved is case of reverse inequality (3.1.2) in theorem 3.1. This was because of lack of a classification of solutions for equation (2.0.8) in the form of that obtained in [48] for the case $p = 2$. Anyway, this classification result is still not available and we hope that it will be tackled in the future. We hope also to generalize this work for more general class of equations like those that contains the fractional p-Laplacian.

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Abstract

In this thesis we study, on compact Riemannian manifolds, a quasi-linear elliptic equation in p-Laplacian operator containing a Hardy term and a critical Sobolev exponent. We first show that Palais-Smale sequences of our equation are submitted to the well known Struwe decomposition formulas. In a second part, we prove some existence results relying on the decomposition results.

Keywords : Riemannian manifolds, Yamabe equation, p-Laplacian, Sobolev exponent, Hardy potential, blow up analysis, bubbles.

Résumé

Dans cette thèse, nous étudions, sur des variétés Riemanniennes compactes, une équation elliptique quasi-linéaire en p-laplacien contenant un terme de Hardy et un exposant critique de Sobolev. Dans une première partie nous démontrons des résultats de décomposition de type Struwe pour les suites de Palais-Smale associées a notre équation. Dans une deuxième partie, nous utilisons les résultats de décomposition obtenus dans la premier partie pour démontrer des résultats d'existence pour notre équation.

Mots clés : Variétés Riemanniennes, équation de Yamabe, p-Laplacien, l'exposant critique de Sobolev, le potentiel de Hardy, analyse de blow up, bulles.

ملخص

في هذه الأطروحة، نهتم بدراسة معادلة شبه خطية بمشغل ب-لابلاسيان و تحتوي على عبارة هاردي والأس الحرج لسوبوليف على متنوعات ريمانية مدمجة. في الجزء الأول نثبت أن متتاليات بالاي-سمايل المرفقة لمعادلتنا تخضع لصيغ تفكيكية مشهورة تُعرف بصيغ تفكيك ستروف. في الجزء الثاني، بالاعتماد علي الصيغ التفكيكية المبرهن عليها في الجزء الأول نثبت بعض نتائج الوجود.

الكلمات مفتاحية: متنوعات ريمانية، معادلة يامابي، أس سوبوليف، عبارة هاردي، تحليل الانفجار، فقاعات.