Democratic and Popular Republic of Algeria
Ministry of Higher Education and Scientific Research
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TITLE

MECHANICS OF THE MATERIAL POINT

(COURSES AND CORRECTED EXERCISES)

Addressed to students level : License (L1)

Domain : Mathematics
Branch: Mathematics
Speciality : Mathematics and Computer Sciences.

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PREFACE

This physics text book is intended for a broad range of students; it covers the fields of Sciences and Technology (ST), Material sciences (SM), and in particularly suitable for first-year LMD (License, Master, and Doctorate) students in Mathematics and Computer Sciences.

This document follows the official curriculum of the Physics 1 module taught in the first year of the LMD programs mentioned earlier. This manuscript is essential for first-year students given the importance of concentration and a thorough understanding of scientific concepts. The combination of these elements, along with the requirement to take notes during the course, can be crucial for effective learning in disciplines such as physics and the sciences. Hence, the necessity to provide such documents to our students.

The proposed work is the result of my experience in courses and supervised work for several years at the University of Tlemcen. This manual compiles all the courses on the mechanics of the material point, with varying levels of detail, along with exercises and problems with solutions and a set of exams from previous years.

This handout comprises six chapters on mechanics. The first one focuses on dimensional analysis. Mathematical reminders and concepts about vectors are presented in the second chapter. The third chapter deals with the kinematics of material point in various coordinate systems. Chapter 4 details relative motion or changes in reference frames. The dynamics of material points is covered in the fifth chapter. Finally, the sixth chapter is dedicated to work and energy. I must stress that this document in no way replaces face-to-face tutorials.

I hope that this collection of exercises and solved test problems in point mechanics will be of great help to the majority of students.

Any comment, proposal or constructive criticism allowing the improvement of the texts will be collected with great interest.

<u>Dr. Hadjou Bélaid Zakia</u>

Summary

Chapter I : Dimensional Analysis And Uncertainty Calculation

Chapter II : Vector Analysis

Chapter III: Kinematics Of Material Point

Chapter IV: Relatif Motion

Chapter V: Dynamics of Particle

Chapter VI: Work and Energy

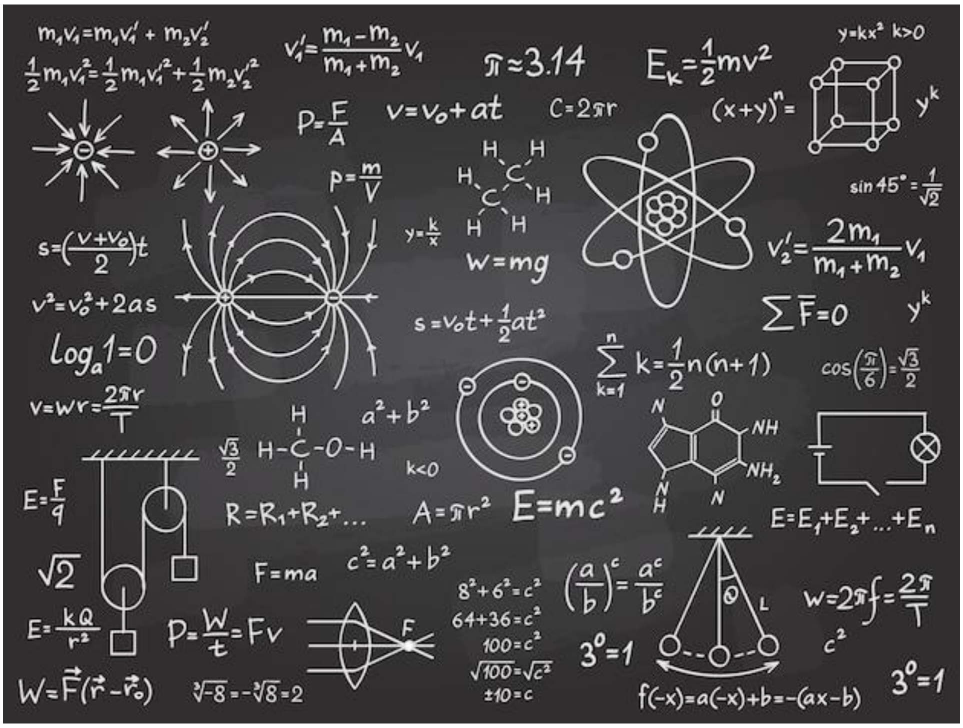
University of Tlemcen
Faculty of Sciences
Department of Mathematics

U.Y:2023/2024

COURSE OF MECHANICS

OF THE MATERIAL POINT

Chapter I: Dimensional Analysis and Uncertainty Calculation



Glossary

In English	In French	In Arabic
Mechanics	La mécanique	مكانيك
Physical quantities	Une grandeur physique	مقدار فيزيائي
Dimensional analysis	Analyse dimensionnelle	التحليل البعدي
Uncertainty Calculation	Calcul d'incertitudes	حساب الارتيابات
Unit	Unité	الوحدة
Force	La force	القوة
Motion	Le Mouvement	الحركة
Velocity (speed)	La vitesse	السرعة
Acceleration	L'accélération	التسارع
Length	Lalongeur	المسافة
Time	Le temps	الزمن
Mass	La masse	الكتلة
Weight	Le poids	الوزن
Momentum	La quantité du mouvement	كمية الحركة
Work	Le travail	العمل
Energy	L'énergie	الطاقة
Power	La puissance	الاستطاعة
Equilibrium	Etat d'équilibre	التوازن
Surface	La surface	المساحة
Volume	Le volume	الحجم
Density	La masse volumique	الكتلة الحجمية
Frequency	La fréquence	التواتر
Linear velocity	La vitesse linéaire	السرعة الخطية
Angular velocity	La vitesse angulaire	السرعة الزاوية
Linear Acceleration	L'accélération linéaire	التسارع الخطي
Angular Acceleration	L'accélération angulaire	التسارع الزاوي
Pressure	La préssion	الضغط
Acceleration of gravity	Accélération de pesanteur	تسارع الجاذبية

Current intensity	L’intensité du courant	شدة التيار الكهربائي
Light intensity	L’intensité de la lumière	شدة الضوء
Quantity of material	La quantité de la matière	كمية المادة
Height	La hauteur	الارتفاع
Dimensionless	Sans dimension	بدون بعد
The period of a pendulum	La période d’une pendule simple	دور نواس بسيط
The sound	Le son	الصوت
Radius	Le rayon	نصف قطر
Relative uncertainty	L’incertitude relative	الارتياب النسبي
Absolute uncertainty	L’incertitude absolue	الارتياب المطلق
Total Differential method	Méthode différentielle totale	طريقة التفاضلية الكلية
Logarithmic Method	Méthode logarithmique	الطريقة اللوغارتمية
Absolute Error	Erreur absolue	الخطا المطلق
Relative Error	Erreur relative	الخطا النسبي
The International System (SI)	Système international SI	النظام الدولي
The CGS system	Système CGS	النظامCGS
Average speed	La vitesse moyenne	السرعة المتوسطة
Instantaneouse speed	La vitesse instantanée	السرعة اللحضية

Part 1: Dimensional analysis

التحليل البعدي

1. Introduction

The observation of physical phenomena is incomplete if it does not lead to quantitative information, which is the measurement of physical quantities. To study a physical phenomenon, one must examine the important variables; the mathematical relationship between these variables constitutes a physical law.

This is possible in certain cases, but for other cases, it is necessary to use a modeling method such as dimensional analysis.*(التحليل البعدي)*

2. Definition of Dimensional Analysis تعريف التحليل البعدي

It is a theoretical tool for interpreting problems based on the dimensions of the involved physical quantities: length, time, mass, and so on.

Dimensional analysis allows for:

- Verifying the validity of dimensioned equations.
- Investigating the nature of physical quantities.
- Exploring the homogeneity of physical laws.
- Determining the unit of a physical quantity based on fundamental units (meter, second, kilogram, etc.).

3. Physical Quantity مقدار فيزيائي

A physical quantity is an observable and measurable property through a specifically designed instrument. Mechanics acknowledges seven fundamental physical quantities: length, time, mass, electric current, temperature, quantity of material, and luminous intensity. Other physical quantities, known as derived quantities, are expressed in terms of these three fundamental quantities, such as velocity, acceleration, force, and more…..

Note :

In general, for first-year students in Mathematics (M), and Computer Science (I), the focus is primarily on the first three fundamental quantities: length, time, and mass.

4. International System of Units الوحدة في النظام العالمي

The value of a physical quantity is given in relation to a standard known as a "unit". The first four fundamental units constitute the MKSA International System (Meter, Kilogram, Second, Ampere). Using these fundamental units, derived units can be constructed: area (m^2), velocity (m/s), force (kg m/s^2)…

Fundamental quantities المقادير الأساسية	Units الوحدة (in the international system MKSA)	Symbols الرمز
Length	**Meter**	**(m)**
Mass	**Kilogram (kg)**	**(kg)**
Time	**Second**	**(s)**
Current intensity	**Ampere**	**(A)**
Temperature	**Kelvin**	**(K)**
Light intensity	**Candela**	**(Cd)**
Quantity of material	**Mole**	**(mol)**

There are specific units such as N (Newton) for force, Hz (Hertz) for frequency, Watt for power, Pascal (Pa) for pressure...

Note: There are two systems of units:

- The International System (SI) known as MKSA (Meter, Kilogram, Second, Ampere), which is the most widely used system.

- The CGS system (Centimeter, Gram, Second), which is less commonly used.

5. Dimensional Equations معادلة ابعاد

Dimension represents the nature of a physical quantity. A physical quantity has only one possible dimension.

The dimension of a quantity G is denoted by: $[\mathbf{G}] = \mathbf{L}$

By denoting M, L, and T as the dimensions of the fundamental quantities mass, length, and time, we can express the dimensions of other derived quantities in terms of these three fundamental dimensions. The resulting equations are the dimension equations for these physical quantities.

The Fundamental Quantities المقادير الأساسية	**Symbols** الرمز	**Dimensions** الأبعاد	**Units** الوحدة **(International System (SI))**
Length	***L***	$[\boldsymbol{l}] = \boldsymbol{L}$	***Meter (m)***
Mass	***M***	$[\boldsymbol{m}] = \boldsymbol{M}$	***Kilogram (kg)***
Time	***T***	$[\boldsymbol{t}] = \boldsymbol{T}$	***Second (s)***
Current intensity	I	$[\mathrm{I}] = \mathrm{I}$	Ampere (A)
Temperature	T	$[\mathrm{T}] = \theta$	Kelvin (K)
Light intensity	J	$[\mathrm{j}] = \mathrm{J}$	Candela (Cd)
Quantity of material	N	$[\mathrm{n}] = \mathrm{N}$	Mole (mol)

Example :

- $[\text{speed}] = [\text{v}] = \frac{[length]}{[\text{time}]} = \frac{[\text{l}]}{[\text{t}]} = \frac{\text{L}}{\text{T}} = \text{LT}^{-1}$ and the unit of speed is (m/s).
- $[\text{acceleration}] = [\text{a}] = \frac{[\text{speed}]}{[\text{time}]} = \frac{[\text{v}]}{[\text{t}]} = \frac{\text{LT}^{-1}}{\text{T}} = \text{LT}^{-2}$ and the unit of acceleration is (m/s^2).
- $[\text{Force}] = [\text{F}] = [\text{mass}][\text{acceleration}] = [\text{m}][\text{a}] = \text{MLT}^{-2}$ and the unit of force is Newton or ($kg.m/s^2$).

Notes :

- The dimension of constants is always equal to 1; we say they are dimensionless.
- Angles and functions like sin, cos, tan, exp, ln, and log are dimensionless functions.

[Numeric value] = 1, [angle] = 1, [cosα] = [sin α] = [tan α] = [cot α] = [ln x] = [e^x] = 1

6. Homogeneity of Dimensional Equations تجانس معادلة ابعاد

The two sides of a dimension equation must have the same dimensions since they represent quantities of the same nature.

G is a physical quantity:

- $G = A \pm B \Rightarrow [G] = [A] = [B]$
- $G = A * B \Rightarrow [G] = [A] * [B]$
- $G = A/B \Rightarrow [G] = [A]/[B]$
- $G = A^n \Rightarrow [G] = [A]^n$

Note:

• A heterogeneous (non-homogeneousغير متجانسة) equation is necessarily False.

• A homogeneous equation is not necessarily true.

• Dimensions cannot be added (or subtracted).

Example 1:

$y = \frac{1}{2}at^2 + v_0t + y_0$ is the equation of a physical law.

- Check that this equation is homogeneous?

This equation is homogeneous if: $[y] = \left[\frac{1}{2}\ at^2\right] = [v_0t] = [y_0]$

We have :

$[y] = [y_0] = L,$

$\left[\frac{1}{2}at^2\right] = \left[\frac{1}{2}\right][a][t]^2 = 1\ LT^{-2}T^2 = L,$

And $[v_0 t] = [v_0][t] = LT^{-1}T = L$

So :
$[y] = \left[\frac{1}{2}at^2\right] = [v_0 t] = [y_0]$ is checked.

Hence the equation $y = \frac{1}{2}at^2 + v_0 t + y_0$ is homogeneous.

Notes:

We can use this property of dimension equations to discover physical laws by knowing the variables involved in the given physical phenomenon and the relationship among them.

Example 2:

The period is given in terms of length and severity by the following relationship:

$$T=k\,.l^x.g^y$$

- Give the physical law of period T*الطور*?

For this it is necessary to determine the exponents x and y.

It is assumed that the equation is homogeneous so: $[\boldsymbol{T}] = [\boldsymbol{k}][\boldsymbol{l}]^{x}[\boldsymbol{g}]^{y}$

The dimensions of all physical quantities in the study relationship are written.

$$[l] = L, [k] = 1 \; and \; \text{T is a time so } [T] = T$$

We have weight force قوة الثقل p=mg with:

$$[\boldsymbol{p}] = [\boldsymbol{m}][\boldsymbol{g}] \Rightarrow [\boldsymbol{g}] = \frac{[\boldsymbol{p}]}{[\boldsymbol{m}]}$$

P : the weight, is a force so it has the dimension of a force:$[p] = [F] = MLT^{-2}$.

So $[g] = \frac{[F]}{[m]} = \frac{MLT^{-2}}{M} = LT^{-2}$

g is the acceleration $[g] = LT^{-2}$

Hence : $T=1\,.L^x.(LT^{-2})^y \Rightarrow M^0L^0T^1 = L^{x+y}.T^{-2y}$

By identification we will have:

$$\begin{cases} x+y=0 \\ -2y=1 \end{cases} \Rightarrow \begin{cases} y=-\frac{1}{2} \\ x=-y=\frac{1}{2} \end{cases} \quad \text{so} \quad T=kl^{\frac{1}{2}}g^{-\frac{1}{2}}$$

$\Rightarrow T=k\sqrt{\frac{l}{g}}$ it's the law of the period.

Example 3:

The average speed of the particles is expressed as a function of the mass **m**, the volume **V**, and the pressure **p** by the fallowing expression:

$v=f(m, V, p) \Rightarrow \ v=k.m^{\alpha}.V^{\beta}.p^{\gamma}$

It is assumed that the equation is homogeneous, therefore: $[v]=k[m]^{\alpha}[V]^{\beta}[p]^{\gamma}$............(1)

with $[m]=M, [v]=LT^{-1}, [V]=L^3, [p]=\frac{[F]}{[s]}=\frac{[m][a]}{[s]}=\frac{MLT^{-2}}{L^2}=ML^{-1}T^{-2}$

(1)$\Rightarrow M^0LT^{-1}=M^{\alpha}L^{3\beta}(ML^{-1}T^{-2})^{\gamma} \ \Rightarrow M^0LT^{-1}=M^{\alpha+\gamma}L^{3\beta-\gamma}T^{-2\gamma}$

By identification we will have:

$$\begin{cases} \alpha+\gamma=0 \\ 3\beta-\gamma=1 \\ -2\gamma=-1 \end{cases} \Rightarrow \begin{cases} \gamma=\frac{1}{2} \\ \alpha=-\gamma=-\frac{1}{2} \\ \beta=\frac{1+\gamma}{3}=\frac{1+1/2}{3}=\frac{1}{2} \end{cases}$$

So: $v=km^{-\frac{1}{2}}V^{\frac{1}{2}}p^{\frac{1}{2}} \quad \Rightarrow v=k\sqrt{\frac{pV}{m}}$

It's a law of the average speed of the particles.

Conclusions:

Dimensional analysis serves the following purposes:

- Verification of the homogeneity of physical formulas.

- Determination of the nature and the unit of a physical quantity.

- Exploration of the general form of physical laws.

2nd part: Uncertainty Calculation

حساب الارتيابات

1. Introduction

In an experiment, exact measurements do not exist. These are always accompanied by more or less significant errors depending on the measurement method used, the quality of the instruments used, and the role of the operator. The measuring instrument, even if built upon a standard, also has a certain precision as provided by the manufacturer. Therefore, measurements are carried out with approximations. Estimating the errors made in measurements and their consequences is essential.

2. Absolute and relative uncertaintyالارتياب المطلق و الارتياب النسبي

2.1. Absolute errorالخطا المطلق

The absolute error of a measured quantity G is the difference δG between the experimental value G_m and a reference value that can be considered as exact, G_e. In reality, since the exact value is inaccessible, it is approximated by taking the average of a series of measurements of the quantity G.

$$\boldsymbol{\delta G = |G_{measured} - G_{exact}|}$$

2.2. Relative errorالخطا النسبي

The relative error is the ratio of the absolute error to the reference value. The relative error is dimensionless; it indicates the quality (precision) of the obtained result. It is expressed in terms of a percentage.

$$\boldsymbol{\frac{\delta G}{G} = \frac{|G_{measured} - G_{exact}|}{G_{measured}}}$$

2.3. Absolute uncertainty الارتياب المطلق

This is the maximum error that can be committed in the evaluation.

$$\Delta G \geq |\delta G| \Rightarrow \Delta G = |G_{ex} - G_m| \Rightarrow G_{ex} = G_m \pm \Delta G$$

$$\Rightarrow G_m - \Delta G \leq G_{ex} \leq G_m + \Delta G$$

The general form is:

$$G_{ex} = G_m \pm \Delta G$$

The absolute uncertainty has the same unit as the measured quantity and is always positive.

Example: m=12,121g and Δm=0,02g

The correct expression for the measurement (the condensed writing) of m is:

m = (12,<u>121</u> $\pm$ 0,<u>020</u>) g

2.4. Relative uncertaintyالارتياب النسبي

The relative uncertainty is the ratio between the absolute uncertainty and the measured value of G. It is also expressed in terms of a percentage and is a convenient way to quantify the precision of a measurement. It is denoted as: ΔG/G

It is given in percentage and it is always smaller than 1.

$$\frac{\Delta m}{m} = \frac{0{,}02}{12{,}121} \approx 0.16\%$$

3. Uncertainty Calculation حساب الارتيابات

Generally, there are two mathematical methods for uncertainty calculation: the total differential method, which is a general approach, and the logarithmic method, which is limited to physical laws expressed as a product or a ratio.

3.1. The total differential methodالطريقة التفاضلية الكلية

Let *f(x, y, z)* be a function that depends on three variables x, y, z:

The total differential of the function (*f*) is expressed by the following equation:

$$df = \left(\frac{\partial f}{\partial x}\right)_{y,z=cst} dx + \left(\frac{\partial f}{\partial y}\right)_{x,z=cst} dy + \left(\frac{\partial f}{\partial z}\right)_{x,y=cst} dz$$

$\left(\frac{\partial f}{\partial x}\right)$ is the partial differential of the function f with respect to x , considering y and z as constants.

$\left(\frac{\partial f}{\partial y}\right)$ is the partial differential of the function f with respect to y, considering x and z as constants.

$\left(\frac{\partial f}{\partial x}\right)$ is the partial differential of the function f with respect to z , considering x and y as constants.

The absolute uncertainty on (f) is generally expressed in the form:

$$\Delta \boldsymbol{f} = \left|\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right| \Delta \boldsymbol{x} + \left|\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\right| \Delta \boldsymbol{y} + \left|\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}\right| \Delta \boldsymbol{z}$$

Example:

Let *f(x, y)* be a physical quantity that depends on two variables x and y.

f is expressed as $f(x,y) = 2xy + x^2y$

The total differential of "*f*" will be given by :$\boldsymbol{df} = \left(\frac{\partial f}{\partial x}\right) \boldsymbol{dx} + \left(\frac{\partial f}{\partial y}\right) \boldsymbol{dy}$

With: $\left(\frac{\partial f}{\partial x}\right)_{y=cst} = 2y + 2xy$ and $\left(\frac{\partial f}{\partial y}\right)_{x=cst} = 2x + x^2$

So: $df = (2y + 2xy)dx + (2x + x^2)dy$

Hence the absolute uncertainty on the quantity « f » is given by:

$$\Delta f = |2y + 2xy|\Delta x + |2x + x^2|\Delta y$$

3.2. Logarithmic method الطريقة اللوقاريتمية

This method is based on the logarithm and its derivative.

Consider a three-variable function, G = f(x, y, z). To calculate the relative uncertainty on the function G using the logarithmic differential method, the following steps should be followed:

1. Introduce the logarithmic function to the function G.

2. Calculate d(logG) = dG / (G ln 10) or d(lnG) = dG / G.

3. $\frac{dG}{G} \leq \frac{\Delta G}{G}$, and deduce the relative uncertainty on G.

Example

Let the function $f(x,y,z)=x^a y^b z^c$

x, y et z are a variables and a, b et c are constants in the exponents.

First, we find the logarithm of "f":

$\log f = \log x^a y^b z^c = \log x^a + \log y^b + \log z^c = a \log x + b \log y + c \log z$

Then, we calculate the derivative of log f:

$$\mathbf{d} \log f = a\, \mathbf{d} \log x + b\, \mathbf{d} \log y + c\, \mathbf{d} \log z$$

$$\frac{df}{f} = a\frac{dx}{x} + b\frac{dy}{y} + c\frac{dz}{z}$$

$$\frac{\Delta f}{f} = \left|\frac{a}{x}\right| \Delta x + \left|\frac{b}{y}\right| \Delta y + \left|\frac{c}{z}\right| \Delta z$$

If x, y, z, a, b and c are positive, we can write Δf/f by :

$$\frac{\Delta f}{f} = a\frac{\Delta x}{x} + b\frac{\Delta y}{y} + c\frac{\Delta z}{z}$$

$\frac{\Delta f}{f}$ Represents relative uncertainty on the quantity « f » الارتياب النسبي.

Proposed exercises about chapter I

Exercise 1

Find the dimension of the following physical quantities:

Surface, Volume, Density, Frequency, Linear Velocity, Angular Velocity, Linear Acceleration, Angular Acceleration, Force, Work, Energy, Power, and Pressure.

Exercise 2

The characteristic equation of a constant temperature fluid is as follows:

$$\left(p+\frac{a}{V^2}\right)(V-b)=c$$

Or p is the pressure and V is the volume.

Determine the dimensions of quantities a, b and c.

Exercise 3

Check the homogeneity of this formula:

$$p = \rho g h_1 + h_2 F$$

Such as: P pression, ρ density, g an acceleration of gravity, h_1 and h_2 are heights and F a force.

Exercise 4

The trajectory y=f(x) of a projectile with an initial velocity($\boldsymbol{v_0}$) from a point (O) located at heigh (h) above the impact plane is given by the following formula

$$y=\frac{g}{2v_0^2}x^2+h$$

Show that this formula is homogeneous.

Exercise 5

Are the following formulas dimensionally valid?

1. $F=\frac{Gm}{r}$, such as: F is a force, G is a constant expressed in $\frac{\mathrm{m}^3}{\mathrm{kg\ s}^2}$, m is a mass, and r is a length.
2. $\mathrm{p} = \rho \mathrm{gh}_1 + \mathrm{h}_2\mathrm{F}$ such as: P is a pressure, g is the acceleration due to gravity, h1 and h2 are heights, and F is a force.

3. $\theta = \frac{\mathrm{b\ sin(a)}}{\mathrm{t\ cos(c)}}$, such as: b and t have dimensions of length.

Exercise 6

1. In a fluid, a ray ball (**نصف القطر**) r animated by a velocity v, is subjected to a friction force given by F=-6.π.η.r.v , where η is the viscosity of the fluid.

   What is the dimension of η?

2. When the ball is dropped without initial speed at the moment t = 0, its speed is written to t > 0: $$v = a\left(1 - exp\left(-\frac{t}{b}\right)\right)$$

Where a and b are two quantities that depend on the characteristics of the fluid. What are the dimensions of a and b?

Exercise 7

Experiments have shown that the speed v of sound in a gas is only dependent on the volumetric mass density ρ and the coefficient of compressibility χ. What is the law that provides the speed v as a function of the gas's characteristics? It is noted that χ has unit equivalent to the inverse of pressure.

Exercise 8

The sound emitted by the wire of a guitar is characterized by its frequency f. This frequency is a function of the force F of the wire tension, the length L and the density ρ of the wire.

Find the expression of frequency f assuming the form:

$$f = K\, F^a L^b \rho^c$$

With K a dimensionless constant and the frequency dimension $[f]=T^{-1}$.

Exercise 9

Let the simple pendulum formed of a ball (sphere) of radius R and mass m. The study of the effect of the air on this pendulum shows that its period T depends on a constant k, the coefficient of the air η, the radius of the ball R and its density ρ.

1- Find the expression of the period assuming the form:

   $T = K\eta^x R^y \rho^z \ \ with\ [\eta] = ML^{-1}T^{-1}$

2- Determine relative uncertainty on T based on$\Delta\eta, \Delta R\ and\ \Delta m$.

Exercise 10

The expression for a physical quantity G is:

$$G = \frac{T^2 ga}{4\pi^2} - a^2.$$

Where **T** represents time, **a** is a length, and **g** is the acceleration due to gravity.

1. Determine the dimension of G and deduce its unit.
2. Calculate the absolute uncertainty ΔG in terms of ΔT and Δa.

Exercise 11

The refractive index n of a substance is given by the relation:

$$\mathrm{n} = \sqrt{\mathrm{N}^2 - \sin^2\alpha}$$

Where N is the prism index and α is an angle.

1. Calculate the absolute uncertainty Δn by considering that Δn = f(N, α).
2. Deduce the relative uncertainty Δn/n.

Exercise 12

The speed limit reached by a weighted parachute is a function of its weight P and its surface S, is given by: $v = \sqrt{\frac{P}{K.S}}$

1) Give the dimension of the constant k.
2) Calculate the speed limit of a parachute having the following characteristics:
3) The weight being known to the nearest 2 % and the surface to 3 %, calculate the relative uncertainty Δv/v on the velocity v, thus the absolute uncertainty Δv and deduce the condensed writing of this velocity.

We give: M=90 kg, S=80 m2, g=9,81 m/s2, and k=1,15 MKS.

Exercise 13

The height H of a liquid of mass M contained in a cylinder of radius R is given by the relation:

$$H = \frac{(2.\sigma.cos\alpha)}{(R.g.\rho)}$$

Where α is the liquid-cylinder contact angle, ρ the density of the liquid and g the gravity acceleration.

1- Using the dimensional equations, find the dimension of σ.
2- Determine relative uncertainty on σ based on absolute uncertainties ΔR, Δg, ΔM and Δα.

Exercise 14

The resonance frequency f of an electric circuit is given by the formula:

$$f = \frac{1}{2\pi\sqrt{L.C}}$$

L and C are known with absolute uncertainties ΔL and ΔC.

Determine as a function of L, C, ΔL and ΔC absolute and relative uncertainties on f with the two differential methods.

Exercise 15

A) The sound emitted by a guitar wire is characterized by its frequency f. This frequency is a function of the force F of the wire tension, the length L and the density ρ of the wire.

Or : $f = K\, F^a\, L^b\, \rho^c$ such K is constant dimensionless.

Determine the relationship of f.

B) The focal length f of a lens is determined from the formula:

$$f = \frac{D^2 - a^2}{4D}$$

Calculate the absolute uncertainty Δf as a function of ΔD and Δa.

Correction of exercises about chapter I

Exercise 1

- Surface :

We have: [l]=L, [t]=T and [m]=M.

[Physical quantities]= $M^x L^y T^z$

$S = l \times l \Rightarrow$ [S]=L.L=$L^2 \Rightarrow$ [S]=L^2 (m^2)

- Volume :

V=l×l×l$\Rightarrow$ [S] =L.L.L=$L^3 \Rightarrow$ [V]=L^3 (m^3)

- Density:

$\rho = \frac{m}{V}$ so $[\rho] = \frac{[m]}{[V]} = \frac{M}{L^3} = ML^{-3} \Rightarrow$ [ρ]=ML^{-3} (kg/m^3)

- Frequency:

$f = \frac{1}{T} \Rightarrow [f] = \frac{1}{[T]} = \frac{1}{T} = T^{-1} \Rightarrow$ [f]=T^{-1} (s^{-1} or Hertz)

(Note: Period $[T] = T$; unit is « s »)

- Linear velocity:

$v = \frac{dx}{dt} \Rightarrow [v] = \frac{[x]}{[t]} = \frac{L}{T} = LT^{-1} \Rightarrow [v] = LT^{-1}$ (m./s)

- Angulaire velocity :

$\omega = \theta^{\cdot} = \frac{d\theta}{dt} = \frac{v}{R} \Rightarrow [\omega] = \frac{[\theta]}{[t]} = \frac{1}{T} = T^{-1} \Rightarrow$ [ω]=T^{-1} (Rd/s)

$[angle] = 1\ i\ and\ its\ unit\ is\ the\ radian\ (rad).$

- Lineair acceleration:

$a = \frac{dv}{dt} \Rightarrow [a] = \frac{[dv]}{[dt]} = \frac{LT^{-1}}{T} = LT^{-2} \Rightarrow [a] = LT^{-2}$ ($m./s^2$)

- Angulaire acceleration :

$\omega^{\cdot} = \theta^{\cdot\cdot} = \frac{d\theta^{\cdot}}{dt} \Rightarrow [\omega^{\cdot}] = \frac{[d\theta^{\cdot}]}{[dt]} = \frac{T^{-1}}{T} = T^{-2} \Rightarrow$ [ω˙]=T^{-2} ($Rd./s^2$)

- Force :

$F = m\times a \Longrightarrow [F]=[m]\times[a]=M.L.T^{-2} \Longrightarrow [F]=MLT^{-2}$ ($kg.m.s^{-2}$ or Newton)

- Work :

$W = F\times d\times \cos\alpha \Longrightarrow [W]=[F]\times[d]\times[\cos\alpha]=MLT^{-2}.L.\ 1=ML^2T^{-2}$ ($kg.m^2.s^{-2}$ or Joule)

- Energy :

$E_C = (½).m.\ v^2 \Longrightarrow [E]=[1/2].[m].[v]^2 = ML^2T^{-2}$ (Joule)

$E_P = m.g.h \Longrightarrow [E]=[m].\ [g].[h]= M.LT^{-2}.L=ML^2T^{-2}$ (Joule)

- Power:

$P = W/t \Longrightarrow [P]=[W]/[t]=(ML^2T^{-2})/T=ML^2T^{-3}$ ($kg.m^2.s^{-3}$ or Watt)

- Pressure:

$P = F/S \Longrightarrow [P] = [F]/[S]=(MLT^{-2})/L^2=ML^{-1}T^{-2}$ ($kg.m^{-1}.s^{-2}$ or Pascal).

Summary :

Physical Quantities	Symbol	Formula used	Dimension	Unit (SI)
Surface	S	l×l	L^2	m^2
Volume	V	l×l×l	L^3	m^3
Density	ρ	m/V	ML^{-3}	$Kg./m^3$
Frequency	F	1/T	T^{-1}	s^{-1} or hertz
Linearvilocity	V	dx/dt	LT^{-1}	m/s^1
AngularVilocity	Ω	dθ/dt	T^{-1}	$Rd./s^1$
LinearAcceleration	γ	dv/dt	LT^{-2}	$m./s^2$
AngularAcceleration	$\dot{\omega}$	$d\dot{\theta}/dt$	T^{-2}	$Rd./s^2$
Force	F	m.a	MLT^{-2}	Newton
Work	W	F.d	$ML^2\ T^{-2}$	Joule
Energy	E	$(½)mv^2$	ML^2T^{-2}	Joule
Power	P	W/t	ML^2T^{-3}	Watt
Pressure	$\wp$	F/S	$ML^{-1}T^{-2}$	Pascal

Exercise 2

We have $\left(P+\frac{a}{V^2}\right)\times(V-b)=C$

G=A+B or G=A-B then [G]=[A]=[B]
[b] = [V] = L^3

$[\frac{a}{V^2}]$ = [P]=$\frac{[a]}{[V]^2}$ ⟹[a] = [P]×[V]^2 = M.$L^{-1}T^{-2}$.L^6 = M.L^5T^{-2}

$$[C]=\left[P+\frac{a}{V^2}\right]\times[V-b]$$

On the other hand: $\left[P+\frac{a}{V^2}\right]=[p]=\left[\frac{a}{V^2}\right] et [V-b]=[V]=[b]$

Et [C]=[P]×[V]=$ML^{-1}T^{-2}$. L^3=ML^2T^{-2}

Exercise 3

Check the homogeneity of this formula: $p=\ \rho g h_1+h_2F$

Such as: *p* a pressure, *g* an acceleration of gravity, h_1 and h_2 are heights and *F* a force.

We have: $\begin{cases}[\mathrm{P}]=\mathrm{M}L^{-1}T^{-2}\\ [\mathrm{g}]=LT^{-2}\\ [h_1]=[h_2]=L\\ [F]=MLT^{-2}\\ [\rho]=ML^{-3}\end{cases}$

This expression is homogeneous if:$[p]=[\rho g h_1]=[h_2F]$

$$[\rho g h_1]=ML^{-3}.L.LT^{-2}=\mathrm{M}L^{-1}T^{-2}=[\mathrm{P}]$$

$$and\quad [h_2F]=\mathrm{M}L^2T^{-2}\neq \mathrm{M}L^{-1}T^{-2}$$

So the equation is heterogeneous (not homogeneous).

Exercise 4

This expression is homogeneous if : $[y]=\left[\frac{g}{2v_0^2}x^2\ \right]=[h]$

We have :

$[y]=[h]=L,\left[\frac{g}{2v_0^2}x^2\ \right]=\left[\frac{1}{2}\right]\left[\frac{g}{v_0^2}\right][x]^2=1\ \frac{LT^{-2}}{(LT^{-1})^2}L^2=L,$

So : $[y] = \left[\frac{g}{2v_0^2}x^2\right] = [h]$ is checked.

Hence the equation $y = \frac{g}{2v_0^2}x^2 + h$ is homogeneous.

Exercise 5

1. $F = \frac{Gm}{r}$ =? such that: F is a force, G a constant expressed in $\left(\frac{m^3}{kg\ s^2}\right)$, m a mass and r a length.

$F = ma \Rightarrow [F] = [m][a] = MLT^{-2}$.

The unit of g is $\left(\frac{m^3}{kg\ s^2}\right)$ so $[G] = \frac{L^3}{MT^2} = M^{-1}L^3T^{-2}$

Then $\frac{G.m}{r} = \frac{[G][m]}{[r]} = \frac{M^{-1}L^3T^{-2}\ M}{L} = L^2T^{-2}$

In conclusion $[F] \neq \left[\frac{Gm}{r}\right]$ therefore the relationship $F = \frac{Gm}{r}$ is not valid

2. $p = \rho g h_1 + h_2 F$ such as: P: a pressure, g: the acceleration of gravity, h_1 and h_2: heights and F: a force.

To demonstrate that the relation $p = \rho g h_1 + h_2 F$ is valid it is necessary that:

$$[p] = [\rho g h_1] = [h_2 F]$$

$$p = \frac{F}{S} \Rightarrow [p] = \frac{[F]}{[S]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

$[\rho g h_1]$=$[\rho][g][h_1]$

And $[\rho] = \frac{[m]}{[V]} = ML^{-3}, [g] = [a] = LT^{-2} et\ [h_1] = L$

So $[\rho g h_1]$=$[\rho][g][h_1]$= $ML^{-3}\ LT^{-2}L = ML^{-1}T^{-2}$

$$[h_2F] = [h_2][F] = MLT^{-2}\ L = ML^2T^{-2}$$

In conclusion $[p] = [\rho g h_1] \neq [h_2 F]$ therefore the relationship $p = \rho g h_1 + h_2 F$ is not valid.

3. $\theta = \frac{b\ \sin(a)}{t\ \cos(c)}$, such as: b and t have a dimension of one length.

To demonstrate that $[\theta] = \left[\frac{b\ \sin(a)}{t\ \cos(c)}\right]$=$\frac{[b\ \sin(a)]}{[t\ \cos(c)]}$ is valid, it is necessary to demonstrate that:

$[\theta] = \left[\frac{b\ \sin(a)}{t\ \cos(c)}\right]$=$\frac{[b\ \sin(a)]}{[t\ \cos(c)]}$?

We have$[\theta] = 1$ and $[\sin(a)]$=$[\cos(c)]$=1

$[b\ \sin(a)] = [b\ ][\sin(a)]$= $[b\ ] = L$

$[t \ \cos(c)] = [t\ ][\cos(c)] = [t] = L$

So $\frac{[b \ \sin(a)]}{[t \ \cos(c)]} = \frac{L}{L} = 1$

In conclusion $[\theta] = \left[\frac{b \ \sin(a)}{t \ \cos(c)}\right] = \frac{[b \ \sin(a)]}{[t \ \cos(c)]}$ is verified therefore the relation $\theta = \frac{b \ \sin(a)}{t \ \cos(c)}$ is valid.

Exercise 6

We have: $F = -6\pi\eta r v$

1- $[\eta] = ?$

$$F = -6\pi\mu r v \implies \eta = -\frac{F}{6\pi r v}$$

$$[\eta] = \frac{[F]}{[r][v]} \text{ with } \begin{cases} [r] = L \\ [F] = MLT^{-2} \\ [v] = LT^{-1} \\ [-6\pi] = 1 \end{cases}$$

Where

$$[\eta] = \frac{MLT^{-2}}{L.LT^{-1}} = ML^{-1}T^{-1}$$

2- We have: $v = a\left(1 - exp\left(-\frac{t}{b}\right)\right)$

we're looking for the dimension of [a] and [b]:

The argument of the exponential is therefore dimensionless:

so $[v] = LT^{-1} = [a] \implies [a] = LT^{-1}$

$$\left[exp\left(-\frac{t}{b}\right)\right] = 1 \Rightarrow \left[-\frac{t}{b}\right] = \left[-1.\frac{t}{b}\right] = [-1]\left[\frac{t}{b}\right] = \left[\frac{t}{b}\right] = 1$$

$$\Rightarrow \left[\frac{t}{b}\right] = \frac{[t]}{[b]} = 1$$

$[b] = [t] = T$

Exercise 7 :

We have : $v = k\rho^x\chi^y$ so $[v] = [k][\rho]^x[\chi]^y$.

$$\text{And}\begin{cases}\text{the speed v : } [v] = LT^{-1} \\ \text{the constante k: } [k] = 1 \\ \text{the density } \rho\text{: } [\rho] = ML^{-3} \\ \chi \text{ is homogeneous in contrast to a pressure: } [\chi] = \frac{1}{[p]} = M^{-1}LT^{+2}\end{cases}$$

$$\Rightarrow [v] = LT^{-1} = (ML^{-3})^x(M^{-1}LT^2)^y$$

$$\Rightarrow M^0LT^{-1} = M^xL^{-3x} \quad M^{-1y}L^yT^{2y}$$

$$\Rightarrow M^0L^1T^{-1} = M^{x-y}L^{-3x+y}T^{2y}$$

By identification:

$$\begin{cases} x - y = 0 \\ -3x + y = 1 \\ 2y = -1 \end{cases} \Rightarrow \begin{cases} x = y = -\frac{1}{2} \\ y = -1/2 \end{cases} \Rightarrow v = k\rho^{-1/2}\chi^{-1/2} = \frac{k}{\sqrt{\rho\chi}}$$

Then:

$$v = \frac{k}{\sqrt{\rho\chi}}$$

Exercise 8:

f=K $F^aL^b\rho^c$; This function is therefore homogeneous $[f] = [k][F]^a[L]^b[\rho]^c$

$$\text{with: }\begin{cases}[F] = [m.a] = [m][a] = M.\ LT^{-2} \\ [L] = Let[k] = 1 \\ [\rho] = \left[\frac{m}{V}\right] = ML^{-3} \\ [f] = T^{-1}\end{cases}$$

so: $[f] = (MLT^{-2})^a(L)^b(ML^{-3})^c = T^{-1}$

$$\Rightarrow M^0L^0T^{-1} = M^{a+c}L^{a+b-3c}T^{-2a}$$

$$By\ identification: \begin{cases} a + c = 0 \\ a + b - 3c = 0 \\ -2a = -1 \end{cases}$$

$$\Rightarrow \begin{cases} a = 1/2 \\ b = -a + 3c = -\frac{4}{2} = -2 \\ c = -a = -1/2 \end{cases}$$

F =K $F^{1/2}L^{-2}\rho^{-1/2} = K\sqrt{F}\frac{1}{L^2}\frac{1}{\sqrt{\rho}}$ So $f = k\frac{\sqrt{F}}{L^2\sqrt{\rho}}$

Exercise 9

1- The period of a pendulum is written :

$T = K\eta^x R^y \rho^z$ such $[\eta] = ML^{-1}T^{-1}$

Suppose the relationship is homogeneous so $[T] = [k][\eta]^x[R]^y[\rho]^z$

With $\begin{cases} [\eta] = ML^{-1}T^{-1} \\ [R] = Let[k] = 1 \\ [\rho] = \left[\frac{m}{V}\right] = \frac{M}{L^3} = ML^{-3} \\ [T] = T \end{cases}$

So $[T] = (ML^{-1}T^{-1})^x L^y (ML^{-3})^z = T$

$$(A^X . A^Y = A^{X+Y})$$

$$\Longrightarrow T = M^x L^{-x} T^{-x} L^y \quad M^z L^{-3z}$$

$$\Longrightarrow M^0 L^0 T^1 = M^{x+z} L^{-x+y-3z} \quad T^{-x}$$

$$by\ identification: \begin{cases} x + z = 0 \\ -x + y - 3z = 0 \\ -x = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} x = -1 \\ y = x + 3z = 2 \\ z = -x = 1 \end{cases}$$

$$\Longrightarrow \quad T = K\eta^{-1}R^2\rho^1$$

$$\text{So } T = k\frac{\rho R^2}{\eta}$$

2- The relative uncertainty on T= f($\Delta\eta, \Delta R, \Delta m$) ?

$$T = \frac{K\rho R^2}{\mu} \text{ With } \rho = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3m}{4\pi R^3} \qquad \text{so } T = \frac{3Km}{4\pi R\mu}$$

$$\Longrightarrow \log T = \log\left(\frac{3mK}{4\pi R\mu}\right) = log3K + \log(m) - \log(4\pi) - \log(R) - \log(\mu)$$

$$\Longrightarrow \frac{dT}{T} = \frac{dm}{m} - \frac{dR}{R} - \frac{d\mu}{\mu}$$

$$\Rightarrow \frac{\Delta T}{T} = \left|\frac{\Delta m}{m}\right| + \left|-\frac{\Delta R}{R}\right| + \left|-\frac{\Delta \mu}{\mu}\right|$$

m, R, and μ are positive quantities, hence:

$$\Rightarrow \frac{\Delta T}{T} = \frac{\Delta m}{m} + \frac{\Delta R}{R} + \frac{\Delta \mu}{\mu}$$

Exercise 10

1- We have $G = \frac{T^2 ga}{4\pi^2} - a^2$, with: $\begin{cases} [T] = T \\ [a] = L \\ [g] = LT^2 \\ [4\pi] = 1 \end{cases}$

The dimension of G :

$$[G] = \left[\frac{T^2 ga}{4\pi^2}\right] = [a^2] = L^2$$

Or $[G] = \left[\frac{T^2 ga}{4\pi^2}\right] = \frac{[T]^2[g][a]}{[4\pi^2]} = \frac{T^2 L T^{-2} L}{1} = L^2$ So [G]=L^2

where G has a unit area (m^2).

2- Calculation of the absolute uncertainty on G as a function of ΔT and Δa:

$$G = \frac{T^2 ga}{4\pi^2} - a^2$$

$$\Rightarrow dG = \left(\frac{\partial G}{\partial T}\right) dT + \left(\frac{\partial G}{\partial a}\right) da$$

$$\Rightarrow dG = \left(\frac{2gaT}{4\pi^2}\right) dT + \left(\frac{gT^2}{4\pi^2} - 2a\right) da$$

So the absolute uncertainty on G is:

$$\Delta G = \left|\frac{gaT}{2\pi^2}\right| \Delta T + \left|\frac{gT^2}{4\pi^2} - 2a\right| \Delta a$$

Exercise 11

$n = \sqrt{N^2 - sin^2\alpha}$:

1. Calculation of the absolute uncertainty on n.

The total differential of n is written:

$$dn = \frac{\partial n}{\partial N}dN + \frac{\partial n}{\partial \alpha}d\alpha$$

The partial differentials of the function "n" with respect to the two variables N and α are:

$$\frac{\partial n}{\partial N} = \frac{2N}{2\sqrt{N^2 - sin\alpha^2}}$$

And

$$\frac{\partial n}{\partial \alpha} = \frac{-2sin\alpha\ .cos\alpha}{2\sqrt{N^2 - sin\alpha^2}}$$

So

$$dn = \frac{N}{\sqrt{N^2 - sin\alpha^2}}dN + \frac{-sin\alpha\ .cos\alpha}{\sqrt{N^2 - sin\alpha^2}}d\alpha$$

Then the absolute uncertainty on n is:

$$\Delta n = \frac{N}{\sqrt{N^2 - sin\alpha^2}}\Delta N + \frac{|-sin\alpha\ .cos\alpha|}{\sqrt{N^2 - sin\alpha^2}}\Delta\alpha$$

Let for $\alpha < \frac{\pi}{2}$ the relative uncertainty on n is:

$$\frac{\Delta n}{n} = \frac{N\Delta N + sin\alpha.cos\alpha.\Delta\alpha}{(|N^2 - sin\alpha^2|)}$$

Exercise 12

1- The dimension of k :

We have $\begin{cases} [p] = M.L.T^{-2} \\ [S] = L^2 \\ [k] = 1 \\ [v] = L.T^{-1} \end{cases}$ $\quad and\ \ k = \frac{p}{v^2.s} \Rightarrow [k] = \frac{[p]}{[v]^2.[s]}$

$$\Rightarrow [k] = [p].[v]^{-2}.[s]^{-1} \Rightarrow [k] = M.L^{-3}$$

2- N.A : $v = \sqrt{\frac{P}{K.S}} = 3.097 m/s$

3- $\frac{\Delta P}{P} = 2\% = 0.02$ and $\frac{\Delta S}{S} = 3\% = 0.03$

The logarithmic method is used to calculate the relative uncertainty on v :

$$v = \sqrt{\frac{P}{K.S}} \Rightarrow \log v = \log\sqrt{\frac{P}{K.S}} = \frac{1}{2}log\,P - \frac{1}{2}logk - \frac{1}{2}\log S$$

$$\Rightarrow d\log v = \frac{1}{2} dlog\, P - \frac{1}{2} dlogk - \frac{1}{2}\mathrm{dlog}\, S$$

$$\frac{dv}{v} = \frac{1}{2}\frac{dp}{p} - \frac{1}{2}\frac{dS}{S} \Rightarrow \frac{\Delta v}{v} = \frac{1}{2}\left|\frac{\Delta p}{p}\right| + \frac{1}{2}\left|-\frac{\Delta S}{S}\right| \Rightarrow \frac{\Delta v}{v} = \frac{1}{2}\frac{\Delta p}{p} + \frac{1}{2}\frac{\Delta S}{S}$$

A.N: $\frac{\Delta v}{v} = 0.025$

Absolute uncertainty on v is given by:

$$\Delta v = v.\frac{\Delta v}{v} = v * \left(\frac{1}{2}\frac{\Delta p}{p} + \frac{1}{2}\frac{\Delta S}{S}\right) = 0.077 m/s$$

Hence the condensed writing of v is given by : v=(3.097±0.077)m/s

Exercise 13

1- $H = \frac{(2.\sigma.cos\alpha)}{(R.g.\rho)} \Longrightarrow \sigma = \frac{HR\rho g}{2cos\alpha}$

$$Hence\ [\sigma] = \frac{[H][R][\rho][g]}{[2][cos]}$$

On a $\begin{cases} [H] = L \\ [R] = L \\ [\rho] = ML^{-3} \\ [g] = LT^{-2} \\ [2] = [cos\alpha] = 1 \end{cases}$

$$\Rightarrow [\sigma] = L.L.M.L^{-3}.LT^{-2} = MT^{-2}$$

So $[\sigma] = MT^{-2}$

2- Let's calculate relative uncertainty onσ, Δσ/σ = f (Δm, ΔR, Δg, Δα) :

We have, $\sigma = \frac{HR\rho g}{2cos\alpha}$ or $\rho = thedensity = \frac{M}{V} = \frac{M}{H\pi R^2}$

So: $\sigma = \frac{HR\frac{M}{H\pi R^2}g}{2cos\alpha} = \frac{Mg}{2\pi Rcos\alpha}$

a- The Logarithmic method :

$$log\sigma = \log\left(\frac{Mg}{2\pi Rcos\alpha}\right) = logMg - \log(2\pi Rcos\alpha)$$

$$\Longrightarrow log\sigma = logM + logg - log2\pi - logR - logcos\alpha$$

$$\Longrightarrow dlog\sigma = dlogM + dlogg - dlog2\pi - dlogR - dlogcos\alpha$$

$=\frac{d\sigma}{\sigma}=\frac{dM}{M}+\frac{dg}{g}-\frac{dR}{R}-\frac{dcos\alpha}{cos\alpha}$

$=\frac{d\sigma}{\sigma}=\frac{dM}{M}+\frac{dg}{g}-\frac{dR}{R}-\frac{sin\alpha}{cos\alpha}d\alpha$

$$\Rightarrow \boldsymbol{\frac{\Delta\sigma}{\sigma}=\frac{\Delta M}{M}+\frac{\Delta g}{g}+\frac{\Delta R}{R}+|tg\alpha|\Delta\alpha}$$

b- The total differential method:

$$d\sigma=\frac{\partial\sigma}{\partial M}dM+\frac{\partial\sigma}{\partial R}dR+\frac{\partial\sigma}{\partial g}dg+\frac{\partial\sigma}{\partial\alpha}d\alpha$$

With $\begin{cases}\frac{\partial\sigma}{\partial g}=\frac{M}{2\pi Rcos\alpha}\\ \frac{\partial\sigma}{\partial M}=\frac{g}{2\pi Rcos\alpha}\\ \frac{\partial\sigma}{\partial R}=\frac{-Mg}{2\pi R^2cos\alpha}\\ \frac{\partial\sigma}{\partial\alpha}=\frac{Mg}{2\pi R}.\frac{sin\alpha}{cos^2\alpha}\end{cases}$

Hence

$$d\sigma=\frac{g}{2\pi Rcos\alpha}dM+\frac{-Mg}{2\pi R^2cos\alpha}dR+\frac{M}{2\pi Rcos\alpha}dg+\frac{Mg}{2\pi R}.\frac{sin\alpha}{cos^2\alpha}d\alpha$$

$$\Rightarrow d\sigma=\underbrace{\left(\frac{Mg}{2\pi Rcos\alpha}\right)}_{=\sigma}\left[\left(\frac{1}{M}\right)dM+\left(\frac{-1}{R}\right)dR+\left(\frac{1}{g}\right)dg+\left(\frac{sin\alpha}{cos\alpha}\right)d\alpha\right]$$

$$\Rightarrow \frac{d\sigma}{\sigma}=\frac{dM}{M}+\frac{-dR}{R}+\frac{dg}{g}+\frac{sin\alpha}{cos\alpha}d\alpha$$

So $\boldsymbol{\frac{\Delta\sigma}{\sigma}=\frac{\Delta M}{M}+\frac{\Delta R}{R}+\frac{\Delta g}{g}+|tg\alpha|\Delta\alpha}$

Exercise 14

We have $f=\frac{1}{2\pi\sqrt{L.C}}$

We will calculate the absolute uncertainty on f.

<u>1st Method: The total differential</u>

$$df=\left(\frac{\partial f}{\partial L}\right)dL+\left(\frac{\partial f}{\partial C}\right)dC$$

With $\begin{cases} \left(\frac{\partial f}{\partial L}\right) = \frac{1}{2\pi}.\frac{-C}{2LC\sqrt{LC}} = \frac{1}{2\pi}.\left(\frac{-1}{2L\sqrt{LC}}\right) \\ \left(\frac{\partial f}{\partial C}\right) = \frac{1}{2\pi}.\frac{-1}{2C\sqrt{LC}} \end{cases}$

So: $df = \frac{1}{2\pi}.\left(\frac{-1}{2L\sqrt{LC}}\right)dL + \frac{1}{2\pi}.\frac{-1}{2C\sqrt{LC}}dC$

$$\Rightarrow df = \left(\frac{1}{2\pi\sqrt{LC}}\right).\left[\left(\frac{-1}{2L}\right)dL + \frac{-1}{2C}dC\right]$$

$$\Rightarrow \frac{df}{f} = \left[\left(\frac{-1}{2L}\right)dL + \frac{-1}{2C}dC\right]$$

$$\Rightarrow \frac{\Delta f}{f} = \left|\frac{-1}{2L}\right|\Delta L + \left|\frac{-1}{2C}\right|\Delta C$$

$\Rightarrow \frac{\Delta f}{f} = \frac{1}{2L}\Delta L + \frac{1}{2C}\Delta C$ is the relative uncertainty on f.

And $\Delta f = \left(\frac{1}{2\pi\sqrt{LC}}\right).(\left|\frac{-1}{2L}\right|\Delta L + \left|\frac{-1}{2C}\right|\Delta C)$

$\boldsymbol{\Delta f = \left(\frac{1}{2\pi\sqrt{LC}}\right).(\frac{1}{2L}\Delta L + \frac{1}{2C}\Delta C)}$ is the absolute uncertainty on f.

2nd Method: The logarithmic differential

We have $f = \frac{1}{2\pi\sqrt{L.C}} \Rightarrow logf = log\frac{1}{2\pi} - \frac{1}{2}logL - \frac{1}{2}logC$

$$\Rightarrow dlogf = dlog\frac{1}{2\pi} - \frac{1}{2}dlogL - \frac{1}{2}dlogC$$

$$\Rightarrow \frac{df}{f} = -\frac{1}{2}\frac{dL}{L} - \frac{1}{2}\frac{dC}{C} \Rightarrow \frac{\Delta f}{f} = \left|-\frac{1}{2}\right|\frac{\Delta L}{L} + \left|-\frac{1}{2}\right|\frac{\Delta C}{C}$$

Hence the relative uncertainty on f is:

$$\boldsymbol{\frac{\Delta f}{f} = \frac{1}{2}\frac{\Delta L}{L} + \frac{1}{2}\frac{\Delta C}{C}}$$

And absolute uncertainty about f:

$$\boldsymbol{\Delta f = f\left(\frac{1}{2}\frac{\Delta L}{L} + \frac{1}{2}\frac{\Delta C}{C}\right)}$$

Exercise 15

A) The expression of the frequency f:

We have: $\begin{cases} [f] = T^{-1} \\ [F] = [m][a] = MLT^{-2} \\ [L] = L \\ [\rho] = \frac{[m]}{[V]} = ML^{-3} \\ [K] = 1 \end{cases}$

We consider that the formula of "f" is homogeneous.

$$[f] = [K][F]^a[L]^b[\rho]^c \Rightarrow T^{-1} = (MLT^{-2})^a(L)^b(ML^{-3})^c$$

$$\Rightarrow T^{-1} = M^{a+c}\, L^{a+b-3c}\, T^{-2a}$$

By identification

$$\begin{cases} a + c = 0 \\ a + b - 3c = 0 \\ -1 = -2a \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2} \\ b = -2 \\ c = -\frac{1}{2} \end{cases} \Rightarrow f = KF^{\frac{1}{2}}L^{-2}\rho^{-\frac{1}{2}}$$

or

$$\mathbf{f = \frac{K}{L^2}\sqrt{\frac{F}{\rho}}}$$

B) Let us calculate the absolute uncertainty Δf as a function of ΔD and Δa.

$$f = \frac{D^2 - a^2}{4D}$$

$$\Rightarrow df = \left(\frac{\partial f}{\partial D}\right) dD + \left(\frac{\partial f}{\partial a}\right) da$$

$$\Rightarrow df = \left(\frac{2D(4D) - 4(D^2 - a^2)}{16D^2}\right) dT + \left(\frac{-2a}{4D}\right) da$$

$$\Rightarrow df = \left(\frac{D^2 - a^2}{4D^2}\right) dT + \left(\frac{-2a}{4D}\right) da$$

So the absolute uncertainty on f is:

$$\Delta f = \left|\frac{D^2 - a^2}{4D^2}\right| \Delta T + \left|\frac{-2a}{4D}\right| \Delta a$$

$$\Rightarrow \Delta f = \left|\frac{D^2 - a^2}{4D^2}\right| \Delta T + \frac{a}{2D} \Delta a$$

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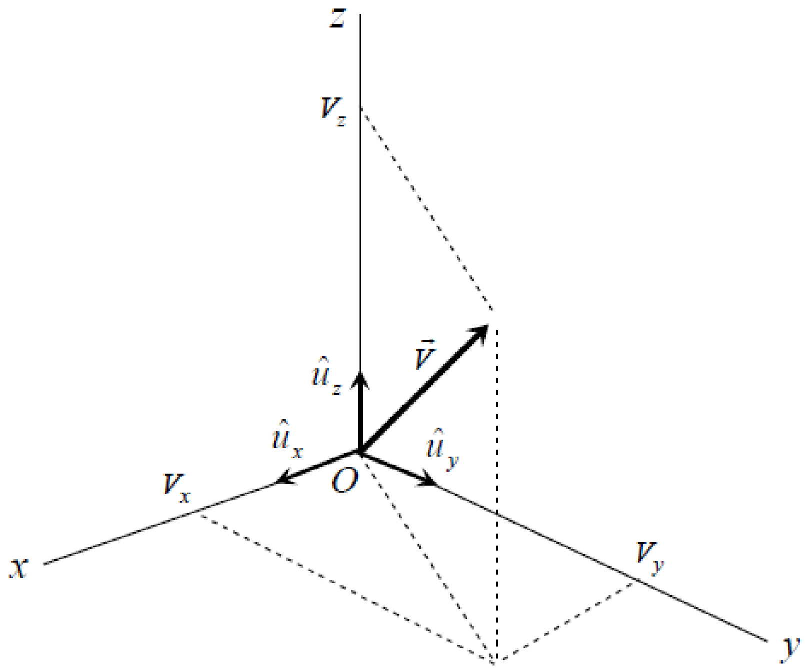
COURSE OF MECHANICS

OF THE MATERIAL POINT

Chapter II: Vector Analysis



Glossary

In English	In French	In Arabic
Vector	Vecteur	شعاع
Vector quantity	Grandeurs vectorielles	مقدار شعاعي
Displacement vectors	Vecteur de déplacement	شعاع التنقل الموضعي
Position vector	Vecteur position	شعاع موضعي
Velocity vector	Le vecteurvitesse	شعاع السرعة
Acceleration vector	Le vecteuraccélération	شعاع التسارع
Unit vector	Vecteur unité	شعاع الوحدة
Magnitude	Module, l'intensité ou la norme du vecteur	طويلة شعاع
Direction	La direction, le sens	اتجاه شعاع
Addition	Addition	جمع
Subtraction	Soustraction	الطرح
Scalar multiplication	Multiplication scalaire	جداء سلمي
Coordinate systems	Systèmes de coordonnées	نظام احداثيات
Force vectors	Le vecteur force	شعاع الفوة
Opposite direction	La direction opposée	اتجاه معاكس
Zero vector	Vecteurnul	شعاع منعدم
Equal vectors	Vecteurségaux	اشعة متساوية
Vector parallel	Vecteur parallel	شعاع موازي
Vector perpendicular	Vecteurperpendiculaire	شعاع معامد
Free vector	Vecteurslibres	اشعة حرة
Sliding vector	Vecteursglissants	اشعة منزلقة
Linked vectors	Vecteursliés	اشعة متصلة
Opposite vector	Vecteursopposes	شعاعان متعاكسان
Algebraic value	Valeur algébrique	قيمة جبرية
Equal magnitudes	Mème amplitude (module)	نفس الطويلة
Graphic representations	Représentationgraphique	تمثيل في الفضاء
The sum of two vectors	la somme de deux vecteurs	مجموع شعاعين
The axis	Un axe	محور

The xy-plane	Un plan (Oxy)	معلم مستوي
The scalar product	Produit scalaire	جداء سلمي
The vector product	Produit vectoriel	جداء شعاعي
The mixed product	Produit mixte	جداء مختلط
The parallelepiped	Un parallélépipède	متوازي المستطيلات
The parallelogramme	Un parallélogramme	متوازي الاضلاع
The norme	La norme	طول الشعاع

1. Introduction

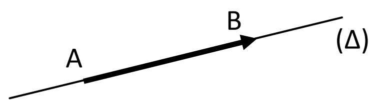
Vectors are fundamental mathematical entities used to represent quantities that have both magnitude and direction. Unlike scalars, which only have magnitude (e.g., distance, time, temperature), vectors provide a more comprehensive description of physical quantities by including information about their orientation or direction.

In other words, in physics, two types of quantities are used: scalar quantities and vector quantities:

- Scalar quantity المقدار السلمي : defined by a number (a scalar) and an appropriate unit such as: volume, mass, temperature, time ...
- Vector quantity المقدار الشعاعي: this is a quantity defined by a scalar, a unit and a direction such as : Displacement vector, velocity $\vec{v}$, weight $\vec{p}$, electric field $\vec{E}$ ...

2. Definition

Vectors are physical or mathematical quantities carrying two properties: magnitude and direction. It is an oriented segment. Symbolically, a vector is usually represented by an arrow.

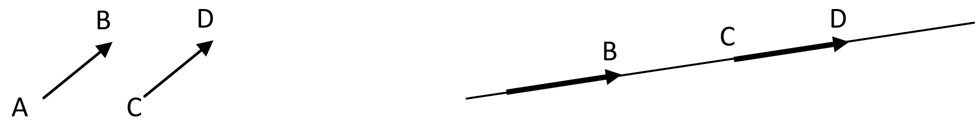


- Origin (المبدأ): presents the point of application "A".
- Support (الحامل): the straight line that carries the vector (Δ).
- Direction (الاتجاه): Vectors have a specific direction or orientation in space, often indicated by angles or coordinate systems (from A to B).
- Modulus (الطويلة): The size or length of a vector represents its magnitude. This is typically represented by a positive numerical value gives the algebraic value of the vector $\overrightarrow{AB}$ noted.

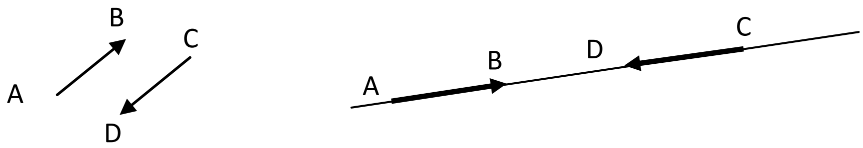
3. Vector types

- **Free vector:** the origin is not fixed.
- **Sliding vector:** the support is fixed, but the origin is not.
- **Linked vectors:** the origin is fixed.

- **Equal vectors**: if they have the same direction, the same support or parallel supports and the same modulus.



- **Opposite vector:** if they have the same support or parallel supports, the same modulus but the direction is opposite.

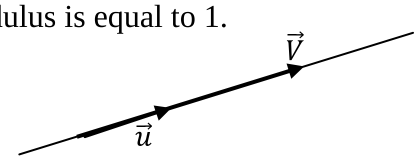


4. Unit Vector شعاع الوحدة

A vector is said to be unitary if its modulus is equal to 1.

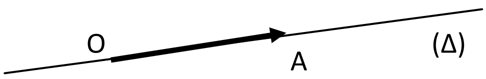
We write: $|\vec{u}|=1$

and $\vec{V} = |\vec{V}|\vec{u}=V.\vec{u}$



5. Algebraic Measurement

Consider an axis (Δ) bearing points O and A. O is the origin, and the abscissa of point A is the algebraic measure of the vector $\overrightarrow{OA}$.

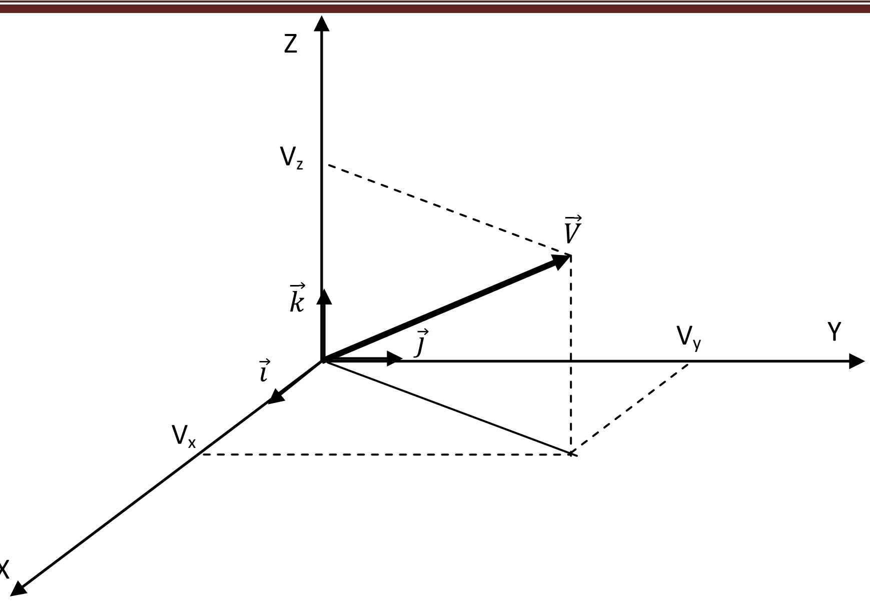


6. Components of a Vector مركبات شعاع

The coordinates of a vector in space, represented in an orthonormal base frameR(O, $\vec{\imath},\vec{\jmath},\vec{k}$) are : V_x, V_y et V_z such that:

$$\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$$

Where a **position vector** $\vec{V} = \overrightarrow{OM}$ is a vector used to determine the position of a point M in space, relative to a fixed reference point O which, typically, is chosen to be the origin of our coordinate system.



The modulus of the vector $\vec{V}$ is : $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$

In cartesian coordinates, a vector is written as:

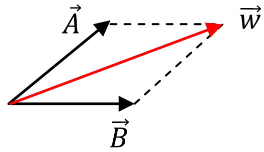
$$\vec{V} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow V = \left\|\vec{V}\right\| = \sqrt{x^2 + y^2 + z^2}$$

7. Elementary Operations on Vectors

7.1. Vector addition

The sum of two vectors $\vec{A}$ and$\vec{B}$ is $\vec{w}$, obtained using the parallelogram:

$$\vec{A} + \vec{B} = \vec{w}$$

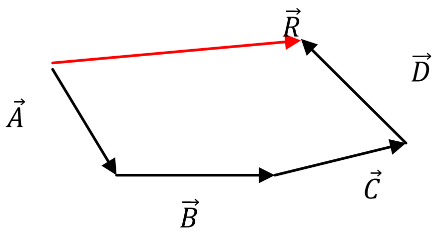


Let two vectors $\vec{A}$ and $\vec{B}$: $\vec{A} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{B} = x'\vec{i} + y'\vec{j} + z'\vec{k}$

$$\vec{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} and\ \vec{B}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} so\ \vec{A} + \vec{B} = \vec{w} = (x + x')\vec{i} + (y + y')\vec{j} + (z + z')\vec{k}$$

<u>Note</u> :

1. For several vectors: $\vec{A} + \vec{B} + \vec{C} + \vec{D} = \vec{R}$



2. Properties :

$\vec{A}+\vec{B}=\vec{B}+\vec{A}$, $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$, $\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

3. Charles relationship:

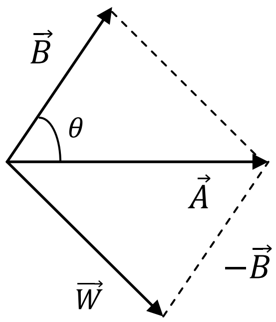
Or the three points: A, B and C, we have: $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

7.2. Subtracting two vectors

This is an anticommutative operation such that: $\vec{W}=\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

Let two vectors: $\vec{A}\ and\ \vec{B}$,$\vec{A}=x\vec{\imath}+y\vec{\jmath}+z\vec{k}$ and $\vec{B}=x'\vec{\imath}+y'\vec{\jmath}+z'\vec{k}$

$\vec{A}\begin{pmatrix}x\\y\\z\end{pmatrix} and \vec{B}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$ so $\vec{A}-\vec{B}=\vec{w}=(x-x')\vec{\imath}+(y-y')\vec{\jmath}+(z-z')\vec{k}$



7.3. Product of a vector and a scalar

The product of a vector $\vec{v}$ by a scalar α is the vector$\alpha\vec{v}$, this vector has the same support as $\vec{v}$.

The two vectors ($\vec{v}$ and $\alpha\vec{v}$) have the same direction if α>0 and they are opposite supports if α<0 .

$$\alpha\vec{v}=\alpha\begin{pmatrix}x\\y\\z\end{pmatrix}=\alpha x\vec{i}+\alpha y\vec{j}+\alpha z\vec{k}$$

Notes:$[\alpha\vec{v}]=|\alpha||\vec{v}|, \alpha(\vec{u}+\vec{v})=\alpha\vec{u}+\alpha\vec{v}$ and $(\alpha+\beta)\vec{u}=\alpha\vec{u}+\beta\vec{u}$

8. Products

8.1. Scalar product الجداء السلمي

Given two vectors $\vec{A}\ and\ \vec{B}$ making an angle θ between them, the scalar product $\boldsymbol{\vec{A}.\vec{B}=m}$ with **m** is a scalar such that:

$$\boldsymbol{\vec{A}.\vec{B}=m=|\vec{A}|.|\vec{B}|\cos(\vec{A},\vec{B})}$$

With :$(\widehat{\vec{A},\vec{B}}) = \theta$

<u>Note</u> : The properties of the scalar product are:

- The scalar product is commutative $\vec{A}.\vec{B} = \vec{B}.\vec{A}$
- The scalar product isn't associative $\vec{V_1}.(\vec{V_2}.\vec{V_3})$,doesn't exist, because the result would be a vector.
- $\vec{A}.\vec{B} = 0$ whenboth vectorsareperpondicular$(\vec{A} \perp \vec{B})$.
- If $\vec{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} and\ \vec{B}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$so $\boldsymbol{\vec{A}.\vec{B} = x.x' + y.y' + z.z'}$

8.2. Vector product الجداء الشعاعي

The vector product of two vectors $\vec{A}\, and\ \vec{B}$ is a vector$\vec{C}$ and is written as:

$$\boldsymbol{\vec{C} = \vec{A}\Lambda\vec{B}}$$

To calculate the vector product of two vectors$\vec{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} and\ \vec{B}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$we have :

$$\vec{A}\Lambda\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix} = \vec{i}\begin{vmatrix} y & z \\ y' & z' \end{vmatrix} - \vec{j}\begin{vmatrix} x & z \\ x' & z' \end{vmatrix} + \vec{k}\begin{vmatrix} x & y \\ x' & y' \end{vmatrix} = \boldsymbol{\vec{C}}$$

$$\vec{A}\Lambda\vec{B} = \vec{i}(yz' - zy') - \vec{j}(xz' - zx') + \vec{k}(xy' - yx') = \boldsymbol{\vec{C}}$$

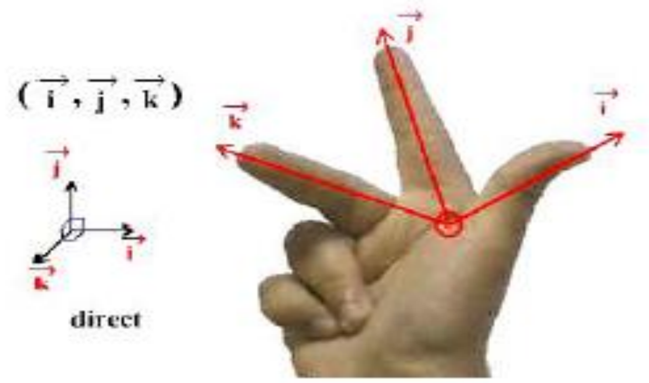
So the modulus of the vector product can be given by another method such as:

$$W = \sqrt{(yz' - zy')^2 + (xz' - zx')^2 + (xy' - yx')^2}$$

Characteristics of vector$\boldsymbol{\vec{C}}$:

The support: $\vec{C}$ is perpondicular to the plane formed by the two vectors $\vec{A}\ and\ \vec{B}$.

The direction: The three vectors $\vec{A}, \vec{B}\ and\ \vec{C}$ form a direct trihedron. The direction is given by the rule of the three fingers of the right hand.



The modulus :

$$|\vec{C}| = |\vec{A}|.|\vec{B}|\sin(\vec{A},\vec{B})$$

The modulus of the vector product corresponds to the area (the surface مساحة) of the parallelogram (متوازي الاضلاع) formed by the two vectors $\vec{A}$ and $\vec{B}$.

<u>Example</u>:

In an orthonormal Cartesian coordinate base $(\vec{\imath}, \vec{\jmath}, \vec{k})$:

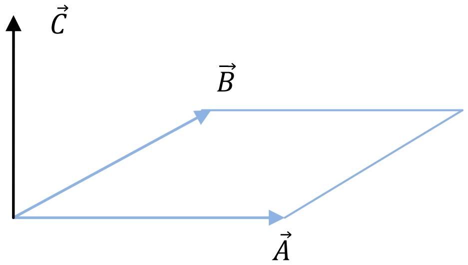
$\vec{\imath} \wedge \vec{\jmath} = \vec{k}$, $\vec{\jmath} \wedge \vec{k} = \vec{\imath}$et $\vec{k} \wedge \vec{\imath} = \vec{\jmath}$.On the other hand $\vec{\imath} \wedge \vec{k} = -\vec{\jmath}$

<u>Notes</u> : The properties of the vector product are:

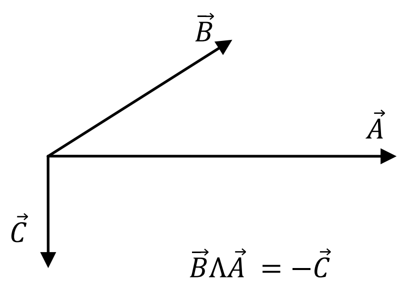
- The vector product is not commutative (Anticommutative).
- Not associative :$\vec{V_1} \wedge (\vec{V_2} \wedge \vec{V_3}) \neq (\vec{V_1} \wedge \vec{V_2}) \wedge \vec{V_3}$.
- Distributive with respect to vector sum:$\vec{A} \wedge (\vec{B_1} + \vec{B_2}) = \vec{A} \wedge \vec{B_1} + \vec{A} \wedge \vec{B_2}$

But: $\vec{V_1} \wedge (\vec{V_2} + \vec{V_3}) \neq (\vec{V_1} \wedge \vec{V_2}) + (\vec{V_1} \wedge \vec{V_3})$

- $\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A}$ because $sin(\vec{A},\vec{B}) = -sin(\vec{B},\vec{A})$



$\vec{A} \wedge \vec{B} = \vec{C}$



$\vec{B} \wedge \vec{A} = -\vec{C}$

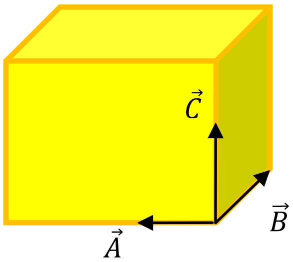
- $\vec{A} \wedge \vec{B} = \vec{0}$ *when the two vectors are parallel* $(\vec{A} \parallel \vec{B})$.

8.3. Mixed product

The mixed product of three vectors $\vec{A}, \vec{B} and \vec{C}$ is a scalar quantity **m** such that:

$$m = (\vec{A} \wedge \vec{B}).\vec{C}$$

Where **m** represents the volume of the parallelepiped (حجم متوازي المستطيلات) constructed by the three vectors :



Note: The mixed productis commutative, $(\vec{A} \wedge \vec{B}).\vec{C} = \vec{A}.(\vec{B} \wedge \vec{C}) = (\vec{C} \wedge \vec{A}).\vec{B}$

9. Derivative of a vector

Let the vector $\vec{A} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ which varies with time:

Its first derivative in relation to time is:

$$\overrightarrow{A'} = \frac{d\vec{A}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

The second derivative is:

$$\overrightarrow{A''} = \frac{d^2\vec{A}}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

Note :

- Derivative of a scalar product$(\vec{A}.\vec{B})' = \overrightarrow{A'}.\vec{B} + \vec{A}.\vec{B}$
- If $\vec{B}$ is constant $(\vec{A}.\vec{B})' = \overrightarrow{A'}.\vec{B}$
- $(\vec{A}^2)' = 0\ because\ (\vec{A}^2)' = 2\overrightarrow{A'}.\vec{A} = 0$
- The derivative vector is perpendicular to the vector.
- A vector is written as $\vec{A} = |\vec{A}|\vec{u} = A\vec{u}$, if $\vec{u}$ is a variable vector, then $\vec{A}' = A'\vec{u} + A\overrightarrow{u'}$.

Example: The position vector on Cartesian Coordinates is written as:

$$\vec{A} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

The velocity vector in Cartesian Coordinatesis written as:

$$\vec{V} = \frac{d\overrightarrow{OM'}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

The acceleration vector in Cartesian Coordinatesis written as:

$$\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

Proposed exercises about chapter II

Exercise 1

$\vec{\imath},\ \vec{\jmath}\ and\vec{k}$ being the unit vectors of the rectangular axes Oxyz, we consider the vectors:

$\vec{r_1}=\vec{\imath}+3\vec{\jmath}-2\vec{k}$, $\vec{r_2}=4\vec{\imath}-2\vec{\jmath}+2\vec{k}$ and $\vec{r_3}=3\vec{\imath}-\vec{\jmath}+2\vec{k}$

1. Show these 3 vectors graphically.
2. Calculate their modulus
3. Calculate products $\vec{r_1}\cdot\vec{r_2}$ and $\vec{r_1}\wedge\vec{r_2}$.

Exercise 2

1. Let the points be M_1 (+1,+1,+1), M_2 (+2,+2,+1) and M_3 (+2,+1,0); calculate the angle $M_1\widehat{M_2}M_3$
2. Determine the equation of the plane (p) passing through the point M2 and perpendicular to the vector $\vec{A}=3\vec{\imath}-2\vec{\jmath}+\vec{k}$

Exercise 3

$\vec{\imath},\ \vec{\jmath}\ et\vec{k}$ being the unit vectors in the orthonormal frame (Oxyz). Let two vectors $\vec{A}$ and $\vec{B}$ be defined by:

$\vec{A}=-\vec{\imath}+\vec{\jmath}-2\vec{k}$ and $\vec{B}=2\vec{\imath}+4\vec{\jmath}-5\vec{k}$

1- Calculate $(\vec{A}.\vec{B})$ and deduce the angle θ= $(\vec{A},\vec{B})$

2- give $(\vec{A}\wedge\vec{B})$, deduce the area of the parallelogram formed by the two vectors

Exercise 4

We give the three vectors $\vec{V_1}(1, 1, 0)$, $\vec{V_2}(0, 1, 0)$ and $\vec{V_3}$ (0, 0, 2).

1. Calculate norms $\|\vec{V_1}\|$, $\|\vec{V_2}\|\ and\|\vec{V_3}\|$, deduce the unit vectors $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ respectively from $\vec{V_1}$, $\vec{V_2}$ and de $\vec{V_3}$.

2. Calculate cos $(\widehat{\vec{v_1},\vec{v_2}})$, knowing that the corresponding angle is between 0 and π.

3. Calculate the mixed product $\vec{v_1}.(\vec{v_2}\wedge\vec{v_3})$. What does this product represent?

Exercise 5

Consider in space, referred to the direct orthonormal reference frame (O,$\vec{\imath},\vec{\jmath},\vec{k}$) the points A(2, 0,0), B(2, −2, 0) and C(2, 3, −1).

1. Calculate the vector product $\overrightarrow{OA}\wedge\overrightarrow{OB}$
2. Calculate the area of triangle OAB.
3. Calculate the mixed product ($\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$), Deduce the volume of the parallelepiped built on the vectors.
4. Between these products, what are the mixed products that can be calculated $(\overrightarrow{OA}\wedge\overrightarrow{OB}).\overrightarrow{OC}$; $\overrightarrow{OA}.(\overrightarrow{OB}\wedge\overrightarrow{OC})$; $(\overrightarrow{OA}\,.\overrightarrow{OB})\wedge\overrightarrow{OC}$; $\overrightarrow{OA}\wedge(\overrightarrow{OB}\,.\overrightarrow{OC})$;

Exercise 6

A) Let $\vec{A}$(1, 2, 1), $\vec{B}$(1, 0, c) be two vectors where c ∈R

1. Calculate the scalar product $\vec{A}.\vec{B}$ and the modulus of the two vectors as a function of c.
2. Determine the values of c for which the angle ($\vec{A}$, $\vec{B}$) is equal to π/3.

B) Consider the points A (3, 5, 4), B (3, 1, 3), C (8, 5, 5) and D (1, 2, 3) in space.

Calculate the mixed product ($\overrightarrow{DA}$, $\overrightarrow{DB}$, $\overrightarrow{DC}$), deduce the volume of the parallelepiped formed by the three vectors.

Exercise 7

Let be three vectors $\overrightarrow{A}$, $\vec{B}$ *and* $\vec{C}$, such as; $\vec{A} = -2\vec{\imath} + \vec{\jmath} + 3\vec{k}$, $\vec{B} = 2\vec{\imath} - \vec{\jmath} + \vec{k}$, $\vec{C} = x\vec{\imath} + 1\vec{\jmath} + z\vec{k}$

1- Calculate x and z so that the vector $\vec{C}$ or :

a- Parallel to$\vec{A}$ b- Parallel to$\vec{B}$

2- If, $\vec{C} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

Calculate x, y and z so that the vector $\vec{C}$ or : Perpendicular to $\vec{A}$ *and* $\vec{B}$ at the same time.

Exercise 8

Let be a vector $\vec{U} = (t\vec{\imath} + 3\vec{\jmath})/(\sqrt{t^2 + 9})$

1. Show that $\vec{U}$ is a unit vector?

2. Calculate its derivative with respect to time?

Exercise 9

Let be the points A (+1,+1,+1), B (+2,+2,+1) and C (+2,+1,0)

1. Calculate the scalar product $\overrightarrow{AB}.\overrightarrow{AC}$ and the vector product $\overrightarrow{AB} \wedge \overrightarrow{AC}$.
2. What do these two products represent? Deduce the angle between the vectors $\overrightarrow{AB}\ et\ \overrightarrow{AC}$.

Exercise 10

$\vec{\imath},\ \vec{\jmath}\ and\ \overrightarrow{k}$ being the unit vectors of an orthonormal reference frame (Oxyz), consider the vectors. $\overrightarrow{r_1}= 2\,\vec{\imath} -2\vec{\jmath} + 3\vec{k}$, $\overrightarrow{r_2} =\vec{\imath} +\vec{\jmath} + \overrightarrow{k}$

1- Calculate the vector product $\overrightarrow{r_1} \Lambda \overrightarrow{r_2}$.

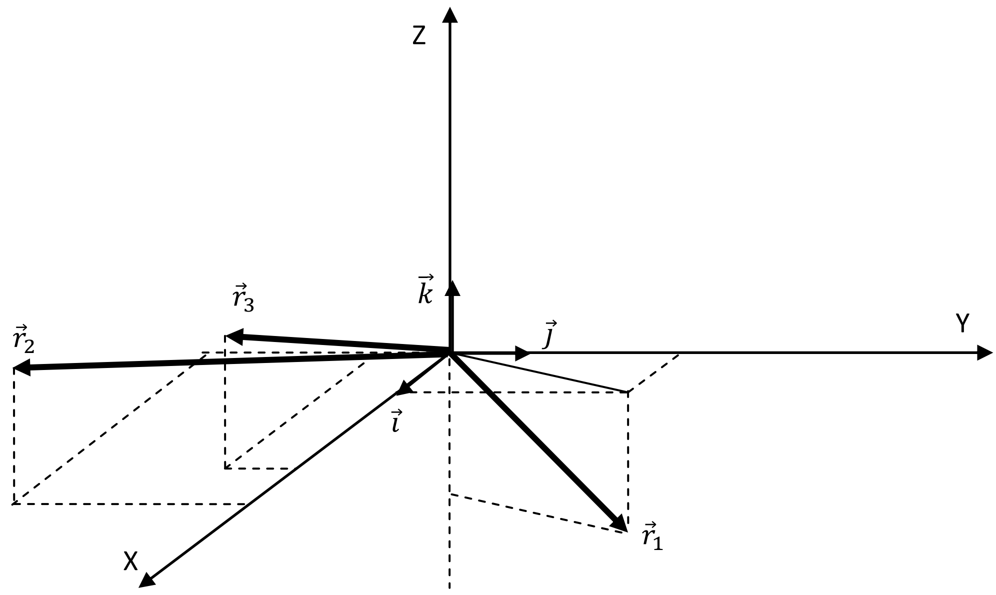
2- Deduce the angle θ formed by the two vectors $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$.

Correction of exercises about chapter II

Exercise 1

We are $\vec{r_1}=\vec{\imath}+3\vec{\jmath}-2\vec{k}\begin{pmatrix}1\\3\\-2\end{pmatrix}\vec{r_2}=4\vec{\imath}-2\vec{\jmath}+2\vec{k}\begin{pmatrix}4\\-2\\2\end{pmatrix}$ and $\vec{r_3}=3\vec{\imath}-\vec{\jmath}+2\vec{k}\begin{pmatrix}3\\-1\\2\end{pmatrix}$

1- Vector representation $\vec{r_1}, \vec{r_2}$ and $\vec{r_3}$:



2- The magnitudes of :

$$\vec{A}\begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow |\vec{A}| = \|\vec{A}\| = \sqrt{x^2+y^2+z^2}$$

$$|\vec{r_1}| = \sqrt{x_1^2+y_1^2+z_1^2} = \sqrt{1+9+4} = \sqrt{14}$$

$$|\vec{r_2}| = \sqrt{x_2^2+y_2^2+z_2^2} = \sqrt{16+4+4} = \sqrt{24}$$

$$|\vec{r_1}| = \sqrt{x_3^2+y_3^2+z_3^2} = \sqrt{9+1+4} = \sqrt{14}$$

3- $\vec{r_1}.\vec{r_2} = x_1x_2+y_1y_2+z_1z_2 = 4-6-4 = -6$

and $\vec{r_1} \wedge \vec{r_2} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 3 & -2 \\ 4 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -2 & 2 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 1 & -2 \\ 4 & 2 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix}\vec{k}$

$$\Rightarrow \vec{r_1} \wedge \vec{r_2} = (3.2 - (-2.(-2)))\vec{\imath} - (1.2 - ((-2).4))\vec{\jmath} + (1.(-2) - (3.4))$$

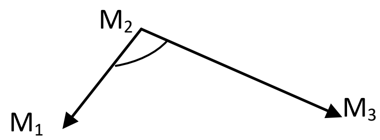
$$\Rightarrow \vec{r_1} \wedge \vec{r_2} = 2\vec{\imath} - 10\vec{\jmath} - 14\vec{k}$$

Exercise 2

1- Let $M_1(1,1,1)$, $M_2(2,2,1)$ et $M_3(2,1,0)$.

Calculate the angle $\widehat{M_1M_2M_3}$ *which is the angle between the two vectors* $\overrightarrow{M_2M_1}$; $\overrightarrow{M_2M_3}$:

We have $\overrightarrow{M_2M_1} = \begin{pmatrix} 1-2 \\ 1-2 \\ 1-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = -\vec{\imath} - \vec{\jmath}$



and $\overrightarrow{M_2M_3} = \begin{pmatrix} 2-2 \\ 1-2 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = -\vec{\jmath} - \vec{k}$

$$\Rightarrow \begin{cases} \overrightarrow{M_2M_1}.\overrightarrow{M_2M_3} = ((-1).0) + (-1.(-1)) + (0.(-1)) = 1 \\ \overrightarrow{M_2M_1}.\overrightarrow{M_2M_3} = \left|\overrightarrow{M_2M_1}\right|.\left|\overrightarrow{M_2M_3}\right|.cos\theta = \sqrt{2}.\sqrt{2}.cos\theta = 2cos\theta \end{cases}$$

$$\Rightarrow 2cos\theta = 1 \Rightarrow cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3} + 2k\pi$$

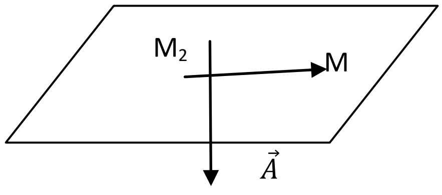
So $\widehat{M_1M_2M_3} = \pm\pi/3 + 2k\pi$

2- The equation of the plane passing through $M_2(2,2,1)$ and perppondicular to the vector $\vec{A} = 3\vec{\imath} - 2\vec{\jmath} + \vec{k}$.

Let X be a point with coordinates (x,y,z) which belongs to the plane (p).

$$\overrightarrow{M_2X} = \begin{pmatrix} x-2 \\ y-2 \\ z-1 \end{pmatrix}$$

We know that $\vec{A}$ perpendicular to this plane therefore:



$\vec{A}.\overrightarrow{M_2X} = (x-2)3 - 2(y-2) + (z-1)1 = 0$

$\Rightarrow 3x - 2y + z - 3 = 0$ (*)

(*)s the equation of the plane passing through $M_2(2,2,1)$ and perpondicular to the vector

$$\vec{A} = 3\vec{\imath} - 2\vec{\jmath} + \vec{k}$$

Exercise 3

A- Let the two vectors $\vec{A}\begin{pmatrix}-1\\1\\-2\end{pmatrix}$ and $\vec{B}\begin{pmatrix}2\\4\\-5\end{pmatrix}$

1- The scalar product

$\vec{A}.\vec{B} = -2 + 4 + 10 = 12$

According to the second writing of the scalar product $\vec{A}.\vec{B} = \|\vec{A}\|\|\vec{B}\| \cos(\vec{A}, \vec{B})$

$\|\vec{A}\| = \sqrt{6}, \|\vec{B}\| = \sqrt{45}$ so $\cos(\vec{A},\vec{B})=\frac{\vec{A}.\vec{B}}{\|\vec{A}\|.\|\vec{B}\|} = \frac{12}{\sqrt{6}.\sqrt{45}}=0.730$

So the angle θ $(\vec{A},\vec{B})$ = 43.08°

2- Vector product

$$\vec{A}\Lambda\vec{B} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ -1 & 1 & -2 \\ 2 & 4 & -5 \end{vmatrix} = (-5-(-8))\vec{\imath} - \big(5-(-4)\big)\vec{\jmath} + (-4-2)\vec{k}$$

$$=3\vec{\imath} - 9\vec{\jmath} - 6\vec{k}$$

The vector product $\vec{A}\Lambda\vec{B}$ gives a vector perpendicular to the plane formed by the two vectors $\vec{A}$ and $\vec{B}$ and the module of this product ($\|\vec{A}\Lambda\vec{B}\|$) presents the surface of the parallelogram formed by the two vectors $\vec{A}$ and $\vec{B}$.

$$\|\vec{A}\Lambda\vec{B}\| = \sqrt{9 + 81 + 36} = \sqrt{126}$$

The area of the parallelogram is $\sqrt{126}$.

Exercise 4

We give the three vectors $\overrightarrow{V_1}(1, 1, 0)$, $\overrightarrow{V_2}(0, 1, 0)$ and $\overrightarrow{V_3}$ (0, 0, 2).

1. Calculates the normes$\|\overrightarrow{V_1}\|$, $\|\overrightarrow{V_2}\|$ and$\|\overrightarrow{V_3}\|$:

Let's calculate the norms of the various vectors and the unit vectors of their respective directions.

$$\|\overrightarrow{V_1}\| = \sqrt{1^2+1^2+0^2} = \sqrt{2} \Rightarrow \overrightarrow{v_1} = \frac{\overrightarrow{V_1}}{\|\overrightarrow{V_1}\|};\ \ \overrightarrow{v_1}(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)$$

$$\|\overrightarrow{V_2}\| = \sqrt{0^2+1^2+0^2} = 1 \Rightarrow \overrightarrow{v_2} = \frac{\overrightarrow{V_2}}{\|\overrightarrow{V_2}\|} = \vec{j};\ \ \overrightarrow{v_2}(0,1,0)$$

$$\|\overrightarrow{V_3}\| = \sqrt{0^2+0^2+2^2} = \sqrt{4} = 2 \Rightarrow \overrightarrow{v_3} = \frac{\overrightarrow{V_3}}{\|\overrightarrow{V_3}\|};\ \ \overrightarrow{v_3}(0,0,1)$$

2. Let's calculate cos $(\widehat{\overrightarrow{v_1},\overrightarrow{v_2}})$ as follows :

$$\overrightarrow{v_1}.\overrightarrow{v_2} = \frac{\sqrt{2}}{2}\ \ \ and\ \overrightarrow{v_1}.\overrightarrow{v_2} = \|\overrightarrow{v_1}\|.\|\overrightarrow{v_2}\|.\cos(\widehat{\overrightarrow{v_1},\overrightarrow{v_2}})$$

$$\Rightarrow \cos\left(\widehat{\overrightarrow{v_1},\overrightarrow{v_2}}\right) = \frac{\sqrt{2}}{2}$$

3. We have a :

$$\overrightarrow{v_1}\,.\overrightarrow{v_2} = \frac{\sqrt{2}}{2}$$

$$\overrightarrow{v_2}\wedge\overrightarrow{v_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\vec{i} + 0\vec{j} + 0\vec{k} = \vec{i}$$

$$\Rightarrow\ \overrightarrow{v_2}\wedge\overrightarrow{v_3} = \vec{i}(1,0,0)$$

$$\overrightarrow{v_1}\,.(\overrightarrow{v_2}\wedge\overrightarrow{v_3}) = 1\times1 + 1\times0 + 0\times0 = 1$$

- The first term represents the scalar product between the vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ is equal to the product of the projection modulus of $\overrightarrow{v_1}$on $\overrightarrow{v_2}$ multiplied by the magnitude of $\overrightarrow{v_2}$.
- The second term is the vector product between $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$.
- The last term is the mixed product between $(\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{v_3})$ and is none other than the volume of the parallelepiped built on the basis of the three vectors.

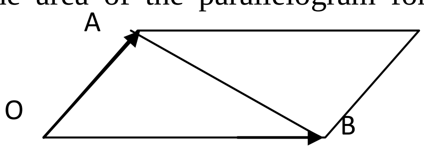
Exercise 5

A(2, 0,0), B(2, −2, 0) and C(2, 3, −1).

1. The vector product $\overrightarrow{OA}\Lambda\overrightarrow{OB}$:

$$\overrightarrow{OA}\begin{pmatrix}2\\0\\0\end{pmatrix};\overrightarrow{OB}\begin{pmatrix}2\\-2\\0\end{pmatrix}so\overrightarrow{OA}\Lambda\overrightarrow{OB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 0 \\ 2 & -2 & 0 \end{vmatrix} = -4\vec{k}$$

The area of the triangle (OAB) is half the area of the parallelogram formed by the two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



S(OAB)=$\frac{|\overrightarrow{OA}\wedge\overrightarrow{OB}|}{2}=\frac{4}{2}=2$

2. The mixed product $(\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC})$, and the volume of the parallelepiped built on the vectors.

$$\left(\overrightarrow{OA}\wedge\overrightarrow{OB}\right).\overrightarrow{OC}=\begin{pmatrix}0\\0\\-4\end{pmatrix}.\begin{pmatrix}2\\3\\-1\end{pmatrix}=4$$

So the volume of the parallelepiped built on the vectors equal 4.

The products that can be calculated are

$\left(\overrightarrow{OA}\wedge\overrightarrow{OB}\right).\overrightarrow{OC} = \overrightarrow{OA}.\left(\overrightarrow{OB}\wedge\overrightarrow{OC}\right)$;

On the other hand, these two products are false because the vector product can only be between two vectors

$\left(\overrightarrow{OA}\,.\overrightarrow{OB}\right)\wedge\overrightarrow{OC}$; $\overrightarrow{OA}\wedge\left(\overrightarrow{OB}\,.\overrightarrow{OC}\right)$;

Exercise 6

A) Let $\vec{A}(1, 2, 1)$, $\vec{B}(1, 0, c)$ be two vectors where c ∈R

1. The scalar product $\vec{A}.\vec{B}$ and the modulus of the two vectors as a function of c.

$$\vec{A}.\vec{B}=1+0+c=c+1$$
$$\|\vec{A}\|=\sqrt{1^2+2^2+1^2}=\sqrt{6}$$
$$\|\vec{B}\|=\sqrt{1^2+0^2+c^2}=\sqrt{1+c^2}$$

2. The values of c for which the angle $(\vec{A}\,,\vec{B}\,)$ is equal to π/3.

According to the second writing of the scalar product $\vec{A}.\vec{B}=\|\vec{A}\|\|\vec{B}\|\cos(\vec{A},\vec{B})$

$(\vec{A}\,,\vec{B}\,)=\pi/3 \Rightarrow \cos(\vec{A},\vec{B})=1/2$

$\cos(\vec{A},\vec{B})=\frac{\vec{A}.\vec{B}}{\|\vec{A}\|.\|\vec{B}\|}=\frac{c+1}{\sqrt{6}.\sqrt{1+c^2}}=\frac{1}{2}$

Then

$$c = 2 \pm \sqrt{3}$$

B) Consider the points A (3, 5, 4), B (3, 1, 3), C (8, 5, 5) and D (1, 2, 3) in space.

Calculate the mixed product $(\overrightarrow{DA}, \overrightarrow{DB}, \overrightarrow{DC})$,

$\overrightarrow{DA}=\begin{pmatrix} 3-1 \\ 5-2 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\overrightarrow{DB}=\begin{pmatrix} 3-1 \\ 1-2 \\ 3-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\overrightarrow{DC}=\begin{pmatrix} 8-1 \\ 5-2 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}$

$$\overrightarrow{DA}\Lambda\overrightarrow{DB} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 3 & 1 \\ 2 & -1 & 0 \end{vmatrix} = \vec{I} + 2\vec{\jmath} - 8\vec{k}$$

$\left(\overrightarrow{DA}\Lambda\overrightarrow{DB}\right) . \overrightarrow{DC}=\begin{pmatrix} 1 \\ 2 \\ -8 \end{pmatrix} . \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}$= -3

The volume of the parallelepiped formed by the three vectors is 3 m^3 (We take the absolute value).

Exercise 7

Let there be three vectors $\overrightarrow{A},\ \ \vec{B}$ and $\vec{C}$ such that

$\vec{A} = -2\vec{\imath} + \vec{\jmath} + 3\vec{k},\ \ \vec{B} = 2\vec{\imath} - \vec{\jmath} + \vec{k},\ \ \ \vec{C} = x\vec{\imath} + 1\vec{\jmath} + z\vec{k}$

1- Calculate x,y and z so that the vector $\vec{C}$ is:

a- $\vec{C}$ Parallel to $\vec{A}$ if $\overrightarrow{A} \wedge \overrightarrow{C} = \vec{0}$

$$\vec{A} \wedge \vec{C} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 1 & 3 \\ x & 1 & z \end{vmatrix} = \vec{0}$$

$$\Rightarrow (z-3)\vec{\imath} - (-2z-3x)\vec{\jmath} + (-2-x)\vec{k} = 0\vec{\imath}+0\vec{\jmath} + 0\vec{k}$$

$$\Rightarrow \begin{cases} z-3 = 0 \\ 2z+3x = 0 \\ -2-x = 0 \end{cases}$$

$$\Rightarrow \begin{cases} z = 3 \\ 2z+3x = 0 \\ x = -2 \end{cases}$$

$$\Rightarrow \overrightarrow{C} = -2\vec{\imath} + \vec{\jmath} + 3\vec{k}$$

b- $\vec{C}$ Parallel to$\vec{B}$ if $\vec{B} \wedge \vec{C} = \vec{0}$

$$\vec{B} \wedge \vec{C} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & -1 & 1 \\ x & 1 & z \end{vmatrix} = \vec{0}$$

$$\Rightarrow (-z-1)\vec{\imath} - (2z - x)\vec{\jmath} + (2 + x)\vec{k} = 0\vec{\imath}+0\vec{\jmath} + 0\vec{k}$$

$$\Rightarrow \begin{cases} -z - 1 = 0 \\ 2z - x = 0 \\ 2 + x = 0 \end{cases}$$

$$\Rightarrow \vec{C} = -2\vec{\imath} + \vec{\jmath} - 1\vec{k}$$

2- If, $\vec{C} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$\vec{C}$ Perpendicular to $\vec{A}$ and$\vec{B}$ at the same time if $\vec{C'} = \vec{A} \wedge \vec{B}$

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 1 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$$\Rightarrow (1+3)\vec{\imath} - (-2-6)\vec{\jmath} + (2-2)\vec{k}$$

$$\Rightarrow \vec{C'} = 4\vec{\imath} + 8\vec{\jmath}$$

Exercise 8

Let be a vector $\vec{U} = (t\vec{\imath} + 3\vec{\jmath})/(\sqrt{t^2+9})$

1- $\vec{U}$ is a unit vector ?

Check that$|\vec{U}| = 1$or$|\vec{U}| = \sqrt{\frac{1}{(t^2+9)}(t^2+9)}$=1

So $\vec{U}$ is an unit vector.

2- The derivative of $\vec{U}$:

$$\frac{d\vec{u}}{dt} = \frac{d}{dt}\left(\frac{t}{(\sqrt{t^2+9})}\right)\vec{\imath} + \frac{d}{dt}\left(\frac{3}{(\sqrt{t^2+9})}\right)\vec{\jmath}$$

$$\Rightarrow \frac{d\vec{u}}{dt} = \left(\frac{t^2 - t^2 + 9}{(t^2+9)^{3/2}}\right)\vec{\imath} + \left(\frac{-3t}{(t^2+9)^{3/2}}\right)\vec{\jmath}$$

$$\Rightarrow \frac{d\vec{u}}{dt} = \left(\frac{9}{(t^2+9)^{3/2}}\right)\vec{\imath} + \left(\frac{-3t}{(t^2+9)^{3/2}}\right)\vec{\jmath}$$

Exercise 9

A. let be the points A (+1,+1,+1), B (+2,+2,+1) and C (+2,+1,0)

The Scalar product $\overrightarrow{AB}.\overrightarrow{BC}$

$$\overrightarrow{AB}\begin{pmatrix}2-1\\2-1\\1-1\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix} and\ \overrightarrow{BC}\begin{pmatrix}2-2\\1-2\\0-1\end{pmatrix}=\begin{pmatrix}0\\-1\\-1\end{pmatrix}$$

$\overrightarrow{AB}.\overrightarrow{BC} = 1X0 + 1X(-1) + 0X(-1) = -1 = \left|\overrightarrow{AB}\right|.\left|\overrightarrow{BC}\right|cos\theta$

The vector product $\overrightarrow{\boldsymbol{AB}} \wedge \overrightarrow{\boldsymbol{AC}}$ with $\overrightarrow{AC}\begin{pmatrix}2-1\\1-1\\0-1\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$$\overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\1 & 1 & 0\\1 & 0 & -1\end{vmatrix} = -\vec{\imath}-(-1)\vec{\jmath}-\vec{k} = -\vec{\imath}+\vec{\jmath}-\vec{k}$$

The angle between $\overrightarrow{AB}.\overrightarrow{BC}$ will; $cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{BC}}{|\overrightarrow{AB}|.|\overrightarrow{BC}|} = \frac{-1}{2}$ so Θ=-π/6

Exercise 10

We have: $\overrightarrow{r_1}= 2\,\vec{\imath} -2\vec{\jmath} + 3\vec{k}$ and $\overrightarrow{r_2} =\vec{\imath} +\vec{\jmath} + \vec{k}$

1- Calculation of vector product $\overrightarrow{r_1}\,\Lambda\,\overrightarrow{r_2}$.

$\overrightarrow{r_1}\,\Lambda\,\overrightarrow{r_2}= -5\vec{\mathrm{i}} + \vec{\mathrm{j}} + 4\vec{\mathrm{k}}$

2- The modulus of vector product $\overrightarrow{r_1}\,\Lambda\,\overrightarrow{r_2}$ is:

$|\overrightarrow{r_1}\,\Lambda\,\overrightarrow{r_2}| = |\overrightarrow{r_1}|.|\overrightarrow{r_1}|.sin\theta$

$$\Rightarrow\ sin\theta = \frac{|\overrightarrow{r_1}\Lambda\overrightarrow{r_2}|}{|\overrightarrow{r_1}|.|\overrightarrow{r_1}|} = 0.9$$

$$\Longrightarrow \theta \sim 64°C$$

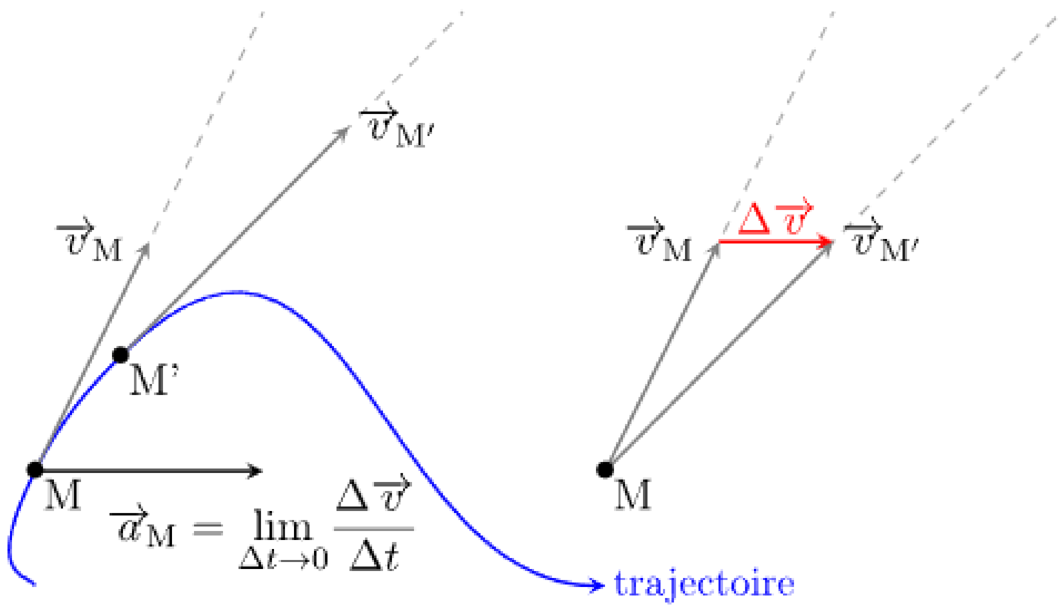
University of Tlemcen
Faculty of Sciences
Mathematics Department

A.Y: 2023/2024

COURSE OF MECHANICS

OF THE MATERIAL POINT

Chapter III: Kinematics of material point



Glossary

In English	In French	In Arabic
Kinematics	La cinématique	الحركيات
Material point	Un point matériel	نقطة مادية
Reference system	Un système référentiel	نظام مرجعي
Velocity (speed)	La vitesse	السرعة
Acceleration	L'accélération	التسارع
Motion characteristics	Caractéristiques d'un mouvement	خصائص الحركة
Position vector	Vecteur position	شعاع الموضعي
Time equation/ Hourly equation	Equation horaire	المعادلة الزمنية
Trajectory	Trajectoire	المسار
Trajectory equation	Equation de la trajetoire	معادلة المسار
Velocity vector	Vecteur vitesse	شعاع السرعة
Acceleration vector	Vecteur accélération	شعاع التسارع
Coordinate systems	Système de coordonnée	نظام الاحداثيات
Cartesian coordinates	Coordonnées cartésiennes	الاحداثيات الكارتيزية
Polar coordinates	Coordonnées polaire	الاحداثيات القطبية
Cylindrical coordinates	Coordonnées cylindriques	الاحداثيات الاسطوانية
Spherical coordinates	Coordonnées sphériques	الاحداثيات الكروية
Intrinsic coordinates	Coordonnées intrinsèques	الإحداثيات الجوهرية
Rectilinear movement	Movement réctiligne	حركة مستقيمة
Uniform rectilinear movement	Movement réctiligne uniforme MRU	حركة مستقيمة منتظمة
Uniformly varied rectilinear movement	Movement réctiligne uniformement varié MRUV	حركة مستقيمة متغيرة بانتظام
Circular movement	Movement circulaire MC	حركة دائرية
Uniform circular movement	Movement circulaire uniforme MCU	حركة دائرية منتظمة
Uniformly varied circular movement	Movement circulaire uniformement varié MCUV	حركة دائرية متغيرة بانتظام

Sinusoidal or harmonic movement	Mouvement sinusoïdal ou harmonique	حركة جيبية أو توافقية
A frame	Un referential	معلم او مرجع
The equation of motion	Equation de mouvement	معادلة الحركة
A mobile	Un mobile	متحرك
Average velocity	La vitesse moyenne	السرعة المتوسطة
Instantaneous velocity	La vitesse instantanée	السرعة اللحظية
Average acceleration	L'accélération moyenne	التسارع المتوسط
Instantaneous acceleration	L'accélération instantanée	التسارع اللحظي
The orthonormal coordinate system	Un système de coordonnées orthogonal	نظام الإحداثيات المتعامد
The Frenet frame	Le repère de Frenet / trièdre de Frenet.	معلم فرينال
The moving point	Un point en movement	نقطة مادية في حالة حركة
The normal acceleration	L'accélération normal	التسارع الناظمي
tangential acceleration	L'accélération tangentielle	التسارع المماسي
Motion	Le Mouvement	الحركة
Weight	Le poids	الوزن
Linear velocity	La vitesse linéaire	السرعة الخطية
Angular velocity	La vitesse angulaire	السرعة الزاوية
Linear Acceleration	L'accélération linéaire	التسارع الخطي
Angular Acceleration	L'accélération angulaire	التسارع الزاوي
Acceleration of gravity	Accélération de pesanteur	تسارع الجادبية
Height	La hauteur	الارتفاع
The period of a pendulum	La pèriode d'une pendule simple	دور نواس بسيط
The sound	Le son	الصوت
Radius	Le rayon	نصف قطر
The abscissa	L'abscisse	الفاصلة
Radius of curvature	Le rayon de courbure	نصف قطر المسار المنحني
The right triangle	Un triangle droit	مثلث قائم
Amplitude	Amplitude	السعة
Frequency	Fréquence	التواتر
Average speed	La vitesse moyenne	السرعة المتوسطة
Instantaneous speed	La vitesse instantanée	السرعة اللحضية

1. Introduction

The theory of General Relativity invented by A. Einstein in 1915 is a relativistic theory of gravitation. This theory challenges the idea of an inert Euclidean space, independent of its material content. Kinematics studies the movement of a material point independently of the causes that give rise to it. It is based on a Euclidean description of space and absolute time. The material point is any material body whose dimensions are theoretically zero and practically negligible in relation to the distance it travels. The state of movement or rest of a body is two essentially relative notions: for example, a mountain is at rest in relation to the earth, but in movement in relation to an observer looking at the earth from afar, for whom the globe (with all that it contains) is in perpetual movement. In this course, we illustrate the notions of velocity and acceleration by restricting ourselves to movements in the plane.

2. Reference Systemمرجع

The concept of motion is relative. A body can be in motion with respect to one object and at rest with respect to another (relative motion), hence the necessity of choosing a reference frame. A reference frame is a system of coordinate axes linked to an observer.

This study of motion is carried out in two forms:

- **Vectorial:** using vectors: position $\overrightarrow{OM}$, velocity $\vec{v}$, and acceleration $\vec{a}$.
- **Algebraic:** by defining the equation of motion along a given trajectory.

3. Characteristics of a movement

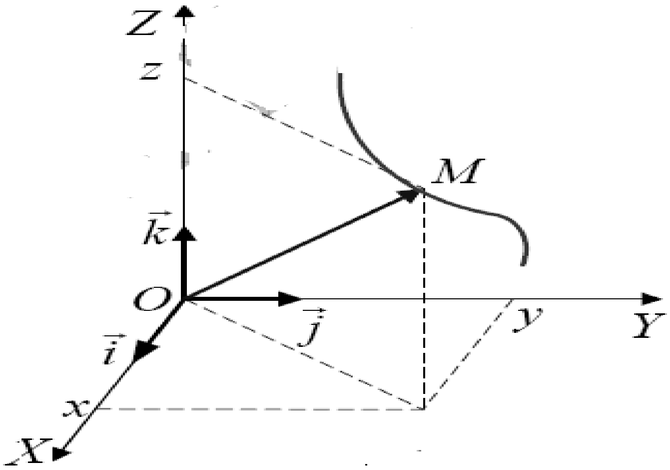
3.1. Vector position and time equationشعاع الموضعي و المعادلة الزمنية للحركة

We define the position of a material point M in a reference frame by the position vector $\overrightarrow{OM}$, where O is a fixed point and serves as the origin of the reference frame. The components of point M or the vector $\overrightarrow{OM}$ are given in the chosen coordinate system's basis (Cartesian coordinates, polar coordinates, etc.).

The point M moves through time, and this movement is described by an equation known as the "time equation" (معادلة زمنية), translated as the "time equation."

3.2.Trajectory المسار

The trajectory is the geometric path of successive positions occupied by the material point over time with respect to the considered reference system.



Example:

The position of a material point M identified by its coordinates (x, y, z) at time t in a coordinate system R(O,$\vec{\imath}$, $\vec{\jmath}$,$\vec{k}$) with a position vector:

$$\overrightarrow{OM} = (t-1)\vec{\imath} + \frac{t^2}{2}\vec{\jmath}$$

$$\overrightarrow{OM} = (t-1)\vec{\imath} + \frac{t^2}{2}\vec{\jmath} \quad \Rightarrow \begin{cases} x = t-1 \\ y = \dfrac{t^2}{2} \end{cases}$$

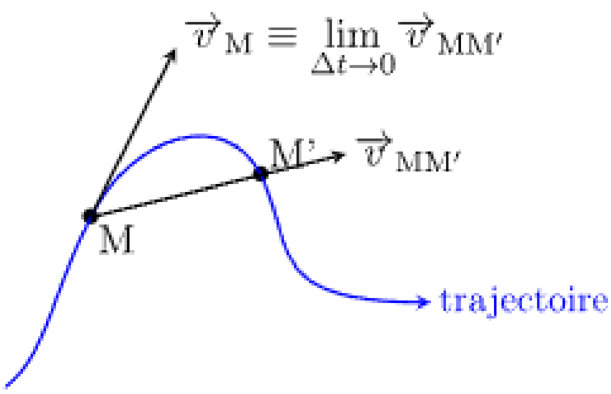
So $t=x+1$

The trajectory equation of the material point is

$$\boldsymbol{y = \frac{(x+1)^2}{2}}$$

3.3. Velocity vector شعاع السرعة

Consider a mobile that is located at position M(t) at time t, and it evolves at the point M’(t+Δt) at instant(t+Δt).



- The **average velocity** السرعة المتوسطة between the two instants t and t+Δt is called:

$$\overrightarrow{v_{moy}} = \frac{\overrightarrow{MM'}}{(t+\Delta t)-t} = \frac{\overrightarrow{MM'}}{\Delta t}$$

- If the time interval Δt is very small (Δt→0), we then refer to it as **instantaneous velocity** السرعة اللحضية**:**

$$\vec{v} = \lim_{\Delta t \to 0} \overrightarrow{v_{moy}} = \lim_{\Delta t \to 0} \frac{\overrightarrow{MM'}}{\Delta t}$$

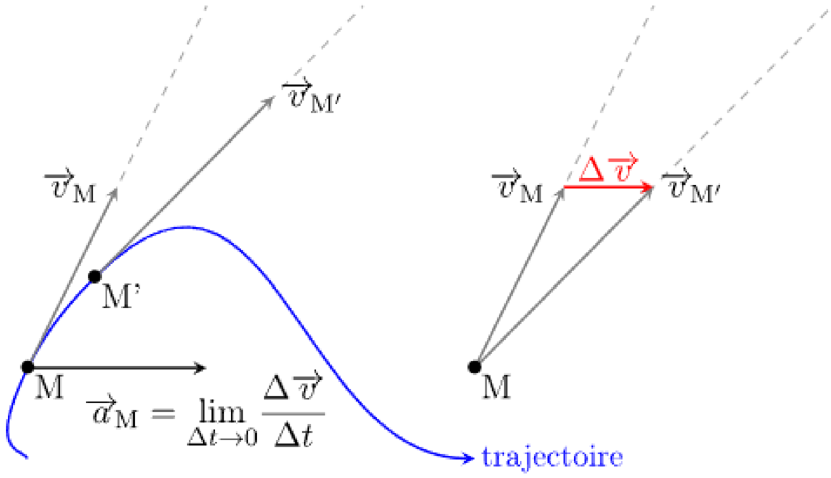
$$\overrightarrow{MM'} = \overrightarrow{MO} + \overrightarrow{OM'} = \overrightarrow{OM'} - \overrightarrow{OM} = \Delta\overrightarrow{OM}$$

So:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{OM}}{\Delta t} \Rightarrow \boldsymbol{\vec{v}} = \frac{\boldsymbol{d\overrightarrow{OM}}}{\boldsymbol{dt}}$$

3.4. Acceleration vector شعاع التسارع

When velocity varies over time v=f(t), point M is subjected to an acceleration.



- The average acceleration*التسارع المتوسط* is written:

$$\overrightarrow{a_{moy}} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{(t+\Delta t)-t} = \frac{\Delta\vec{v}(t)}{\Delta t}$$

- When the time is very small $\Delta t \to 0$ instantaneous acceleration *التسارع اللحضي* is written by :

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{OM}}{\Delta t}$$

$$\Rightarrow \boldsymbol{\vec{a}} = \frac{\boldsymbol{d\vec{v}(t)}}{\boldsymbol{dt}} = \frac{\boldsymbol{d^2\overrightarrow{OM}}}{\boldsymbol{dt^2}}$$

4. Expression of velocity and acceleration in different coordinate systems

To solve a problem in physics, we must locate the position of the moving point M in space $\overrightarrow{OM}(t)$.

The position must be located from a frame of reference (reference), we are required to choose the appropriate reference to use it according to the problem we want to solve

Generally, we use Cartesian, Polar, Cylindrical or Spherical coordinates

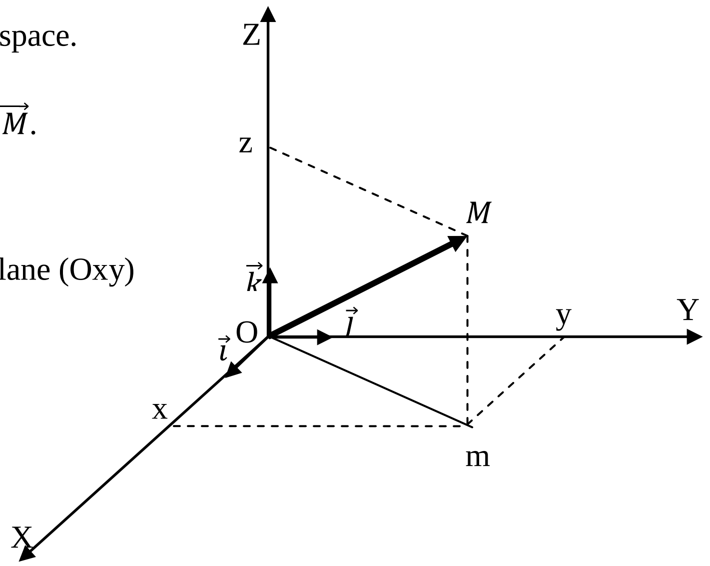
4.1 Cartesian coordinates

Let the frame be R(O,x,y,z) with the unit vectors $\vec{\imath}, \vec{\jmath}$ *and* $\vec{k}$. With x,y and z are the coordinates of point M which gives its position in space.

They are also the vector components $\overrightarrow{OM}$.

x: abscissa; y: ordinate and z: height

m is the projection of point M in the plane (Oxy)



- **Vecteur position**

$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

The unit vectors $\vec{\imath}, \vec{\jmath}$ *et* $\vec{k}$ constitute a basis linked to the axes (Ox), (Oy) and (Oz)

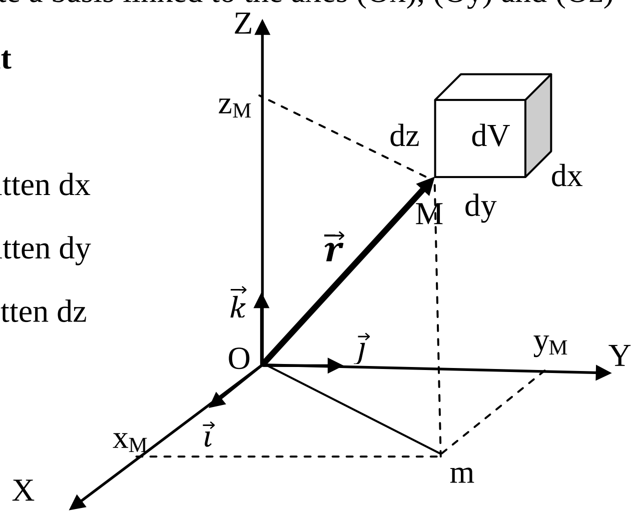
- **Elementary displacement**

The elementary displacement dl:

Next (Ox) the displacement is written dx

Next (Oy) the displacement is written dy

Next (Oz) the displacement is written dz



By fixing y and z, M moves along $\vec{\imath}$, the elementary displacement is then written $\overrightarrow{dl_1} = dx\vec{\imath}$.

By fixing x and z, M moves along $\vec{\jmath}$, the elementary movement is then written $\overrightarrow{dl_2} = dy\vec{\jmath}$.

By fixing x and y, M moves along $\vec{k}$, the elementary displacement is then written $\overrightarrow{dl_3} = dz\vec{k}$.

The total displacement of point M is:

$$\overrightarrow{dl} = \overrightarrow{dl_1} + \overrightarrow{dl_2} + \overrightarrow{dl_3} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

Or mathematically :

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} \Rightarrow d\overrightarrow{OM} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

The elementary volume $dV=dl_1.dl_2.dl_3=dx.dy.dz$

- **Velocity vector**

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k} \Rightarrow \begin{cases} v_x = \dfrac{dx}{dt} \\ v_y = \dfrac{dy}{dt} \\ v_z = \dfrac{dz}{dt} \end{cases}$$

The velocity module is written: $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Note: The magnitude of the velocity, equal to |v|, is called the speed. In S.I. units, v is expressed in (m/s) or ($m.s^{-1}$).

- **Acceleration vector**:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} \Rightarrow \begin{cases} a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \\ a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \\ a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} \end{cases}$$

The acceleration module is written:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

The unit of acceleration in S.I units is (m/s^2) or ($m.s^{-2}$).

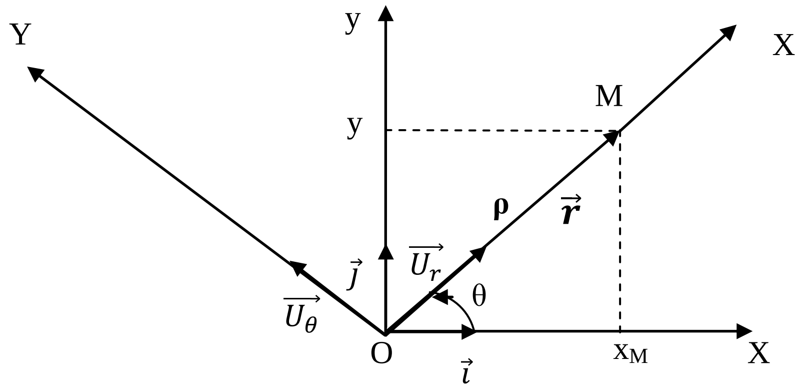
4.2. Polar coordinates الاحداثيات القطبية

When the motion is in a plane, it's also possible to locate the position of point M using its polar coordinates (ρ, θ).

ρ: polar radius $\rho = \left|\overrightarrow{OM}\right|$ ($0 \leq \rho \leq R$)

θ: polar angle $\theta = (ox, \overrightarrow{OM})$ ($0 \leq \theta \leq 2\pi$)

Let's consider point M moving in space, identified by its polar coordinates (ρ, θ) in the orthonormal coordinate system (OXY) with unit vectors $\overrightarrow{u_r}$, $\overrightarrow{u_\theta}$.



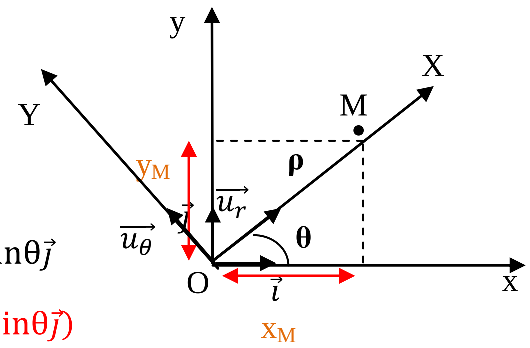
- **Position Vector**

The position vector of a material point M in polar coordinates is written: $\boldsymbol{\vec{r}} = \boldsymbol{\overrightarrow{OM}} = \boldsymbol{\rho \vec{U}_r}$

The unit vectors $\vec{U}_r$ is following $\overrightarrow{OM}$ and $\vec{U}_\theta$ is perpendicular to $\vec{U}_r$ ($\vec{U}_r \perp \vec{U}_\theta$).

Transit relations between cartesian coordinates and polar coordinates

We project the point M into the plane (Oxy)



$$\begin{cases} \boldsymbol{x_M = |\overrightarrow{OM}| cos\theta = \rho cos\theta} \\ \boldsymbol{y_M = |\overrightarrow{OM}| sin\theta = \rho sin\theta} \end{cases}$$

$\overrightarrow{OM} = x_M\vec{\imath} + y_M\vec{\jmath} \Rightarrow \overrightarrow{OM}/cart = \rho\cos\theta\vec{\imath} + \rho\sin\theta\vec{\jmath}$

$\overrightarrow{OM}/pol = \rho\vec{U}_r$ and $\overrightarrow{OM}/cart = \rho(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath})$

By identification: $\overrightarrow{u_r} = \cos\theta\vec{\imath} + \sin\theta\vec{\jmath}$

Rule: Note: The derivative of a unit vector with respect to an angle is a unit vector perpendicular to the angle in the positive direction.

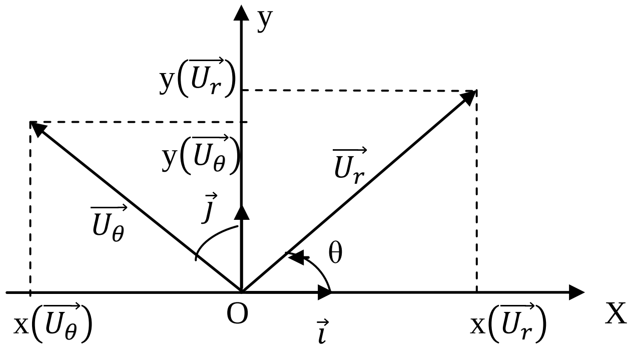
مشتقة شعاع وحدة بالنسبة إلى الزاوية هي شعاع وحدة عمودي على هذا الاخير في الاتجاه الموجب

The vector $\overrightarrow{u_\theta} \perp \overrightarrow{u_r}$ in the direction of θ which corresponds to the direct direction therefore

$$\overrightarrow{u_\theta} = \frac{d\overrightarrow{u_r}}{d\theta}$$

So $\overrightarrow{u_\theta} = -sin\theta\vec{\imath} + \cos\theta\vec{\jmath}$

By projecting the unit vectors we will have the same results



$\overrightarrow{u_r} = x(\overrightarrow{u_r})\vec{\imath} + y(\overrightarrow{u_r})\vec{\jmath} \Rightarrow \overrightarrow{u_r} = |\overrightarrow{u_r}|\cos\theta\vec{\imath} + |\overrightarrow{u_r}|\sin\theta\vec{\jmath}$

with $|\overrightarrow{u_r}| = 1$ since it is a unit vector therefore $\overrightarrow{u_r} = 1\cos\theta\vec{\imath} + 1\sin\theta\vec{\jmath}$

$\overrightarrow{u_\theta} = x(\overrightarrow{u_\theta})\vec{\imath} + y(\overrightarrow{u_\theta})\vec{\jmath} \Rightarrow \overrightarrow{u_\theta} = -|\overrightarrow{u_\theta}|\sin\theta\vec{\imath} + |\overrightarrow{u_\theta}|\cos\theta\vec{\jmath}$

with $|\overrightarrow{u_\theta}| = 1$ since it is a unit vector therefore $\overrightarrow{u_\theta} = -\sin\theta\vec{\imath} + \cos\theta\vec{\jmath}$

To write the unit vectors $\overrightarrow{u_r}$ and $\overrightarrow{u_\theta}$ as a function of $\vec{\imath}$ and $\vec{\jmath}$ we use the passage table

	$\vec{\imath}$	$\vec{\jmath}$
$\overrightarrow{u_r}$	cosθ	sinθ
$\overrightarrow{u_\theta}$	- sinθ	cosθ

$\vec{\imath} = \cos\theta\overrightarrow{u_r} - \sin\theta\overrightarrow{u_\theta}$ and $\vec{\jmath} = \sin\theta\overrightarrow{u_r} + cos\theta\overrightarrow{u_\theta}$

Example :

Write the vector $\vec{A} = 2x\vec{\imath} + y\vec{\jmath}$

$\begin{cases} x = \rho\ cos\theta \\ y = \rho\ sin\theta \end{cases}$ and $\begin{cases} \vec{\imath} = \cos\theta\overrightarrow{u_r} - \sin\theta\overrightarrow{u_\theta} \\ \vec{\jmath} = \sin\theta\overrightarrow{u_r} + cos\theta\overrightarrow{u_\theta} \end{cases}$

So $\vec{A} = 2\ \rho\ cos\theta\ (\cos\theta\overrightarrow{u_r} - \sin\theta\overrightarrow{u_\theta}) + \rho\ sin\theta\ (\sin\theta\overrightarrow{u_r} + \cos\theta\overrightarrow{u_\theta})$

$\Rightarrow \vec{A} = 2\ \rho\ cos^2\theta\overrightarrow{u_r} - 2\rho(\cos\theta\sin\theta)\overrightarrow{u_\theta} + \rho\ sin^2\theta\ \overrightarrow{u_r} + \rho(\sin\theta\ \cos\theta)\overrightarrow{u_\theta}$

$\Rightarrow \vec{A} = \rho\ (2cos^2\theta + sin^2\theta)\overrightarrow{u_r} - \rho(\cos\theta\sin\theta)\overrightarrow{u_\theta}$

$\Rightarrow \vec{A} = (\rho\ cos^2\theta + 1)\overrightarrow{u_r} - \rho(\cos\theta\sin\theta)\overrightarrow{u_\theta}$

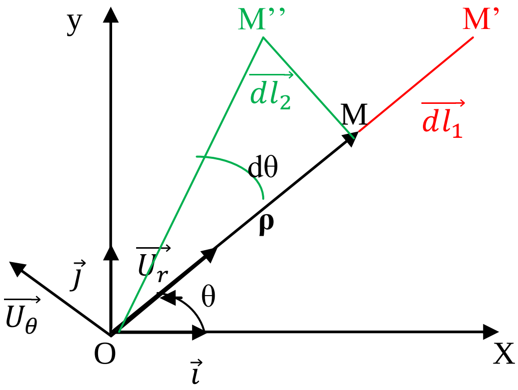
- **Elementary displacement**

The variables ρ and θ are independent: we fix one and change the other

- We fix θ, and we change ρ, then the moving point moves from the point M(ρ, θ) to the point M'(ρ+dρ, θ)

$$\overrightarrow{dl_1} = \overrightarrow{MM'} = d\rho\overrightarrow{u_r}$$

- We fix ρ, and we change θ, then the moving point moves from the point M(ρ, θ) to the point M'(ρ, θ+d θ)



The angle θ varies by dθ, causing a linear displacement of point M towards point M'' $\overrightarrow{U_\theta}$, $\left(\overrightarrow{MM''} \perp \overrightarrow{u_\theta}\right)$

In the right triangle OMM'', MM'' = ρsinθdθ.

Since dθ is very small, we can approximate Sin(dθ) as dθ.

Therefore, MM'' = ρdθ, so

$$\overrightarrow{dl_2} = \overrightarrow{MM''} = \rho d\theta\overrightarrow{u_\theta}$$

so

$$\overrightarrow{dl} = \overrightarrow{dl_1} + \overrightarrow{dl_2} = d\rho\overrightarrow{u_r} + \rho d\theta\overrightarrow{u_\theta}$$

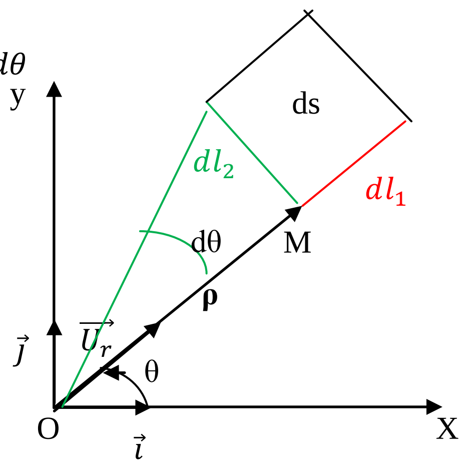
We can obtain the same result mathematically:

$$\overrightarrow{OM} = \rho\vec{U}_r \Rightarrow d\overrightarrow{OM} = d\rho\vec{U}_r + \rho d\vec{U}_r$$

To make the derivative of a unit vector $d\vec{U}_r$, we must bring out the derivative with respect to an angle $\frac{d\vec{U}_r}{d\theta}$ for this we multiply and divide by dθ

$$d\vec{U}_r = \frac{d\vec{U}_r}{d\theta}\, d\theta = \vec{U}_\theta d\theta$$

With $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$ so $d\overrightarrow{OM} = d\rho\vec{U}_r + \rho\, d\theta\vec{U}_\theta$



Calculation of the surface:

$$ds = |\overrightarrow{dl_1}|.|\overrightarrow{dl_2}| = d\rho.\rho d\theta \Rightarrow s = \iint d\rho.\rho d\theta$$

We can separate the variables since they are independent

$s = \int_0^R \rho d\rho \,.\int_0^{2\pi} d\theta = \frac{R^2}{2} 2\pi$

with ρ varies from 0 to R and θ varies from 0 to 2π ⇒ $s = \pi R^2$

- **Velocity vector**

$$\vec{\boldsymbol{v}} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\,\frac{d\vec{U}_r}{dt}$$

We have: $\frac{d\vec{U}_r}{dt} = \frac{d\vec{U}_r}{dt}\frac{d\theta}{d\theta} = \frac{d\vec{U}_r}{d\theta}\frac{d\theta}{dt}$

With : $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$ donc $\frac{d\vec{U}_r}{dt} = \frac{d\theta}{dt}\vec{U}_\theta$ so $\vec{\boldsymbol{v}} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\frac{d\theta}{dt}\vec{U}_\theta$

$\Rightarrow \vec{\boldsymbol{v}} = \rho^{\cdot}\vec{U}_r + \rho\theta^{\cdot}\vec{U}_\theta$ with $\rho^{\cdot} = \frac{d\rho}{dt}$ and $\theta^{\cdot} = \frac{d\theta}{dt}$

- **Acceleration vector**

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\frac{d\vec{U}_r}{dt} + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\,\frac{d^2\theta}{dt^2}\vec{U}_\theta + \rho\,\frac{d\theta}{dt}\frac{d\vec{U}_\theta}{dt}$

$$\Rightarrow \vec{a} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\,\frac{d^2\theta}{dt^2}\vec{U}_\theta - \rho\,\left(\frac{d\theta}{dt}\right)^2\vec{U}_r$$

With : $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta et \frac{d\vec{U}_\theta}{d\theta} = -\vec{U}_r$

$$\text{So : } \vec{a} = \rho^{\cdot\cdot}\vec{U}_r + 2\rho^{\cdot}\theta^{\cdot}\vec{U}_\theta + \rho\theta^{\cdot\cdot}\vec{U}_\theta - \rho(\theta^{\cdot})^2\vec{U}_r$$

4.3. Cylindrical Coordinates الاحداثيات الاسطوانية

If the spatial trajectory involves ρ and z playing a specific role in determining the position vector ($\overrightarrow{OM}$); for example, the movement of air molecules in a whirlwind; it is preferable to use cylindrical coordinates (ρ, θ, z). With:

ρ: polar radius

θ: polar angle

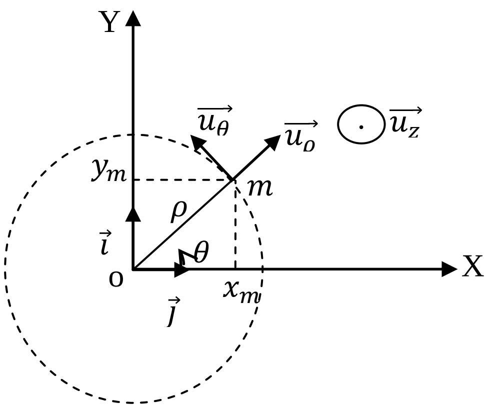
z: altitude or height

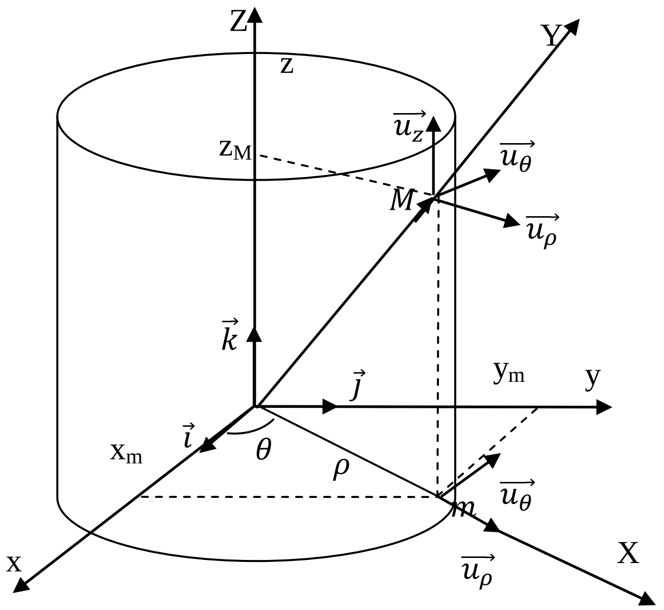
and $\begin{cases} \rho = |\overrightarrow{Om}|, \ \ 0 < \rho < R \\ \theta = \left((ox), \overrightarrow{Om}\right), 0 < \theta < 2\pi \\ z = z_M \,, \ \ 0 < z < H \end{cases}$

Where m is the projection of point M onto the plane (Oxy), and R is the radius of the cylinder, and H is the height of the cylinder.

If we add the 'z' component to polar coordinates in space, we obtain what is known as cylindrical coordinates. Consider R(Oxyz) and a point M belonging to a cylinder.

Point M is identified by three coordinates ρ, θ (polar coordinates), and z.





- **Position vector**

The position vector in cylindrical coordinates (ρ, θ, z) in the orthonormal frame R'$(O, \vec{u_\rho}, \vec{u_\theta}, \vec{u_z})$ is written:

$$\begin{cases} \vec{r} = \overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM} \ (Relation\ de\ \text{Charles}) \\ \overrightarrow{Om} = \rho\vec{u_\rho} \ (Coordonnées\ polaires) \\ \overrightarrow{mM} = z\vec{u_z} \ (\ hauteur\ \ du\ cylindre) \end{cases} \Rightarrow \vec{r} = \overrightarrow{OM} = \rho\vec{u_\rho} + z\vec{u_z}$$

- **Unit vectors**

The unit vectors $\vec{\boldsymbol{U}}_{\boldsymbol{\rho}}$ is following $\overrightarrow{Om}$ (m is the projection of the point M on the plane (Oxy)) and $\vec{\boldsymbol{U}}_{\boldsymbol{\theta}}$ is perpendicular to $\vec{U}_r$ and $\overrightarrow{Om}$ in the direction of θ **($\vec{\boldsymbol{U}}_{\boldsymbol{\rho}} \perp \vec{\boldsymbol{U}}_{\boldsymbol{\theta}}$)** and $\vec{\boldsymbol{U}}_{\boldsymbol{z}}$ is following (Oz), **($\vec{\boldsymbol{U}}_{\boldsymbol{z}} \parallel \vec{\boldsymbol{k}}$)** and it is perpendicular to the plane formed by the two other unit vectors ($\vec{\boldsymbol{U}}_{\boldsymbol{\rho}}$ and $\vec{\boldsymbol{U}}_{\boldsymbol{\theta}}$).

Transit relations between cylindrical coordinates and Cartesian coordinates:

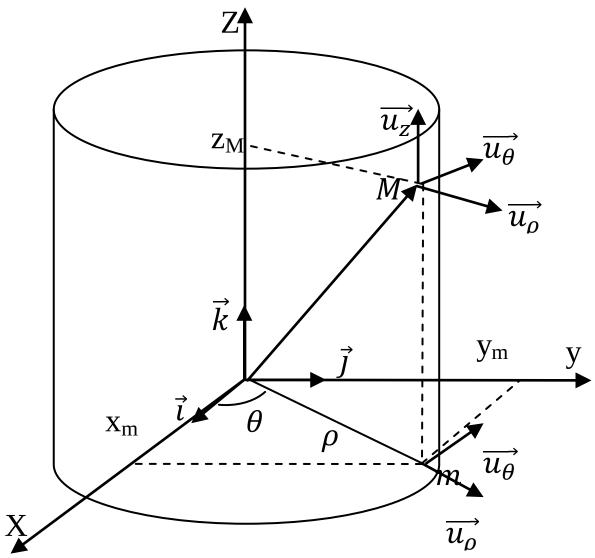
By projecting the point m onto the axes (Ox) and (Oy) (like polar coordinates) z is the height

$$\begin{cases} \boldsymbol{x} = \rho\ cos\theta \\ \boldsymbol{y} = \rho\ sin\ \theta \\ \boldsymbol{z} = \boldsymbol{z} \end{cases}$$

$\overrightarrow{OM}/_{cylin} = \overrightarrow{Om} + \overrightarrow{mM} = \rho\overrightarrow{u_\rho} + z\overrightarrow{u_z}$

$\overrightarrow{OM}/_{cart} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$\overrightarrow{OM}/_{cart} = \rho\,(cos\theta\,\vec{\imath} + sin\,\theta\,\vec{\jmath}) + z\vec{k}$



By identification

$$\begin{cases} \overrightarrow{u_\rho} = cos\theta\,\vec{\imath} + sin\,\theta\,\vec{\jmath} \\ \overrightarrow{u_\theta} = \dfrac{d\overrightarrow{u_\rho}}{d\theta} = -sin\theta\,\vec{\imath} + cos\,\theta\,\vec{\jmath} \\ \overrightarrow{u_z} = \vec{k} \end{cases}$$

Using the passage table :

	$\vec{\imath}$	$\vec{\jmath}$	$\vec{k}$
$\overrightarrow{u_\rho}$	Cosθ	Sinθ	0
$\overrightarrow{u_\theta}$	- sinθ	Cosθ	0
$\overrightarrow{u_z}$	0	0	1

$\vec{\imath} = \cos\theta\overrightarrow{u_\rho} - \sin\theta\overrightarrow{u_\theta}$

$\vec{\jmath} = \sin\theta\overrightarrow{u_\rho} + cos\theta\overrightarrow{u_\theta}$ and $\vec{k} = \overrightarrow{u_z}$

- **Elementary displacement**

The variables ρ, θ and z are independent: we fix one and change the other

- We fix θ, z and we change ρ, then the moving point moves from the point M(ρ, θ,z) to the point M'(ρ+dρ, θ,z)

$$\overrightarrow{dl_1} = \overrightarrow{MM'} = d\rho\overrightarrow{u_\rho}$$

- We fix ρ, z and we change θ, then the moving point moves from the point M(ρ, θ,z) to the point M'(ρ, θ+d θ,z)

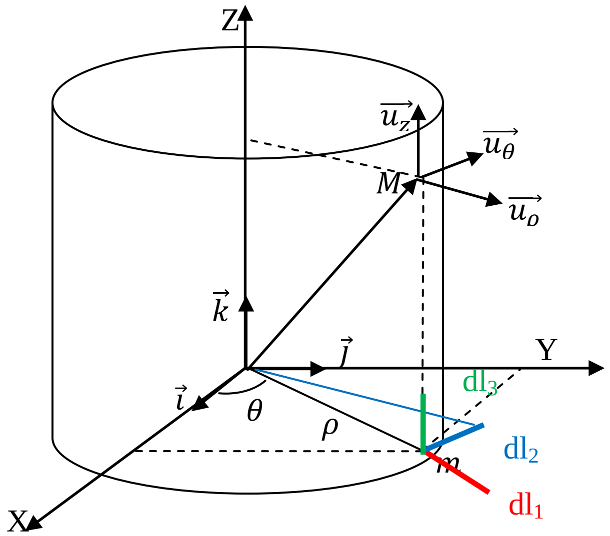
The angle θ varies by dθ, this leads to a linear movement from point M towards point M" following $\overrightarrow{U_\theta}$, $\left(\overrightarrow{MM''} \parallel \overrightarrow{u_\theta}\right)$

In the right triangle OMM'', MM''=ρ sindθ dθ is so small then sin dθ≈dθ.

Then MM''=ρdθ therefore $\overrightarrow{dl_2} = \overrightarrow{MM''} = \rho d\theta \overrightarrow{u_\theta}$

- We fix ρ, θ and we change z, then the moving point moves from the point M(ρ, θ,z) to the point M'''(ρ, θ,z+dz)

$$\overrightarrow{dl_3} = \overrightarrow{MM'''} = dz\overrightarrow{u_z}$$



So

$$\overrightarrow{dl} = \overrightarrow{dl_1} + \overrightarrow{dl_2} + \overrightarrow{dl_3} = d\rho\overrightarrow{u_\rho} + \rho d\theta\overrightarrow{u_\theta} + dz\overrightarrow{u_z}$$

We can obtain the same result mathematically

$\overrightarrow{OM} = \rho\overrightarrow{u_\rho} + z\overrightarrow{u_z} \Rightarrow d\overrightarrow{OM} = d\rho\vec{U}_\rho + \rho d\vec{U}_\rho + dz\overrightarrow{u_z} + zd\overrightarrow{u_z}$

$d\overrightarrow{u_z} = 0$ car $\overrightarrow{u_z} = \vec{k}$ it's a vector fix.

$$d\vec{U}_\rho = \frac{d\vec{U}_\rho}{d\theta}\, d\theta = \vec{U}_\theta d\theta$$

With $\frac{d\vec{U}_\rho}{d\theta} = \vec{U}_\theta$ so $d\overrightarrow{OM} = d\rho\vec{U}_\rho + \rho\, d\theta\vec{U}_\theta + dz\overrightarrow{u_z}$

- **The cylinder volume**

$$dV = |\overrightarrow{dl_1}|.|\overrightarrow{dl_2}|.|\overrightarrow{dl_3}| = d\rho.\rho d\theta.dz \Rightarrow V = \iiint d\rho.\rho d\theta \text{ dz}$$

We can separate the variables since they are independent

$V = \int_0^R \rho d\rho \,.\int_0^{2\pi} d\theta \int_0^H dz = \frac{R^2}{2} 2\pi H \Rightarrow V = \pi R^2 H$

(with ρ varies from 0 to R and θ varies from 0 to 2π and z varies from 0 to H)

- **The surface of the base of cylinder**

$ds_{base} = |\overrightarrow{dl_1}|.|\overrightarrow{dl_2}| = d\rho.\rho d\theta \Rightarrow s_{base} = \iint d\rho.\rho d\theta$

We can separate the variables since they are independent

$s = \int_0^R \rho d\rho \,.\int_0^{2\pi} d\theta = \frac{R^2}{2} 2\pi \Rightarrow s_{base} = \pi R^2$

- **The lateral surface of cylinder**

$ds_{lat} = |\overrightarrow{dl_2}|.|\overrightarrow{dl_3}| = d\rho.\rho d\theta \Rightarrow s_{base} = \iint \rho d\theta.dz$

The radius is constant ρ=R, the variables are independent so we can separate them

$$s = R\int_0^{2\pi} d\theta \,.\int_0^H dz = R2\pi H \Rightarrow s_{base} = 2\pi RH$$

- **Velocity vector**

The velocity in this case is written by: $\vec{\boldsymbol{v}} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\frac{d\vec{U}_r}{dt} + \frac{dz}{dt}\vec{U}_z + z\frac{d\vec{U}_z}{dt}$

$$\frac{d\vec{U}_r}{dt} = \frac{d\vec{U}_r}{dt}\frac{d\theta}{d\theta} = \frac{d\vec{U}_r}{d\theta}\frac{d\theta}{dt}$$

With: $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$ so $\frac{d\vec{U}_r}{dt} = \frac{d\theta}{dt}\vec{U}_\theta$ and $\frac{d\vec{U}_z}{dt} = \vec{O}$

$$\vec{\boldsymbol{v}} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\,\frac{d\theta}{dt}\vec{U}_\theta + \frac{dz}{dt}\vec{U}_z$$

$$\Rightarrow \vec{\boldsymbol{v}} = \rho^{\cdot}\vec{U}_r + \rho\,\theta^{\cdot}\vec{U}_\theta + z^{\cdot}\overrightarrow{U_z}$$

With: $\rho^{\cdot} = \frac{d\rho}{dt}$, $\theta^{\cdot} = \frac{d\theta}{dt}$ and $z^{\cdot} = \frac{dz}{dt}$

- **Acceleration vector**

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\frac{d\vec{U}_r}{dt} + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\frac{d^2\theta}{dt^2}\vec{U}_\theta + \rho\frac{d\theta}{dt}\frac{d\vec{U}_\theta}{dt} + \frac{d^2z}{dt^2}\vec{U}_z + \frac{dz}{dt}\frac{d\vec{U}_z}{dt}$

$$\Rightarrow \vec{a} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\,\frac{d^2\theta}{dt^2}\vec{U}_\theta - \rho\left(\frac{d\theta}{dt}\right)^2\vec{U}_r + \frac{d^2z}{dt^2}\vec{U}_z$$

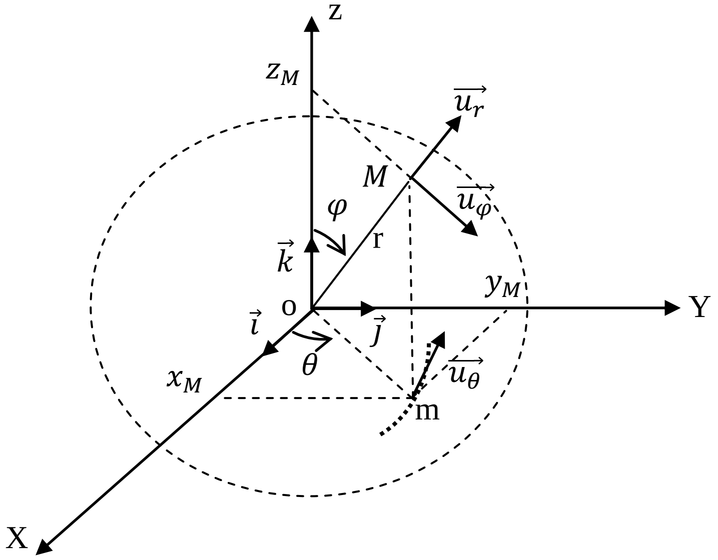
With : $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$, $\frac{d\vec{U}_\theta}{d\theta} = -\vec{U}_r$ and $\frac{d\vec{U}_z}{dt} = \vec{O}$

$$\Rightarrow \vec{a} = \rho^{\cdot\cdot}\vec{U}_r + 2\rho^{\cdot}\theta^{\cdot}\vec{U}_\theta + \rho\theta^{\cdot\cdot}\vec{U}_\theta - \rho(\theta^{\cdot})^2\vec{U}_r + z^{\cdot\cdot}\overrightarrow{U_z}$$

4.4. Spherical coordinates الاحداثيات الكروية

When the point O and the distance r separating M and O play a characteristic role, the use of spherical coordinates (r,θ,φ) are the best suited in the orthonormed base ($\overrightarrow{u_r}, \overrightarrow{u_\theta}, \overrightarrow{u_\varphi}$) with:

$$\begin{cases} r = |\overrightarrow{OM}|, \ 0 < r < R \\ \theta = \left((ox), \overrightarrow{Om}\right) \ 0 < \theta < 2\pi \\ \varphi = \left((oz), \overrightarrow{OM}\right) \ 0 < \varphi < \pi \end{cases}$$



With m is the projection of M in the plane (Oxy).

- **Position Vector** :

The position vector in spherical coordinates (r,θ,φ) is written as: $\vec{\boldsymbol{r}} = \overrightarrow{\boldsymbol{OM}} = \boldsymbol{r}\overrightarrow{\boldsymbol{U_r}}$

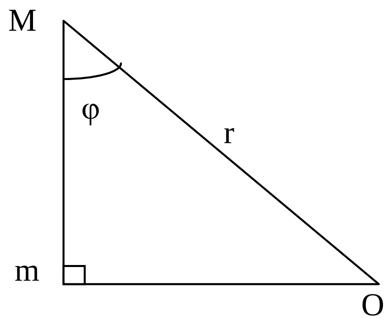
- **The unit vectors**

The unit vectors $\vec{\boldsymbol{U}}_{\boldsymbol{r}}$ is following $\overrightarrow{OM}$ and $\vec{\boldsymbol{U}}_{\boldsymbol{\varphi}}$ is perpendicular to $\vec{U}_r$ and $\overrightarrow{OM}$ in the direction of φ **($\vec{\boldsymbol{U}}_{\boldsymbol{\varphi}} \perp \vec{\boldsymbol{U}}_{\boldsymbol{r}}$)** and $\vec{\boldsymbol{U}}_{\boldsymbol{\theta}}$ is perpendicular to $\overrightarrow{Om}$ **($\vec{\boldsymbol{U}}_{\boldsymbol{\theta}} \perp \overrightarrow{Om}$).**

Transit relations between spherical coordinates and Cartesian coordinates

By projecting m onto the axes (Ox) and (Oy)

$$\begin{cases} \boldsymbol{x} = |\overrightarrow{Om}|\ cos\theta \\ \boldsymbol{y} = |\overrightarrow{Om}|\ \ sin\ \theta \\ \quad \boldsymbol{z} = |\overrightarrow{mM}| \end{cases}$$



By taking the right triangle (OmM):

We have Om=r sinφ and mM=r cosφ, replacing them in passing relationships we will have:

$$\begin{cases} \boldsymbol{x} = r\ sin\varphi\ cos\theta \\ \boldsymbol{y} = r\ sin\varphi\ \ sin\ \theta \\ \quad \boldsymbol{z} = r \cos\varphi \end{cases}$$

$$\overrightarrow{OM}\Big/_{sph} = r\overrightarrow{u_r}$$

$$\overrightarrow{OM}/_{cart} = r\ sin\varphi\ cos\theta\vec{\imath} + r\ sin\varphi\ \ sin\ \theta\vec{\jmath} + r\cos\varphi\,\vec{k}$$

$$\overrightarrow{OM}/_{cart} = r\left(\ sin\varphi\ cos\theta\vec{\imath} +\ sin\varphi\ \ sin\ \theta\vec{\jmath} + \cos\varphi\,\vec{k}\right)$$

By identification

$$\overrightarrow{u_r} = sin\varphi\ cos\theta\vec{\imath} +\ sin\varphi\ \ sin\ \theta\vec{\jmath} + \cos\varphi\,\vec{k}$$

$$\overrightarrow{u_\varphi} = \frac{-d\overrightarrow{U_r}}{d(-\varphi)} = \frac{d\overrightarrow{U_r}}{d\varphi} =\ cos\varphi\ cos\theta\vec{\imath} + cos\ \varphi\ \ sin\ \theta\vec{\jmath} - \sin\varphi\,\vec{k}$$

$$\overrightarrow{U_\theta} = \overrightarrow{U_r}\Lambda\overrightarrow{U_\varphi} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ sin\varphi\ cos\theta & sin\varphi\ \ sin\ \theta & \cos\varphi \\ cos\varphi\ cos\theta & cos\ \varphi\ \ sin\ \theta & -\sin\varphi \end{vmatrix}$$

$$= \vec{\imath}(-sin^2\varphi\ \ sin\ \theta - cos\ ^2\varphi\ \ sin\ \theta) - \vec{\jmath}(-sin^2\varphi\ cos\theta - cos^2\varphi\ cos\theta)$$

$$+ \vec{k}(sin\varphi\ cos\theta cos\ \varphi\ \ sin\ \theta - sin\varphi\ \ sin\ \theta cos\varphi\ cos\theta)$$

$$\overrightarrow{U_\theta} = -\sin\theta\vec{\imath} + cos\theta\vec{\jmath}$$

By using the pasage table :

	$\vec{\imath}$	$\vec{\jmath}$	$\vec{k}$
$\overrightarrow{u_r}$	$sin\varphi\ cos\theta$	$sin\varphi\ \ sin\ \theta$	$\cos\varphi$
$\overrightarrow{u_\varphi}$	$cos\varphi\ cos\theta$	$cos\ \varphi\ \ sin\ \theta$	-sin φ
$\overrightarrow{u_\theta}$	$-\sin\theta$	$cos\theta$	1

$$\vec{\imath} =\ sin\varphi\ cos\theta\,\overrightarrow{u_r} + cos\varphi\ cos\theta\,\overrightarrow{u_\varphi} - \sin\theta\,\overrightarrow{u_\theta}$$
$$\vec{\jmath} =\ sin\varphi\ sin\theta\,\overrightarrow{u_r} + cos\varphi\ sin\theta\,\overrightarrow{u_\varphi} + \cos\theta\,\overrightarrow{u_\theta}$$
$$\vec{k} =\ cos\varphi\ \ \overrightarrow{u_r} -\ sin\varphi\,\overrightarrow{u_\varphi}$$

- **Elementary displacement**

The variables r, φ and θ are independent: we fix one and change the other:

- We fix φ,θ and we change r, then the moving point moves from the point M(r, φ,θ) to the point M'(r+dr, φ,θ) so $\overrightarrow{dl_1} = \overrightarrow{MM'} = dr\overrightarrow{u_r}$
- We fix r, θ and we change φ, then the moving point moves from the point M(r, φ ,θ) to the point M''(r, , φ+dφ ,θ)

The angle φ varies by dφ, this leads to a linear movement from point M towards point M" following $\overrightarrow{U_\varphi}$, $\left(\overrightarrow{MM''} \perp \overrightarrow{u_\varphi}\right)$.

In the right triangle OMM'', MM''= r.sindφ

dφ is so small then sin dφ≈dφ

then MM"=r.dφ therefore $\overrightarrow{dl_2} = \overrightarrow{MM''} = r\, d\varphi\, \overrightarrow{u_\varphi}$

- We fix r, φ and we change θ, then the moving point moves from the point M(r, φ, θ) to the point M''(r, φ, θ+dθ)

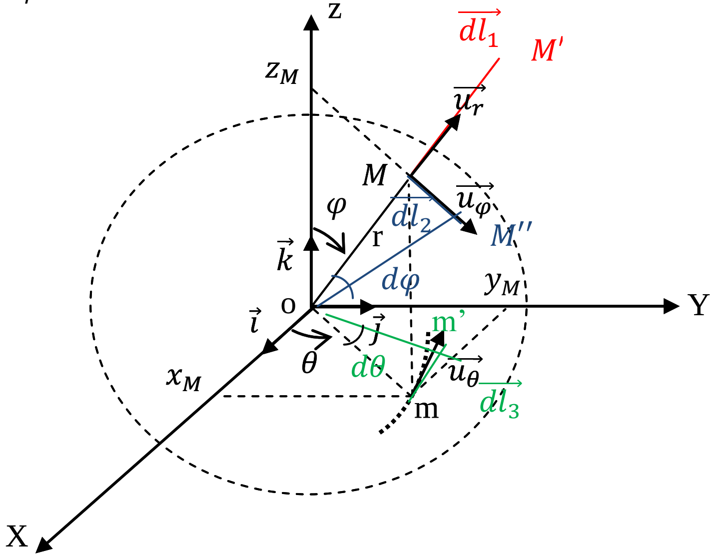
The angle φ varies by dφ, this leads to a linear displacement of the point m (projection of the point M in the plane (Oxy)) towards the following point m' $\overrightarrow{U_\theta}$, $\left(\overrightarrow{mm'} \perp \overrightarrow{u_\theta}\right)$.

In the right triangle Omm', mm'= Om.sindθ

dθ is so small then sin dθ≈dθ

So *mm'*=Omdθ and Om=r sinφ therefore $\overrightarrow{dl_2} = \overrightarrow{mm'} = \mathrm{r}\sin\varphi\, d\theta\, \overrightarrow{u_\theta}$

$\overrightarrow{dOM} = dr\overrightarrow{u_r} + rd\varphi\overrightarrow{u_\varphi} + \mathrm{r}\sin\varphi\, d\theta\, \overrightarrow{u_\theta}$



Or mathematically :

$$\overrightarrow{OM} = r\overrightarrow{U_r} \Rightarrow d\overrightarrow{OM} = d\left(r\overrightarrow{U_r}\right) = dr\overrightarrow{U_r} + rd\overrightarrow{U_r}$$

$$d\overrightarrow{U_r} = \frac{\partial U_r}{\partial \theta} d\theta + \frac{\partial U_r}{\partial \varphi}\ d\varphi$$

$$\overrightarrow{U_r} = sin\varphi\ cos\theta\vec{\imath} +\ sin\varphi\ \ sin\ \theta\vec{\jmath} + \cos\varphi\ \vec{k}$$

$$\frac{\partial\overrightarrow{U_r}}{\partial\theta} = -sin\varphi\ sin\theta\vec{\imath} +\ sin\varphi \cos\theta\vec{\jmath} = \sin\varphi\left(-sin\theta\vec{\imath} +\ \ cos\ \theta\vec{\jmath}\right)$$

$$\Rightarrow \frac{\partial\overrightarrow{U_r}}{\partial\theta} = sin\varphi\overrightarrow{U_\theta}$$

$$\frac{\partial\overrightarrow{U_r}}{\partial\varphi} = cos\varphi\ cos\theta\vec{\imath} + cos\varphi\ \sin\theta\vec{\jmath} - sin\varphi\vec{k}$$

$$\Rightarrow \frac{\partial\overrightarrow{U_r}}{\partial\varphi} = \overrightarrow{U_\varphi}$$

$$d\overrightarrow{OM} = dr\overrightarrow{U_r} + rd\varphi\overrightarrow{U_\varphi} + rsin\varphi d\theta\overrightarrow{U_\theta}$$

- **Volume of sphere**

$dV=dl_1dl_2dl_3=dr\ \ r\ sin\varphi\ d\theta\ \ r\ d\varphi \Rightarrow V = \iiint r^2 dr \sin\varphi\ d\varphi\ d\theta$

$$\Rightarrow V = \int_0^R r^2 dr \int_0^\pi \sin\varphi \int_0^{2\pi} d\theta = \left.\frac{r^3}{3}\right] (-\ cos\varphi)]\ \ \theta]$$

$$\Rightarrow V = \frac{4}{3}\ \pi R^3$$

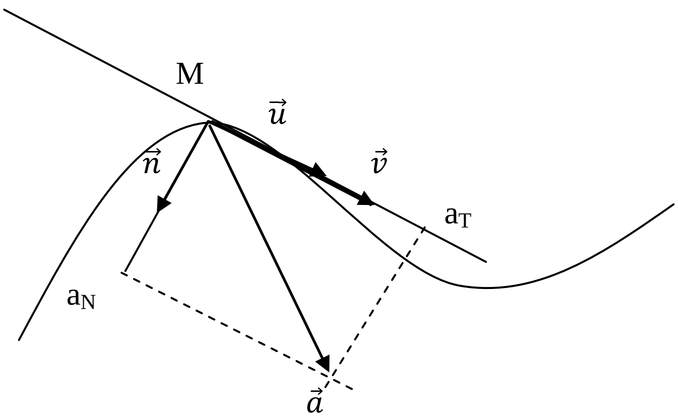
- **Velocity vector**

The velocity vector is written in spherical coordinates (r,θ,φ) by:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dr}{dt}\overrightarrow{U_r} + r\frac{d\varphi}{dt}\overrightarrow{U_\varphi} + rsin\varphi\frac{d\theta}{dt}\overrightarrow{U_\theta}$$

4.5. Intrinsic coordinates (Frenet frame) احداثيات الحركة المنحنية

We used to work in a fixed frame, but in this case, we study the motion in a moving frame that travels with the moving point "M". This frame is the Frenet frame.



We study the motion in the Frenet frame:

The Frenet frame is a two-dimensional reference frame.

- $\vec{u}$ is the unit vector along the tangent to the trajectory.
- $\vec{n}$ is the unit vector normal to the trajectory and perpendicular to $\vec{u}$, directed towards the center of curvature.
- The position remains unchanged (the frame moves with point M).
- The velocity vector is tangent to the trajectory, and it is written as: $\boldsymbol{\vec{v} = |\vec{v}|\vec{u}}$
- The acceleration vector :

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d|\vec{v}|\vec{u}}{dt} = \frac{d|\vec{v}|}{dt}\vec{u} + |\vec{v}|\frac{d\vec{u}}{dt}$$

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{d\theta}.\frac{d\theta}{dt} = \vec{n}.\omega with \quad \vec{n} = \frac{d\vec{u}}{d\theta} and \ \omega = \frac{d\theta}{dt}$$

The acceleration vector is written by: $\vec{a} = a_T\vec{u} + a_N\vec{n}$

So : $\vec{a} = \frac{d|\vec{v}|}{dt}\vec{u} + |\vec{v}|.\vec{n}.\omega$

(the perimeter of a circle (محيط دائرة) $l = 2\pi R$*, for the length of a segment (طول قوس)*

$x = \theta R$ *; from angular velocity to linear velocity by* $\frac{dx}{dt} = R\frac{d\theta}{dt} \Rightarrow v = R\omega$*)*

Hence:

$\omega = \frac{v}{R}$ with R is the radius of the curvature of the trajectory.

So $\vec{a} = \frac{d|\vec{v}|}{dt}\vec{u} + \frac{v^2}{R}\vec{n}$

The normal acceleration (التسارع الناظمي) and tangential acceleration (التسارع المماس) are written by : $\begin{cases} a_T = \frac{d|\vec{v}|}{dt} \\ a_N = \frac{v^2}{R} \end{cases}$

$|\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{a_N^2 + a_T^2}$

Note :

- $R \to \infty$: so the trajectory is a line .
- R is constant: so the trajectory is circular.

5. Study of some movements

5.1. Rectilinear motion حركة خطية

We have linear motion if the trajectory is a straight line.

We choose a point O as the origin on the trajectory and a unit vector $\vec{\imath}$.

The position of the mobile M, as a function of time, is identified by its abscissa:

$x(t) = \overline{OM(t)}$.

The position vector will be:$\overrightarrow{r(t)} = \overrightarrow{OM(t)} = x(t)\vec{\imath}$

5.1.1. Uniform rectilinear motion حركة مستقيمة منتظمة URM

We have uniform rectilinear motion if the trajectory is a straight line and the velocity vector is constant. This is a motion with zero acceleration $\overrightarrow{a(t)} = \vec{0}$.

The initial conditions to t=0 ; x=x_0.

- ***The velocity***

$$a = \frac{dv}{dt} = 0 \Rightarrow \int_{v_0}^{v} dv = \int_0^t 0.dt = cte$$

So $\boldsymbol{v=v_0=cte}$

- ***The position***

$$v = \frac{dx}{dt} = v_0 \Rightarrow \int_{x_0}^{x} dx = \int_0^t v_0 d\,t = [v_0 t]_0^t = v_0 t$$

So : $\boldsymbol{x=v_0\,t+x_0}$ ***This is the hourly equation of the motion. URM***

5.1.2. Uniformly varied rectilinear motion حركة مستقيمة متغيرة بانتظام UVRM

One has a uniformly varied rectilinear movement if the trajectory is a straight and the acceleration is constant.

The initial conditions to t=0 ; $v=v_0$ and x=x_0

- ***The velocity***

$$a = \frac{dv}{dt} = a_0 \Rightarrow \int_{v_0}^{v} dv = \int_0^t a_0 d\,t = [a_0 t]_0^t$$

So $\boldsymbol{v=a_0\,t+v_0}$

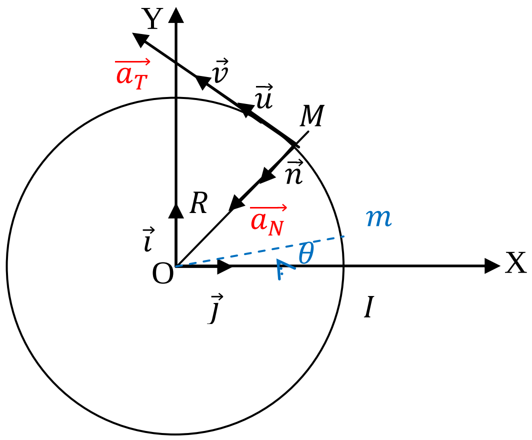
- ***The position***

$$v = \frac{dx}{dt} = a_0 t + v_0 \quad \Rightarrow \int_{x_0}^{x} dx = \int_0^t (a_0 t + v_0) d\,t = \left[\frac{1}{2} a_0 t^2 + v_0 t\right]_0^t$$

So $\boldsymbol{x = \frac{1}{2} a_0 t^2 + v_0 t + x_0}$ ***this is the hourly equation of the motion UVRM***

5.2. Circular motion حركة دائرية

Circular motion is plane motion with constant radius of curvature ρ=R. The trajectory of the moving object is a circle of radius R .



- ***The position***

The moving point travels from point I to point M, thus the trajectory forms an arc$\widehat{IM}$.

By considering an elementary displacement of the moving point from point I to point m, we would have a displacement in the form of an elementary arc Im.

In the right triangle OIm, $\widehat{Im}$=R sinθ

In the right triangle. If θ is so small then$sin\theta \approx \theta$.

So $\boldsymbol{\widehat{Im}=R.\theta}$

- ***The speed***

$$v = \frac{d\widehat{Im}}{dt} = R\frac{d\theta}{dt}$$

R is constant, the speed is following the trajectory, so it is written $\vec{v} = v\vec{u}$ so the vector $\vec{u}$ would be following the tangent.

$\frac{d\theta}{dt} = \boldsymbol{\theta}^{\cdot} = \boldsymbol{\omega}$ ***is the angular velocity*** **السرعة الزاوية**

$$v = R\frac{d\theta}{dt} = R.\boldsymbol{\theta}^{\cdot} = \boldsymbol{R.\omega}$$

Note: The relationship between linear velocity and angular velocity is: $\mathbf{v = R\omega}$

- ***The acceleration***

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{u} + v\,\frac{d\vec{u}}{dt}$

$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{d\theta}\frac{d\theta}{dt}$ *with* $\frac{d\vec{u}}{d\theta} = \vec{n}$

(with$(\vec{u},\vec{n})$ the unit vectors in the Fresnet farme and $\frac{d\theta}{dt} = \omega$)

5.2.1. Uniforme circular motionحركة دائرية منتظمة

In this case the angular velocity ω is constant and therefore the linear velocity v is also constant, then $a_T = 0$.

The acceleration in this case is :$\vec{a} = \overrightarrow{a_N} = \frac{v^2}{R}\vec{n}$

5.2.2. Uniformly variable circular motion حركة دائرية متغيرة بانتظام

In this case the angular velocity ω is not constant and therefore the velocity v is not constant also, then $\vec{a} = a_T\vec{u} + a_N\vec{n}$.

The acceleration in this case is: $\vec{a} = \frac{dv}{dt}\vec{u} + \frac{v^2}{R}\vec{n}$ =$R\frac{d\omega}{dt}\vec{u} + R\omega^2\vec{n}$

5.3. Sinusoidal or harmonic motion حركة جيبية

The movement is called sinusoidal or harmonic if its evolution over time is written by the equation:

$$x(t) = A\sin(\omega t + \varphi)$$

A: amplitude, ω: angular frequency, and φ: phase.

$$\omega = \frac{2\pi}{T} = 2\pi f$$

T: period and f: frequency

- ***The speed***

$$v(t) = \frac{dx(t)}{dt} = A\omega \cos(\omega t + \varphi)$$

- ***The acceleration***

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} = -A\,\omega^2 \sin(\omega t + \varphi) \Rightarrow a(t) = \frac{d^2x(t)}{dt^2} = -\,\omega^2\,x(t)$$

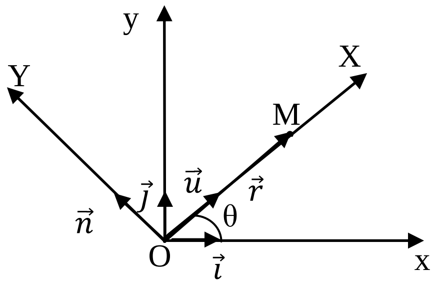
Note:

Another type of movement which is relative movement will be detailed in the next chapter.

Proposed exercises about chapter III

Exercise 1

We consider a vector $\vec{r}$, of module r = OM= a θ, carried by an axis OX making with Ox the variable angle θ. We denote by $\vec{u}$ the unit vector of OX and by $\vec{n}$ the unit vector directly perpendicular to $\vec{u}$.

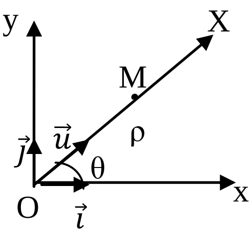


1. Calculate the express of $\frac{d\vec{r}}{d\theta}$ in terms of a, θ, $\vec{u}$ and $\vec{n}$.
2. Represent this vector $\frac{d\vec{r}}{d\theta}$.

Exercise 2

A) A material point M is marked by its cartesian coordinates (x,y).

1. Write x and y in terms of the polar coordinates ρ and θ.
2. Give the expression of the unit vector $\vec{u}$ as a function of the unit vectors $\vec{\imath}\ and\ \vec{\jmath}$.
3. Calculate $d\vec{u}/_{d\theta}$, what does this vector represent?



B) If the position of point M is given by $\begin{cases} \overrightarrow{OM} = t^2\vec{u} \\ \theta = \omega t \end{cases}$ (ω constant)

Find the expression of the velocity vector $\vec{v}$ in polar coordinates.

Exercise 3

Consider a polar coordinate system with the origin O and unit vectors $\overrightarrow{u_\rho}$ and $\overrightarrow{u_\theta}$.

Let M be a point with coordinates (ρ, θ).

1. Using a detailed diagram, provide the expression for the position vector $\overrightarrow{OM}$ in polar coordinates
2. Give the conversion relationships between polar and Cartesian coordinates.
3. Express the vector $\vec{A} = 2x\vec{\imath} + y\vec{\jmath}$ in polar coordinates.
4. Write the elementary displacement vector in polar coordinates.
5. Provide the velocity vector and the acceleration vector in polar coordinates.
6. Find the expression for the elementary area in this coordinate system and deduce the area of a disk with radius R.

Exercise 4

A material point M is identified by its cartesian coordinates (x, y, z).

1. Write down the relationship between cartesian coordinates and cylindrical coordinates (using a diagram).
2. Write the position vector in cylindrical coordinates and deduce the velocity vector in the same coordinate system.
3. If the position of the point is represented in cylindrical coordinates by $\begin{cases} \rho = 4t^2 \\ \theta = \omega t \\ z = \sqrt{t} \end{cases}$

Find the expression of the velocity vector $\vec{v}$ in cylindrical coordinates.

Exercise 5

The differential of the vector $\vec{r}$, $d\vec{r} = d\vec{l} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$ can be expressed in cylindrical coordinates as $d\vec{r} = \frac{\partial \vec{r}}{\partial \rho} d\rho + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial z} dz$.

1. Using the formulas for switching between the two coordinate systems, evaluate the vectors$\frac{\partial \vec{r}}{\partial \rho}$,$\frac{\partial \vec{r}}{\partial \theta}$ et $\frac{\partial \vec{r}}{\partial z}$.
2. Derive the unit vectors $\overrightarrow{U_\rho}, \overrightarrow{U_\theta}$ et$\overrightarrow{U_z}$(cylindrical coordinates) as a function of $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ (Cartesian coordinates), check that they are orthogonal.
3. Write $\vec{A} = 2x\vec{\imath} + y\vec{\jmath} - 2z\vec{k}$ in cylindrical coordinates.

Exercise 6

Write the vector $\boldsymbol{\vec{A} = x\,.\vec{\imath} - 2.y.\vec{\jmath} + z.\vec{k}}$ in cylindrical coordinates i.e. as a function of **ρ, θ ,** $\boldsymbol{z}$ $\overrightarrow{\mathbf{u_\rho}}$, $\overrightarrow{\mathbf{u_\theta}}$, $\overrightarrow{\mathbf{u_z}}$. (using passing relations)

If the position of point M is given by $\begin{cases} \overrightarrow{OM} = t^3\overrightarrow{u_\rho} + 5t^2\overrightarrow{u_z} \\ \theta = \omega t \end{cases}$ (ω constant)

Find the expression of the vectors: speed $\vec{v}$ and acceleration $\vec{a}$ in cylindrical coordinates.

Exercise 7

A body moves along the x axis according to the relation $x(t)=2t^3+5t^2+5$.

1. Determine the velocity v(t) and acceleration a(t) at each instant t.

2. Calculate the body's position, velocity and instantaneous acceleration for t_1=2s and t_2=3s.
3. Deduce the average velocity and acceleration of the body between $t_1$and t_2.

Exercise 8

The x and y coordinates of a moving point M in the (oxy) plane vary with time t according to the following relationships: x=t+1 and $y=(t^2 /2)+2$.

Find :

1. The equation of the trajectory
2. The components of speed and acceleration and their modules.
3. Accelerations: normal a_N and tangential a_T and deduce the radius of curvature.
4. The nature of the movement

Exercise 9

A particle is launched with an initial horizontal speed v0 according to the time-dependent equations:

$$\begin{cases} x = v_0 t \\ y = \frac{1}{2} g t^2 \end{cases}$$

Determine:

1. The trajectory equation.
2. The components of speed and its module.
3. The components of acceleration and its module.
4. Tangential and normal accelerations.
5. The radius of curvature R of the particle's trajectory.

Exercise 10

A comet is moving through the solar system. His position is expressed:

$$\overrightarrow{OM} = (t-1)\vec{\imath} + \frac{t^2}{2}\vec{\jmath}$$

Where O is the origin of the landmark (the sun) and t represents the time expressed in seconds. We assume that the comet remains in the plane (Oxy)

1. Write the equation of the trajectory
2. Determine the components of the velocity vector $\vec{v}$ and the acceleration vector $\vec{a}$ and give the nature of the movement.

3. Express the expressions for the tangential a_T and normal a_N accelerations and deduce the radius of curvature.

Exercise 11

A particle moves on a trajectory whose trajectory equation is $y=x^2$ such that at each instant $v_x=v_0=cst$. If t=0, x_0, y_0 =0.

Determine :

1- The x(t) and y(t) coordinates of the particle.

2- The speed and acceleration of the particle.

3- The normal and tangential accelerations as well as the radius of curvature.

Exercise 12

A body moves on a straight line with an acceleration such that

1. a) $a=-kv$; b) $a=-kv^2$ where k is a constant

If at t=0; $v=v_0$ and $x=x_0$

2. Find for both cases its speed and its displacement in time as well as v as a function of x

Exercise 13

A body whose motion is defined by the following velocity components: $\mathbf{v_x=1}$ and $\mathbf{v_y=2/(t+1)}$

Knowing that at t=0 x=0 and y=2.

1- What is the equation of the trajectory y=f(x).

2- Calculate the components of the acceleration.

Exercise 14

From the ground, a balloon rises with a constant initial speed v_0 (according to y). The wind gives the balloon horizontal speed $\mathbf{V_x=a.y}$ (a constant).

1. Determine the equations of motion x(t) and y(t), Deduce the equation of the trajectory y=f(x)

2. Calculate the accelerations a, a_N and a_T. Deduce the radius of curvature.

Exercise 15

A stone is thrown from the top of a 20 m high building, with a horizontal speed of 10m/s.

1. What time does it take for the stone to reach the ground?

2. At what distance from the building will the stone reach the ground?

b- With what speed will the stone reach the ground?

Exercise 16

The motion of a body is defined by the following velocity components:

$$\begin{cases} v_x = R\omega\cos(\omega t) \\ v_y = R\omega\sin(\omega t) \end{cases}$$

Knowing that ω is constant and at t=0, the moving body is at point M (0, R).

Determine :

1. The components of the acceleration vector and its modulus.
2. The tangential and normal components of acceleration, and deduce the radius of curvature.
3. The components of the position vector and deduce the equation of the trajectory.
4. What is the nature of the motion?

Exercise 17

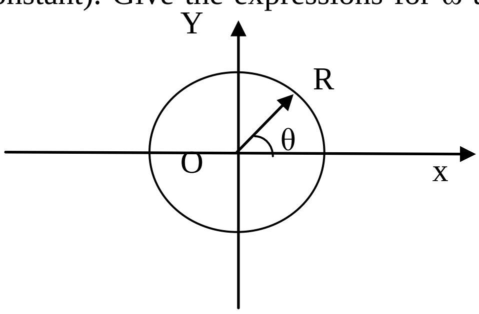
A material point M moves along the OX axis with acceleration $\vec{a} = a\vec{\imath}$ with a > 0.

1. Determine the velocity vector knowing that v (t=0)= v_0 .
2. Determine the position vector $\overrightarrow{OM}$ given thatx(t=0)=x_0 .
3. Check that: $v_f^2 \quad - v_i^2 = 2a(x - x_0)$.
4. What is the condition that$\vec{a}.\vec{v}$ so that the motion is uniformly accelerated and retarded?

Exercise 18

Consider a moving point M describing a circle of radius R and center O with an angular speed $\omega = \frac{d\theta}{dt}$. At time t=0 point M is at A.

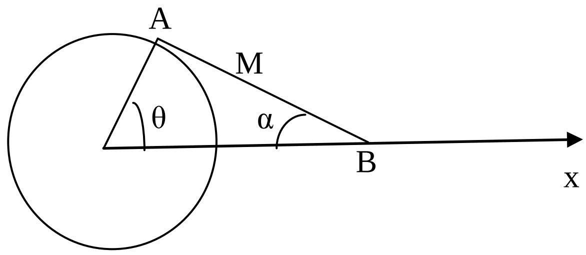
1. Write the coordinates of M as a function of R and θ.
2. Calculate the modulus of the speed of point M.
3. Determine the components of the acceleration on the axes Ox and Oy (Cartesian coordinates) on the one hand and on the axes parallel and perpendicular to OM on the other hand (polar coordinates).
4. We assume that $\alpha = \frac{d\omega}{dt}$ (α is a non-zero constant). Give the expressions for ω and θ as a function of time.
5. We recall that at t=0, θ_0=0 et $\omega=\omega_0$.
   What relationship exists between ω and θ.



Exercise 19

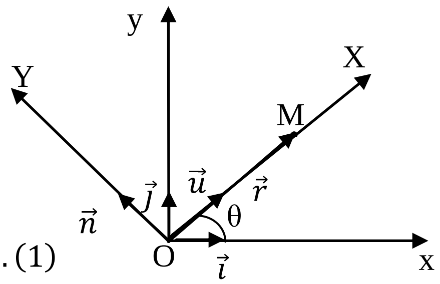
OA is a rod of length L animated by a uniform circular movement of angular speed ω around the point O. AB is another rod of length R, articulated at A to OA such that B can move on Ox.

1. Establish the time equations of M (middle of AB)
2. Determine the abscissa of B. Is its movement sinusoidal?
3. Calculate the speed of M
4. Show that if r=R, the movement of B becomes sinusoidal



Correction of exercises about chapter III

Exercise 1



We have $|\vec{r}| = r = OM = a\theta$

1- $\frac{d\vec{r}}{d\theta} = f(\theta, a, \vec{u}, \vec{n}) = ?$

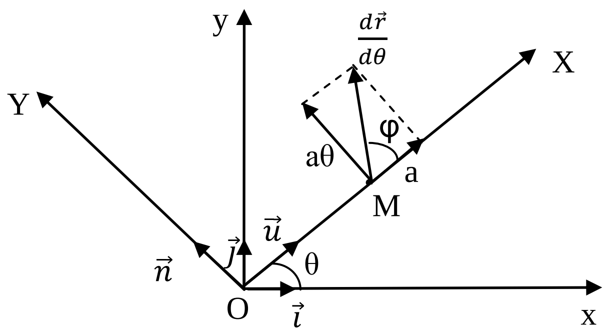
According to the diagram $\begin{cases} \vec{r} = \overrightarrow{OM} = a\theta\vec{u} \\ \vec{u} = cos\theta\vec{\imath} + sin\theta\vec{\jmath} \quad \dots\dots(1) \\ \vec{n} = -sin\theta\vec{\imath} + cos\theta\vec{\jmath}. \end{cases}$

$$\frac{d\vec{r}}{d\theta} = \frac{d}{d\theta}(\theta a\,\vec{u}) = a\vec{u} + a\theta\frac{d\vec{u}}{d\theta}$$

$$(1) \Rightarrow \frac{d\vec{u}}{d\theta} = -sin\theta\vec{\imath} + cos\theta\vec{\jmath} = \vec{n}$$

$$\Rightarrow \frac{d\vec{r}}{d\theta} = a\vec{u} + a\theta\vec{n}$$

2- $\frac{d\vec{r}}{d\theta} = a\vec{u} + a\theta\vec{n}$ which originates from point M presents as follows:



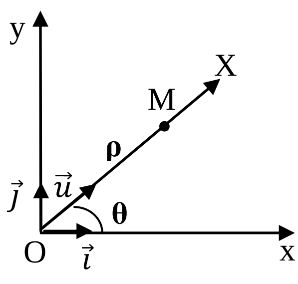
3- $\left|\frac{d\vec{r}}{d\theta}\right| = \sqrt{a^2 + (a\theta)^2} = a\sqrt{1+\theta^2}$

Exercise 2

A) A material point M is identified by its Cartesian coordinates (x,y):

Find x and y in terms of polar coordinates ρ and θ

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} \quad (1)$$



In the othor hand $\overrightarrow{OM}$ is written by projection as:

$$\overrightarrow{OM} = \rho cos\theta\vec{\imath} + \rho sin\theta\vec{\jmath} \quad (2)$$

(1) and (2) $\Rightarrow \begin{cases} x = \rho cos\theta \\ y = \rho sin\theta \end{cases}$

1- The unit vector $\vec{u}$ as a function of the unit vectors $\vec{\imath}\ and\ \vec{\jmath}$:

we have $\overrightarrow{OM} = |\overrightarrow{OM}|\vec{u} = \rho\vec{u} = \rho cos\theta\vec{\imath} + \rho sin\theta\vec{\jmath}$

so $\vec{u} = cos\theta\vec{\imath} + sin\theta\vec{\jmath}$ and $\vec{n} = -sin\theta\vec{\imath} + cos\vec{\jmath}$

$\vec{n}$ and $\vec{u}$ represent the unit vectors of the polar coordinate basis.

2- Calculate the expression of $d\vec{u}/d\theta$, which this vector represents?

$$\frac{d\vec{u}}{d\theta} = \frac{d(cos\theta\vec{\imath} + sin\theta\vec{\jmath})}{d\theta} = -sin\theta\vec{\imath} + cos\vec{\jmath} = \vec{n}$$

$\boldsymbol{\frac{d\vec{u}}{d\theta}}$ represents a unit vector perpendicular to $\vec{u}$ in the direct direction .

2- The position of point M is given by $\begin{cases} \overrightarrow{OM} = t^2\vec{u} \\ \theta = \omega t \end{cases}$ (ω constant)

The expression of the velocity vector $\vec{v}$ in polar coordinates is :

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(t^2\vec{u})}{dt} = 2t\vec{u} + t^2\frac{d\vec{u}}{dt}$$

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{d\theta}.\frac{d\theta}{dt} = \vec{n}.\omega$$

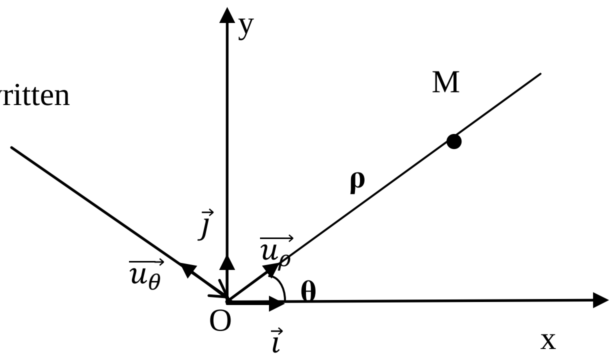
$$\boldsymbol{\vec{v} = 2t.\vec{u} + t^2.\omega.\vec{n}}$$

Exercise 3

B- The polar coordinates are ρ and θ ; with $\rho = \|\overrightarrow{OM}\|$; 0<ρ<R and the $\theta = (\overrightarrow{Ox}, \overrightarrow{OM})$ with 0<θ<2π.

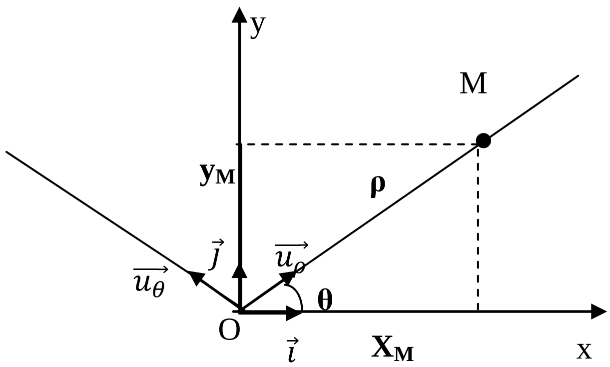
1- The vector $\overrightarrow{OM}$ in polar coordinates is written

as following :$\overrightarrow{OM} = \rho\overrightarrow{u_\rho}$



2- the transition relationships between polar and Cartesian coordinates.

$$\begin{cases} cos\theta = \frac{x_M}{\rho} \\ Sin\theta = \frac{y_M}{\rho} \end{cases} \Rightarrow \begin{cases} x_M = \rho cos\theta \\ y_M = \rho sin\theta \end{cases}$$



So the vector $\overrightarrow{OM}$ in coordinates cartesian

is written $\overrightarrow{OM} = x_M\vec{\imath} + y_M\vec{\jmath}$

We had : $\overrightarrow{OM} = \rho\overrightarrow{u_\rho}$ (in polar coordinates)

Then $\overrightarrow{OM} = \rho(cos\theta\vec{\imath} + sin\theta\vec{\jmath})$

By identification $\overrightarrow{u_\rho} = cos\theta\vec{\imath} + sin\theta\vec{\jmath}$ and $\overrightarrow{u_\theta} = \frac{d\,\overrightarrow{u_\rho}}{d\theta} = -sin\theta\vec{\imath} + cos\theta\vec{\jmath}$

3- The writing of the vector $\vec{A} = 2x\vec{\imath} + y\vec{\jmath}$ in polar coordinates

we have $\begin{cases} x_M = \rho cos\theta \\ y_M = \rho sin\theta \end{cases}$ and $\begin{cases} \overrightarrow{u_\rho} = cos\theta\vec{\imath} + sin\theta\vec{\jmath} \\ \overrightarrow{u_\theta} = -sin\theta\vec{\imath} + cos\theta\vec{\jmath} \end{cases}$

By using the passage table

	$\overrightarrow{u_\rho}$	$\overrightarrow{u_\theta}$
$\vec{\imath}$	Cosθ	-sin θ
$\vec{\jmath}$	Sinθ	cosθ

So $\vec{\imath} = \cos\theta\,\overrightarrow{u_\rho} - \sin\theta\,\overrightarrow{u_\theta}$ and $\vec{\jmath} = \sin\theta\,\overrightarrow{u_\rho} + \cos\theta\,\overrightarrow{u_\theta}$

The vector $\vec{A}$ is then written as;

$$\vec{A} = 2\rho cos\theta\left(\cos\theta\,\overrightarrow{u_\rho} - \sin\theta\,\overrightarrow{u_\theta}\right) + \rho sin\theta\left(\sin\theta\,\overrightarrow{u_\rho} + \cos\theta\,\overrightarrow{u_\theta}\right)$$

$$\Rightarrow \vec{A} = \rho(1 + cos^2\theta)\overrightarrow{u_\rho} - \rho\sin\theta\cos\theta\,\overrightarrow{u_\theta}$$

4- The vector of elementary displacement in polar coordinates

$d\overrightarrow{OM} = d\left(\rho\overrightarrow{u_\rho}\right) = d\rho\overrightarrow{u_\rho} + \rho d\overrightarrow{u_\rho}$ with $d\overrightarrow{u_\rho} = \frac{d\overrightarrow{u_\rho}}{d\theta}d\theta = \overrightarrow{u_\theta}d\theta$

So $d\overrightarrow{OM} = d\rho\overrightarrow{u_\rho} + \rho d\theta\overrightarrow{u_\theta}$

5- The velocity vector and the acceleration vector in polar coordinates.

The velocity vector in polar coordinates: $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\overrightarrow{u_\rho} + \rho\frac{d\theta}{dt}\overrightarrow{u_\theta}$

The acceleration vector in polar coordinates:

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\rho}{dt^2}\overrightarrow{u_\rho} + \frac{d\rho}{dt}\frac{d\overrightarrow{u_\rho}}{dt} + \frac{d\rho}{dt}\frac{d\theta}{dt}\overrightarrow{u_\theta} + \rho\frac{d^2\theta}{dt^2}\overrightarrow{u_\theta} + \rho\frac{d\theta}{dt}\frac{d\overrightarrow{u_\theta}}{dt}$

With $\frac{d\overrightarrow{u_\rho}}{dt} = \frac{d\overrightarrow{u_\rho}}{d\theta}.\frac{d\theta}{dt} = \frac{d\theta}{dt}\overrightarrow{u_\theta}$ et $\frac{d\overrightarrow{u_\theta}}{dt} = \frac{d\overrightarrow{u_\theta}}{d\theta}.\frac{d\theta}{dt} = -\frac{d\theta}{dt}\overrightarrow{u_\rho}$

so $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\rho}{dt^2}\overrightarrow{u_\rho} + \frac{d\rho}{dt}\frac{d\theta}{dt}\overrightarrow{u_\theta} + \frac{d\rho}{dt}\frac{d\theta}{dt}\overrightarrow{u_\theta} + \rho\frac{d^2\theta}{dt^2}\overrightarrow{u_\theta} - \rho\frac{d\theta}{dt}\frac{d\theta}{dt}\overrightarrow{u_\rho}$

$\Rightarrow \vec{a} = \frac{d^2\rho}{dt^2}\overrightarrow{u_\rho} + 2\frac{d\rho}{dt}\frac{d\theta}{dt}\overrightarrow{u_\theta} + \rho\frac{d^2\theta}{dt^2}\overrightarrow{u_\theta} - \rho\left(\frac{d\theta}{dt}\right)^2\overrightarrow{u_\rho}$

6- The expression of the elementary surface in the polar frame:

$ds = dl_1.dl_2$ and $d\overrightarrow{OM} = d\rho\overrightarrow{u_\rho} + \rho d\theta\overrightarrow{u_\theta} = dl_1\overrightarrow{u_\rho} + dl_2\overrightarrow{u_\theta}$

with dl_1 is the variation of ρ along $\overrightarrow{u_\rho}$ which is $d\rho$ and dl_2 is the variation of θ along $\overrightarrow{u_\theta}$

$$ds = d\rho.\rho d\theta$$

The surface of a disk with radius R.

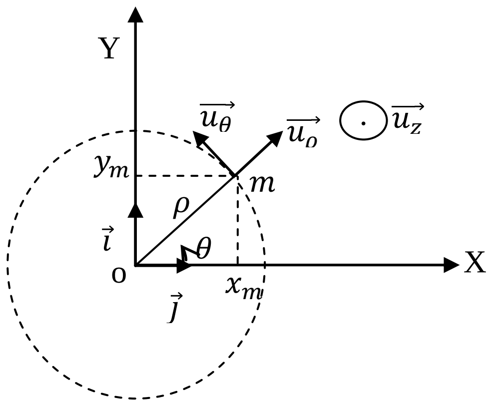
S=$\iint d\rho.\rho d\theta = \int_0^R \rho d\rho \int_0^{2\pi} d\theta = \frac{R^2}{2}2\pi = \pi R^2$

Exercise 4

1. A material point M is identified by its Cartesian coordinates (x, y, z).

Write the relationship between Cartesian coordinates and polar coordinates.

$$\begin{cases} x = \rho cos\theta \\ y = \rho sin\theta \\ z = z_M \end{cases}$$



2. Find the expression of the position vector and deduce the velocity $\vec{v}$ of point M in cylindrical coordinates.

$$\overrightarrow{OM} = \rho\overrightarrow{U_\rho} + z\overrightarrow{U_z}$$

$$\Rightarrow \vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\overrightarrow{U_\rho} + \rho\frac{d\overrightarrow{U_\rho}}{dt} + \frac{dz}{dt}\overrightarrow{U_z} + z\frac{d\overrightarrow{U_z}}{dt}$$

We have $\frac{d\overrightarrow{U_z}}{dt} = 0 \Rightarrow \vec{v} = \dot{\rho}\overrightarrow{U_\rho} + \rho\frac{d\theta}{dt}\frac{d\overrightarrow{U_\rho}}{d\theta} + \dot{z}\overrightarrow{U_z}$

$$\Rightarrow \vec{v} = \dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\theta}\overrightarrow{U_\theta} + \dot{z}\overrightarrow{U_z}$$

3. A velocity vector $\vec{v}$ of point M in cylindrical coordinates:

We have $\begin{cases} \rho = 4t^2 \\ \theta = \omega t \\ z = \sqrt{t} \end{cases}$ Hence $\begin{cases} \frac{d\rho}{dt} = 8t \\ \frac{d\theta}{dt} = \omega \\ \frac{dz}{dt} = \frac{1}{2\sqrt{t}} \end{cases}$

$$\Rightarrow \vec{v} = \dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\theta}\frac{d\overrightarrow{U_\rho}}{d\theta} + \dot{z}\overrightarrow{U_z} = 8t\overrightarrow{U_\rho} + 4t^2.\omega.\overrightarrow{U_\theta} + \frac{1}{2\sqrt{t}}\overrightarrow{U_z}$$

Exercise 5

The differential of vector $\vec{r}$, $d\vec{r} = d\vec{l} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$ can be expressed in cylindrical coordinates as $d\vec{r} = \frac{\partial\vec{r}}{\partial\rho}d\rho + \frac{\partial\vec{r}}{\partial\theta}d\theta + \frac{\partial\vec{r}}{\partial z}dz.$

1. We are looking for the vectors $\frac{\partial\vec{r}}{\partial\rho}$, $\frac{\partial\vec{r}}{\partial\theta}$ et $\frac{\partial\vec{r}}{\partial z}$.

We are $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

- The displacement vector in cartesian coordinates (x, y, z) :

$$d\vec{r} = d\vec{l} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

- The displacement vector in cylindrical coordinates (ρ, θ, z) :

$$d\vec{r} = \frac{\partial\vec{r}}{\partial\rho}\, d\rho + \frac{\partial\vec{r}}{\partial\theta}\, d\theta + \frac{\partial\vec{r}}{\partial z}\, dz$$

Relationships between cartesian coordinates (x, y, z) and cylindrical coordinates (ρ, θ, z) are :

$$\begin{cases} x = \rho cos\theta \\ y = \rho sin\theta \\ z = z_M \end{cases} \Rightarrow \begin{cases} dx = d\rho.cos\theta - \rho\,.\sin\theta.d\theta \\ dy = d\rho.sin\theta + \rho.cos\theta.d\theta \\ dz = dz_M \end{cases}$$

$$\Rightarrow d\vec{r} = d\vec{l} = (d\rho.cos\theta - \rho\,.\sin\theta.d\theta)\vec{\imath} + (d\rho.sin\theta + \rho.cos\theta.d\theta)\vec{\jmath} + dz\vec{k}$$

$$\Rightarrow d\vec{r} = (cos\theta.\vec{\imath} + sin\theta.\vec{\jmath})d\rho + (-\rho sin\theta\vec{\imath} + \rho.cos\theta.\vec{\jmath})d\theta + dz\vec{k} \ldots\ldots\ldots\ldots(1)$$

$$\Rightarrow d\vec{r} = \left(\frac{\partial\vec{r}}{\partial\rho}\right) d\rho + \left(\frac{\partial\vec{r}}{\partial\theta}\right) d\theta + \left(\frac{\partial\vec{r}}{\partial z}\right) dz \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

With identification between (1) and (2) we'll have :

$$\Rightarrow \begin{cases} \dfrac{\partial \vec{r}}{\partial \rho} = cos\theta.\vec{\imath} + sin\theta.\vec{\jmath} \\ \dfrac{\partial \vec{r}}{\partial \theta} = -\rho sin\theta\vec{\imath} + \rho.cos\theta.\vec{\jmath} \\ \dfrac{\partial \vec{r}}{\partial z} = \vec{k} \end{cases}$$

2. Deduce Unit Vectors $\overrightarrow{U_\rho}, \overrightarrow{U_\theta}$ and $\overrightarrow{U_z}$ (cylindrical coordinats) as function of $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ (Cartesian coordinates) :

The displacement vector in cylindrical coordinates is written:

$$d\vec{r} = d\rho\overrightarrow{U_\rho} + \rho d\theta\overrightarrow{U_\theta} + dz\vec{k} \text{……………………………..(3)}$$

$$\text{(1) and (3)} \Rightarrow \begin{cases} \overrightarrow{U_\rho} = \frac{\partial \vec{r}}{\partial \rho} = cos\theta.\vec{\imath} + sin\theta.\vec{\jmath} \\ \overrightarrow{U_\theta} = \frac{1}{\rho}\frac{\partial \vec{r}}{\partial \theta} = -sin\theta\vec{\imath} + cos\theta.\vec{\jmath} \\ \overrightarrow{U_z} = \frac{\partial \vec{r}}{\partial z} = \vec{k} \end{cases}$$

<u>Note</u> :

The unit vectors of the Cartesian coordinates base can be written as a function of the unit vectors of the cylindrical coordinates base from the table below:

	$\vec{\imath}$	$\vec{\jmath}$	$\vec{k}$
$\overrightarrow{u_\rho}$	Cosθ	Sinθ	0
$\overrightarrow{u_\theta}$	-sinθ	Cosθ	0
$\overrightarrow{u_z}$	0	0	1

$$\Rightarrow \begin{cases} \vec{\imath} = \cos\theta\overrightarrow{u_\rho} - \sin\theta\overrightarrow{u_\theta} \\ \vec{\jmath} = \sin\theta\overrightarrow{u_\rho} + \cos\theta\overrightarrow{u_\theta} \\ \vec{k} = \overrightarrow{u_z} \end{cases}$$

3. Checking that they are orthogonal?

$$\Rightarrow \begin{cases} \left|\overrightarrow{U_\rho}\right| = \sqrt{cos\theta^2 + sin\theta^2} = 1 \\ \left|\overrightarrow{U_\theta}\right| = \sqrt{(-sin\theta)^2 + cos\theta^2 = 1} \\ \left|\overrightarrow{U_z}\right| = \left|\vec{k}\right| = 1 \end{cases}$$

Hence $\overrightarrow{U_\rho}, \overrightarrow{U_\theta}$et $\overrightarrow{U_z}$, are the unit vectos.

We have $\overrightarrow{U_\rho}.\overrightarrow{U_\theta} = 0, \overrightarrow{U_\rho}.\overrightarrow{U_z} = 0 \; and \; \overrightarrow{U_z}.\overrightarrow{U_\theta} = 0$

So $\overrightarrow{U_\rho}, \overrightarrow{U_\theta}$, and $\overrightarrow{U_z}$ are orthogonal vectors.

Therefore the vectors $\overrightarrow{U_\rho}, \overrightarrow{U_\theta}, \overrightarrow{U_z}$ form an orthonormal reference frame.

4. Write $\vec{A} = 2x\vec{\imath} + y\vec{\jmath} - 2z\vec{k}$ in cylindrical coordonates.

We have $\begin{cases} x = \rho cos\theta \\ y = \rho sin\theta \\ z = z_M \end{cases}$ and $\begin{cases} \vec{\imath} = \cos\theta\overrightarrow{u_\rho} - \sin\theta\overrightarrow{u_\theta} \\ \vec{\jmath} = \sin\theta\overrightarrow{u_\rho} + \cos\theta\overrightarrow{u_\theta} \\ \vec{k} = \overrightarrow{u_z} \end{cases}$

So $\vec{A} = 2x\vec{\imath} + y\vec{\jmath} - 2z\vec{k}$ is wretten by :

$$\Rightarrow \vec{A} = 2\rho cos\theta\left(\cos\theta\overrightarrow{u_\rho} - \sin\theta\overrightarrow{u_\theta}\right) + \rho sin\theta\left(\sin\theta\overrightarrow{u_\rho} + \cos\theta\overrightarrow{u_\theta}\right) - 2z\vec{k}$$

$$\Rightarrow \vec{A} = (2\rho cos\theta^2 + \rho sin\theta^2)\overrightarrow{u_\rho} + (-2\rho\cos\theta\sin\theta + \rho os\theta\sin\theta)\overrightarrow{u_\theta} - 2z\vec{k}$$

$$\Rightarrow \boldsymbol{\vec{A} = (cos\theta^2 + 1)\rho\overrightarrow{u_\rho} - \rho\cos\theta\sin\theta\overrightarrow{u_\theta} - 2z\overrightarrow{u_z}}$$

Exercise 6

Writing the vector $\boldsymbol{\vec{A} = x\,.\vec{\imath} - 2.y.\vec{\jmath} + z.\vec{k}}$ in cylindrical coordinates:

Transit relations between cylindrical and Cartesian coordinates

$$\begin{cases} x_M = \rho cos\theta \\ y_M = \rho sin\theta \\ z_M = mM \end{cases}$$

So the vector $\overrightarrow{OM}$ in Cartesian coordinates is written $\overrightarrow{OM} = x_M\vec{\imath} + y_M\vec{\jmath} + z_M$
$\overrightarrow{OM} = \rho(cos\theta\vec{\imath} + sin\theta\vec{\jmath}) + z\vec{k}$

We have: $\overrightarrow{OM} = \rho\overrightarrow{u_\rho} + z\overrightarrow{u_z}$ (in cylindrical coordinates)

By identification $\overrightarrow{u_\rho} = cos\theta\vec{\imath} + sin\theta\vec{\jmath}$, $\overrightarrow{u_z} = \vec{k}$

and $\overrightarrow{u_\theta} = \frac{d\,\overrightarrow{u_\rho}}{d\theta} = -sin\theta\vec{\imath} + cos\theta\vec{\jmath}$

By using the table passage

	$\overrightarrow{u_\rho}$	$\overrightarrow{u_\theta}$	$\overrightarrow{u_z}$
$\vec{\imath}$	Cosθ	-sin θ	0
$\vec{\jmath}$	Sinθ	cosθ	0
$\vec{k}$	0	0	1

$\vec{\imath} = \cos\theta\,\overrightarrow{u_\rho} - \sin\theta\,\overrightarrow{u_\theta}$

$\vec{\jmath} = \sin\theta\,\overrightarrow{u_\rho} + \cos\theta\,\overrightarrow{u_\theta}$

$\vec{k} = \overrightarrow{u_z}$

The vector $\vec{A}$ is written by:

$$\vec{A} = \rho cos\theta\left(\cos\theta\,\overrightarrow{u_\rho} - \sin\theta\,\overrightarrow{u_\theta}\right) - 2\,\rho sin\theta\left(\sin\theta\,\overrightarrow{u_\rho} + \cos\theta\,\overrightarrow{u_\theta}\right) + z\,\overrightarrow{u_z}$$

$$\Rightarrow \vec{A} = \rho(cos^2\theta - 2\,sin^2\theta)\overrightarrow{u_\rho} - 3\rho\sin\theta\cos\theta\ \overrightarrow{u_\theta}\,V + z\,\overrightarrow{u_z}$$

B. The position of point M is given by $\begin{cases}\overrightarrow{OM} = t^3\overrightarrow{\mathbf{u_\rho}} + 5t^2\overrightarrow{\mathbf{u_z}} \\ \theta = \omega t\end{cases}$ (ω constant)

1. Speed is written : $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = 3t^2\overrightarrow{u_\rho} + t^3\frac{d\overrightarrow{u_\rho}}{dt} + 10\,t\,\overrightarrow{\mathbf{u_z}} + 10t\,\frac{\mathbf{d\,\overrightarrow{u_z}}}{\mathbf{dt}}$

With $\frac{d\overrightarrow{u_\rho}}{dt} = \frac{d\overrightarrow{u_\rho}}{d\theta}.\frac{d\theta}{dt} = \omega\overrightarrow{u_\theta}\ and\ \frac{d\vec{U}_z}{dt} = \vec{O}$

$$\Rightarrow \vec{v} = 3\,t^2\overrightarrow{u_\rho} + t^3\omega\,\overrightarrow{u_\theta} + 10t\,\overrightarrow{\mathbf{u_z}}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = 6\,t\,\overrightarrow{u_\rho} + 3\,t^2\ \frac{d\overrightarrow{u_\rho}}{dt} + 3\,t^2\,\omega\,\vec{U}_\theta + +t^3\omega\frac{d\vec{U}_\theta}{dt} + 10\,\vec{U}_z + 10t\,\frac{\mathbf{d\,\overrightarrow{u_z}}}{\mathbf{dt}}$$

$$\Rightarrow \vec{a} = 6\,t\,\overrightarrow{u_\rho} + 3\,t^2\ \omega\overrightarrow{u_\theta} + 3\,t^2\,\omega\,\vec{U}_\theta - t^3\omega\,\omega\vec{U}_\rho + 10\,\vec{U}_z$$

With $\frac{d\overrightarrow{u_\rho}}{dt} = \frac{d\overrightarrow{u_\rho}}{d\theta}.\frac{d\theta}{dt} = \omega\overrightarrow{u_\theta}$, $\frac{d\overrightarrow{u_\theta}}{dt} = \frac{d\overrightarrow{u_\theta}}{d\theta}.\frac{d\theta}{dt} = -\omega\vec{U}_\rho$

$$\Rightarrow \vec{a} = 6\,t\,\overrightarrow{u_\rho} + 6\,t^2\ \omega\,\overrightarrow{u_\theta} - t^3\omega\,^2\vec{U}_\rho + 10\,\vec{U}_z$$

So $\vec{a} = (6\,t - t^3\omega\,^2)\,\overrightarrow{u_\rho} + 6\,t^2\ \omega\,\overrightarrow{u_\theta} + 10\,\vec{U}_z$

Exercise 7

a- we have $x(t)=2t^3+5t^2+5$ so :

The velocity: $v(t) = \frac{dx}{dt} = 6t^2+10t$

The acceleration: $a(t)=\frac{dv(t)}{dt}=12t+10$

b- The body's position at time t_1=2s, as well as its instantaneous velocity and acceleration:

The position : $x(2)= 2(2)^3+5(2)^2+5=41$m

Instantaneous speed: $v(2)=6(2)^2+10(2)=44$m/s

Instantaneous acceleration: $a(2)=12(2)+10=34$m/s^2

-The body's position at time t2=3s, as well as its instantaneous velocity and acceleration:

Position : $x(3)= 2(3)^3+5(3)^2+5=104$m

Instantaneous speed: $v(3)=6(3)^2+10(3)=84$m/s

Instantaneous acceleration: $a(3)=12(3)+10=46m/s^2$

c- We deduce the speed and average acceleration of the body between t_1 and t_2:

Average speed: $v_{moy} = \frac{\Delta x}{\Delta t} = \frac{x(t_2)-x(t_1)}{t_2-t_1} \Rightarrow v_{moy} = \frac{104-41}{3-2} = 63m/s$

Average acceleration :

$$a_{moy} = \frac{\Delta v}{\Delta t} = \frac{v(t_2)-v(t_1)}{t_2-t_1} \Rightarrow a_{moy} = \frac{84-44}{3-2} = 40m/s^2$$

Exercise 8

The coordinates of a moving point M in the plane (oxy) are written as:

$$x(t)=t+1 \text{ and } y(t)=(t^2/2)+2$$

a- The equation of the trajectory is then written :

To find the equation of the trajectory, simply find the relationship between x(t) and y(t).

To do this, deduce the time from one equation, x(t) or y(t), and replace it in the other equation).

Here, we'll write t as a function of x :

$t=x-1$ so $y = \frac{(x-1)^2}{2} + 2 = \boldsymbol{\frac{x^2}{2} - x + \frac{5}{2}}$

The equation of the trajectory is : $\mathbf{y(x) = \frac{x^2}{2} - x + \frac{5}{2}}$

b- Components of velocity and acceleration vectors:

- The velocity :$\overrightarrow{v(t)} = v_x(t)\vec{i} + v_y(t)\vec{j}$

$$\begin{cases} v_x(t) = \dfrac{dx(t)}{dt} = 1 \\ v_y(t) = \dfrac{dy(t)}{dt} = t \end{cases}$$

The velocity is written by $\overrightarrow{\mathbf{v(t)}} = \vec{\mathbf{i}} + \mathbf{t}\vec{\mathbf{j}}$

The velocity module:$|\vec{v}(t)| = \sqrt{1+t^2}$

- The acceleration: $\overrightarrow{a(t)} = a_x(t)\vec{i} + a_y(t)\vec{j}$

$$\begin{cases} a_x(t) = \dfrac{dv_x(t)}{dt} = 0 \\ a_y(t) = \dfrac{dv_y(t)}{dt} = 1 \end{cases}$$

So $\overrightarrow{\mathbf{a(t)}} = \vec{\mathbf{j}}$

The acceleration module is: $|\vec{a}(t)| = 1$

c- Normal and tangential acceleration:

- Tangential acceleration

$a_T = \frac{d|\overrightarrow{v(t)}|}{dt}$ with $|\overrightarrow{v(t)}| = \sqrt{v_x^2 + v_y^2} = \sqrt{1+t^2}$

$$a_T = \frac{d(\sqrt{1+t^2})}{dt} = \frac{2t}{2\sqrt{1+t^2}}$$

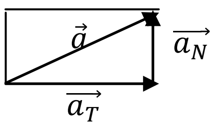
$$a_T = \frac{t}{\sqrt{1+t^2}} \quad \text{because} \quad \boldsymbol{(U^n)' = nU'\,U^{n-1}}$$

- Normal acceleration :

The accelerations a_N and a_T are the normal and tangential components of the acceleration.$\vec{a}$

$(\vec{a} = a_T\overrightarrow{U_T} + a_N\overrightarrow{U_N} \Rightarrow |\vec{a}| = \sqrt{\boldsymbol{a_T^2} + \boldsymbol{a_N^2}})$

We have the shape of a right triangle, by applying Pitagort's relation.



$\boldsymbol{a^2 = a_T^2 + a_N^2}$

So $a_N^2 = a^2 - a_T^2$ or $|\vec{a}| = \sqrt{\boldsymbol{a_T^2} + \boldsymbol{a_N^2}}$

$$a_N^2 = 1 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 = 1 - \frac{t^2}{1+t^2}$$

$$a_N^2 = \frac{1}{1+t^2}$$

So $a_N = \frac{1}{\sqrt{1+t^2}} = \frac{1}{v}$

- The radius of curvature: $a_N = \frac{v^2}{R} = \frac{1}{v} \Rightarrow R = v^3 = (1+t^2)^{\frac{3}{2}}$

c- The nature of movement

$$\overrightarrow{a(t)}.\overrightarrow{v(t)} = \begin{pmatrix}0\\1\end{pmatrix}.\begin{pmatrix}1\\t\end{pmatrix} = 1(0) + t(1) = t > 0$$

The motion is then uniformly accelerated.

Exercise 9

The x and y coordinates of a mobile point M in the (xy) plane vary with time t according to the following relationships: $\begin{cases} x = v_0 t \\ y = \frac{1}{2}gt^2 \end{cases}$

1- The equation of the trajectory is then written as follows:

Here, we will express t as a function of x: $t = \frac{x}{v_0}$ So $y = \frac{1}{2}g\left(\frac{x}{v_0}\right)^2 = \frac{g}{2v_0^2}\boldsymbol{x^2}$

The equation of the trajectory is: $\mathbf{y(x)} = \frac{g}{2v_0^2}\boldsymbol{x^2}$

The components of the velocity

$$\begin{cases} v_x(t) = \frac{dx(t)}{dt} = v_0 \\ v_y(t) = \frac{dy(t)}{dt} = gt \end{cases}$$

The velocity is expressed as: $\overrightarrow{\mathbf{v(t)}} = v_0\vec{\mathbf{i}} + gt\,\vec{\mathbf{j}}$

The magnitude of the velocity: $|\vec{v}(t)| = \sqrt{v_0^2 + (gt)^2} = \sqrt{v_0^2 + g^2t^2}$

The components of the acceleration

$$\begin{cases} a_x(t) = \frac{dv_x(t)}{dt} = 0 \\ a_y(t) = \frac{dv_y(t)}{dt} = g \end{cases}$$

The acceleration is expressed as: $\overrightarrow{\mathbf{a(t)}} = \mathbf{g}\vec{\mathbf{j}}$

The magnitude of the acceleration $|\vec{a}(t)| = g$

2- The nature of the movement

$$\overrightarrow{a(t)}.\overrightarrow{v(t)} = v_0(0) + gt(g) = g^2t > 0$$

The movement in this case is uniformly accelerated.

3- Normal and tangential accelerations.

- Tangential acceleration:

$a_T = \frac{d|\overrightarrow{v(t)}|}{dt}$ avec $|\overrightarrow{v(t)}| = \sqrt{v_0^2 + g^2t^2}$ so $a_T = \frac{d(\sqrt{v_0^2+g^2t^2})}{dt} = \frac{2g^2t}{2\sqrt{v_0^2+g^2t^2}}$

$$a_T = \frac{g^2t}{\sqrt{v_0^2 + g^2t^2}} = \frac{g^2t}{v}$$

- Normal acceleration

The accelerations a_N and a_T are the normal and tangential components of the acceleration vector $\vec{a}$.

$(\vec{a} = a_T\overrightarrow{U_T} + a_N\overrightarrow{U_N}) \Rightarrow a^2 = a_T^2 + a_N^2$ so $a_N^2 = a^2 - a_T^2$

$$a_N^2 = g^2 - \left(\frac{g^2t}{\sqrt{v_0^2 + g^2t^2}}\right)^2 = g^2 - \frac{g^4t^2}{v_0^2 + g^2t^2} \Rightarrow a_N^2 = \frac{g^2v_0^2 + g^4t^2 - g^4t^2}{v_0^2 + g^2t^2}$$

So $a_N = \sqrt{\frac{g^2v_0^2}{v_0^2+g^2t^2}} = \frac{gv_0}{v}$

The radius of curvature

$a_N = \frac{v^2}{R} = \frac{gv_0}{v} \Rightarrow R = \frac{v^3}{gv_0} = \frac{(v_0^2+g^2t^2)^{\frac{3}{2}}}{gv_0}$

Exercise 10

$$\overrightarrow{OM} = (t-1)\vec{\imath} + \frac{t^2}{2}\vec{\jmath}$$

1- The equation of the trajectory

$$\overrightarrow{OM} = (t-1)\vec{\imath} + \frac{t^2}{2}\vec{\jmath} \Rightarrow \begin{cases} \mathrm{x = t - 1} \\ \mathrm{y} = \frac{t^2}{2} \end{cases}$$

$$\mathrm{t=x+1} \Rightarrow \mathrm{y} = \frac{(\mathrm{x}+1)^2}{2}$$

2- The components of velocity and acceleration, and their magnitudes :

- Velocity

$$\begin{cases} v_x = \frac{dx}{dt} \\ v_y = \frac{dy}{dt} \end{cases} \Rightarrow \begin{cases} v_x = 1 \\ v_y = t \end{cases} \qquad \vec{v} = v = \vec{\imath} + t\vec{\jmath} \Rightarrow |\vec{v}| = \sqrt{1+t^2}$$

- Acceleration

$$\begin{cases} a_x = \frac{dv_x}{dt} \\ a_y = \frac{dv_y}{dt} \end{cases} \Rightarrow \begin{cases} a_x = 0 \\ a_y = 1 \end{cases} \quad \vec{a} = 1\vec{\jmath} \Rightarrow |\vec{a}| = a = 1$$

3- The nature of the movement

$\vec{a}.\vec{v} = t > 0$ so The movement in this case is uniformly accelerated.

Normal and tangential accelerations.

- Tangential acceleration

$$a_T = \frac{d|\vec{v}|}{dt} = \frac{d\left(\sqrt{1+t^2}\right)}{dt} = \frac{t}{\sqrt{t^2+1}}$$

- Normal acceleration

  We have $a^2 = a_T^2 + a_N^2$ so $a_N^2 = a^2 - a_T^2$

$$a_N^2 = 1 - \frac{t^2}{t^2+1} \Rightarrow a_N^2 = \frac{1}{v^2} \Rightarrow a_N = \frac{1}{v}$$

4- The radius of curvature

$$a_N = \frac{v^2}{R} \Rightarrow R = \frac{v^2}{a_N} = \frac{v^3}{1} = v^3$$

Exercise 11

A particle moves along a trajectory with the equation $y = x^2$ in such a way that at each moment $v_x = v_0$=const. If at t=0, x_0, y_0=0.

a- Let's find the coordinates x(t) and y(t) of the particle.

We have the following (Ox):

$$v_x = v_0 = \frac{dx}{dt} \Rightarrow \int_0^x dx = \int_0^t v_0 dt$$

$$\Rightarrow \boldsymbol{x(t)} = v_0 t$$

On the other hand: $y = x^2 \Rightarrow y(t) = v_0^2 t^2$

$$\text{So } \begin{cases} \boldsymbol{x(t)} = v_0 t \\ y(t) = v_0^2 t^2 \end{cases}$$

5- The components of velocity and acceleration,

- Velocity

$$\begin{cases} v_x = \dfrac{dx}{dt} = v_0 \\ v_y = \dfrac{dy}{dt} = 2v_0^2 t \end{cases} \Rightarrow \overrightarrow{v(t)} = v_0 \vec{\imath} + 2v_0^2 t \vec{\jmath}$$

The velocity module $\left|\overrightarrow{v(t)}\right| = \sqrt{v_0^2 + 4v_0^4 t^2}$

- The acceleration

$$\begin{cases} a_x = \dfrac{dv_x}{dt} = 0 \\ a_y = \dfrac{dv_y}{dt} = 2v_0^2 \end{cases} \Rightarrow \overrightarrow{v(t)} = 2v_0^2 \vec{\jmath}$$

The acceleration module $\left|\overrightarrow{a(t)}\right| = 2v_0^2$

b- Normal and tangential acceleration

- The tangential acceleration

$$a_T = \frac{d\left|\overrightarrow{v(t)}\right|}{dt} = \frac{4v_0^4 t}{\sqrt{v_0^2 + 4v_0^4 t^2}}$$

- The normal acceleration

$$a_N^2 = a^2 - a_T^2 \Rightarrow a_N^2 = 4v_0^4 - \frac{16v_0^8 t^2}{v_0^2 + 4v_0^4 t^2}$$

$$\Rightarrow a_N^2 = \frac{4v_0^6}{v_0^2 + 4v_0^4 t^2}$$

So $a_N = \frac{2v_0^3}{\sqrt{v_0^2 + 4v_0^4 t^2}} = \frac{2v_0^3}{v}$

- The radius of curvature

$$a_N = \frac{v^2}{R} = \frac{2v_0^3}{v} \Rightarrow R = \frac{v^3}{2v_0^3}$$

Exercise 12

- ***1st case : a=-kv***

K=constant, at t=0, $v=v_0$ et $x=x_0$

We have the acceleration, seeking the velocity as a function of time.

$$a = \frac{dv}{dt} = -kv \Rightarrow \frac{dv}{v} = -kdt$$

So $\int_{v_0}^{v} \frac{dv}{v} = \int_0^t -kdt \Rightarrow lnv - lnv_0 = -kt$

$$\Rightarrow \quad ln\frac{v}{v_0} = -kt$$

Then $\frac{v}{v_0} = e^{-kt}$ and $v(t)=v_0 e^{-kt}$

We have found the velocity, now seeking the position as a function of time.

$$v = \frac{dx}{dt} = v_0 e^{-kt} \Rightarrow dx = v_0 e^{-kt}\, dt$$

So $\int_{x_0}^{x} dx = \int_0^t v_0 e^{-kt}\, dt \Rightarrow x - x_0 = \frac{v_0}{k}(1 - e^{-kt})$

Then $x(t) = x_0 + \frac{v_0}{k}(1 - e^{-kt})$

We need to find the relationship between velocity and position.

We have $\frac{v}{v_0} = e^{-kt}$ and $x = x_0 + \frac{v_0}{k}(1 - e^{-kt})$

So $x = x_0 + \frac{v_0}{k}\left(1 - \frac{v}{v_0}\right) = x_0 + \frac{v_0}{k}\left(\frac{v_0 - v}{v_0}\right)$

$$x = x_0 + \frac{v_0 - v}{k} \Rightarrow v_0 - v = k(x - x_0)$$

Then $v = k(x - x_0) + v_0$

- ***2nd case : a=-kv²***

Like the first case, we first seek the velocity as a function of time.

$$a = \frac{dv}{dt} = -kv^2 \Rightarrow \frac{dv}{v^2} = -kdt$$

So $\int_{v_0}^{v} \frac{dv}{v^2} = \int_0^t -kdt \Rightarrow \frac{-1}{v} + \frac{1}{v_0} = -kt \quad so \quad \frac{1}{v} = kt + \frac{1}{v_0}$

Then $\frac{1}{v} = \frac{ktv_0 + 1}{v_0}$ (*)

The velocity is written as: $v(t) = \frac{v_0}{ktv_0 + 1}$

We have found the velocity, now seeking the position as a function of time.

$v = \frac{dx}{dt} = \frac{v_0}{ktv_0+1}$ so $dx = \frac{v_0}{ktv_0+1}dt$

$$\int_{x_0}^{x} dx = \int_0^t \frac{v_0}{ktv_0 + 1} dt \Rightarrow x - x_0 = \frac{1}{k}\ln(ktv_0 + 1)$$

The position as a function of time is given by $x(t) = x_0 + \frac{1}{k}\ln(ktv_0 + 1)$

The relationship between velocity and position.

$(*) \Rightarrow ktv_0 + 1 = \frac{v_0}{v}$ $donc\ x = x_0 + \frac{1}{k}\ln\frac{v_0}{v}$

$\Rightarrow ln\frac{v_0}{v} = k(x - x_0) \Rightarrow \frac{v_0}{v} = e^{k(x-x_0)}$ so $v = v_0 e^{k(x_0-x)}$

Exercise 13

Let's find the equation of the trajectory of a body whose motion is defined by

$v_x=1$ and $v_y=2/(t+1)$

At (t=0) x=0 and y=2

$$\begin{cases} v_x = 1 \Rightarrow \frac{dx}{dt} = 1 \\ v_y = \frac{2}{t+1} \Rightarrow \frac{dy}{dt} = \frac{2}{t+1} \end{cases} \text{so} \begin{cases} dx = dt \Rightarrow x = t \\ dy = \frac{2}{t+1}dt \Rightarrow \int_2^y dy = \int_0^t \frac{2}{t+1}dt \ (*) \end{cases}$$

$(*) \Rightarrow y - 2 = 2\ln(t + 1)$

By replacing t by x we have the equation of the trajectory of the form:

Y=2+2.ln(x+1)

a- The components of acceleration

b- $\begin{cases} a_x = \frac{dv_x}{dt} = \frac{d(1)}{dt} = 0 \\ a_y = \frac{dv_y}{dt} = \frac{d(\frac{2}{t+1})}{dt} = \frac{-2}{(t+1)^2} \end{cases}$

So $\overrightarrow{a(t)} = \frac{-2}{(t+1)^2}\vec{j}$

Its modulus: $|\vec{a}| = \frac{2}{(t+1)^2}$

Exercise 14

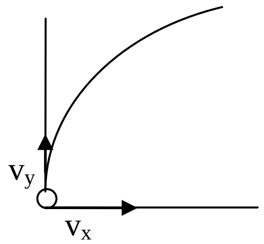
A balloon rises with a horizontal velocity v_0 (v_y=v_0) and the wind gives it a horizontal velocity (v_x =ay)

a- Let's determine the equations x(t) and y(t), with v_x=ay and v_y=v_0

We take at t=0 x=0 and y=2

We have v_y=$v_0 \Rightarrow \frac{dy}{dt} = v_0$

$\Rightarrow dy = v_0 dt \quad donc\ \int_0^y dy = \int_0^t v_0 dt$



$\Rightarrow y = v_0 t$

on the other hand

$v_x = ay = av_0 t \Rightarrow \frac{dx}{dt} = av_0 t$

$\Rightarrow dx = \ av_0 t dt \quad donc\ \int_0^x dx = \int_0^t av_0 t dt \Rightarrow x = av_0 \frac{t^2}{2}$

So $\ y = v_O t \Rightarrow t = \frac{y}{v_0}$

$$\Rightarrow \ x = av_0 \frac{\left(\frac{y}{v_0}\right)^2}{2} = \frac{a}{2v_0} y^2$$

The equation of the trajectory y=f(x) is of the form: $\boldsymbol{y} = \sqrt{\frac{\boldsymbol{2xv_0}}{\boldsymbol{a}}}$

b- Normal and tangential accelerations:

$$\vec{v} = v_0 \vec{\imath} + ay\vec{\jmath} = v_0 \vec{\imath} + av_0 t\vec{\jmath}$$

The acceleration is of the form: $\vec{a} = \frac{d\vec{v}}{dt} = av_0 \vec{\jmath}$

And $\ |\vec{v}| = \sqrt{v_0^2 + a^2 v_0^2 t^2}$

The tangential acceleration

$a_T = \frac{d|\vec{v}|}{dt} = \frac{d\sqrt{v_0^2 + a^2 v_0^2 t^2}}{dt} \ \Rightarrow a_T = \frac{2a^2 v_0^2 t}{2\sqrt{v_0^2 + a^2 v_0^2 t^2}}$

So the tangential acceleration is written $\ \boldsymbol{a_T} = \frac{\boldsymbol{a^2 v_0^2 t}}{\boldsymbol{v}}$

The normal acceleration

$a^2 = a_T^2 + a_N^2 \Rightarrow a_N^2 = a^2 - a_T^2 \Rightarrow a_N^2 = a^2 v_0^2 - \frac{a^4 v_0^4 t}{v_0^2 + a^2 v_0^2 t^2} \ \Rightarrow a_N^2 = \frac{a^2 v_0^4}{v^2}$

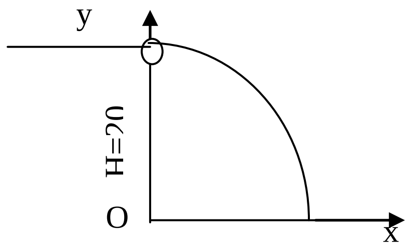
So the normal acceleration is written $\ \boldsymbol{a_N} = \frac{\boldsymbol{a\, v_0^2}}{\boldsymbol{v}}$

Radius of curvature :

$$a_N = \frac{v^2}{R} = \frac{a\ v_0^2}{v} \Rightarrow \boldsymbol{R} = \frac{\boldsymbol{v^3}}{\boldsymbol{av_0^2}}$$

Exercise 15

a-A stone is thrown from the top of a building 20 meters high with a velocity v_x=10 m/s



Calculating the time taken by the stone to reach the ground:

Considering oy direction, the acceleration is gravity g.

So a_y = -g (because the motion is in the opposite direction of oy, and a_x = 0 since the velocity v_x in the direction of Ox is constant).

$$a_y = \frac{dv_y}{dt} = -g \Rightarrow dv_y = -gdt$$

So $v_y = -gt$

At t=0, the stone has zero velocity in the y direction.

$$v_y = \frac{dy}{dt} = -gt \Rightarrow dy = -gtdt$$

So $y = \frac{-1}{2}gt^2 + y_0$

At t=0, the stone is at the top of the building, so y_0=H=20.

When the stone reaches the ground, its y component will be zero, so y=0.

Therefore, we are looking for the time at which y=0.

$Y=-0{,}5\ g\ t^2+20 = 0 \Rightarrow gt^2=40$

We take g=10m/s, so the time taken by the stone to reach the ground is t=2s.

b- Along (Ox), $v_x = \frac{dx}{dt} = v_0 \Rightarrow dx = v_0 dt$

At t=0, x=0, so $x=v_0t=10t$

The stone reaches the ground at a distance of 20 m.

c- Along (oy), v_y=-g.t=-20m/s, and along (ox), v_x=10m/s,

So $\vec{v} = -20\vec{\imath} + 10\vec{\jmath} \Rightarrow |\vec{v}| = \sqrt{400 + 100} = \sqrt{500}$ m/s

Exercise 16

$$\begin{cases} v_x = R\omega\cos(\omega t) \\ v_y = R\omega\sin(\omega t) \end{cases}$$

Knowing that at t=0, the moving body is at the origin O (0,0),

1. The components of the acceleration vector and its modulus

$$\cdot\begin{cases} a_x = \frac{dv_x}{dt} = -R\omega^2 sin\,(\omega t) \\ a_y = \frac{dv_y}{dt} = R\omega^2 \cos(\omega t) \end{cases}$$

$$[\vec{a}] = \sqrt{(-R\omega^2 sin\,(\omega t))^2 + (R\omega^2 cos\,(\omega t))^2} = R\omega^2$$

2. The tangential and normal components of acceleration and deduce the radius of curvature.

- Tangential acceleration:

$$[\vec{v}] = \sqrt{(R\omega cos(\omega t))^2 + (R\omega sin\,(\omega t))^2} = R\omega$$

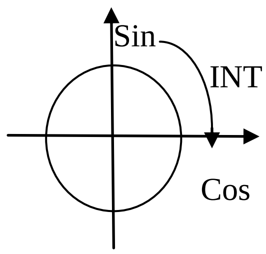
$$a_T = \frac{dv}{dt} = \frac{dR\omega}{dt} \Rightarrow a_T = 0$$

- Normal acceleration:

$a_N = \frac{v^2}{R} = a = \; R\omega^2 \quad and \quad a_T = 0$ so $R = \frac{v^2}{a_N} = \frac{R^2\omega^2}{R\omega^2}$=R

Radius of curvature is R.

3. The components of the position vector



$$\begin{cases} v_x = R\omega\cos(\omega t) \\ v_y = R\omega\sin(\omega t) \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{dt} = R\omega\cos(\omega t) \\ \dfrac{dy}{dt} = R\omega\sin(\omega t) \end{cases}$$

$$\Rightarrow \begin{cases} dx = R\omega\cos(\omega t)dt \\ dy = R\omega\sin\,(\omega t)\,dt \end{cases}$$

$$\Rightarrow \begin{cases} \displaystyle\int dx = R\int \omega\cos(\omega t)dt \\ \displaystyle\int dy = R\int \omega\sin\,(\omega t)\,dt \end{cases}$$

$$\Rightarrow \begin{cases} x = R\sin(\omega t) \\ y = -R\cos\,(\omega t) \end{cases}$$

The trajectory equation.

$$x^2 + y^2 = R^2 sin^2\omega t + R^2 cos^2 \omega t \Rightarrow x^2 + y^2 = R^2$$

4. The nature of movement

The acceleration a=a_N and the equation of the trajectory is $x^2 + y^2 = R^2$, so the motion is uniformly circular.

Exercise 17

A material point M moves along the OX axis with acceleration $\vec{a} = a\vec{\imath}$ with a > 0.

1- Determine the velocity vector knowing that v (t=0)= v_0 .

$$a = \frac{dv}{dt} \Rightarrow \int_{v_0}^{v} dv = a\int_0^t d\,t$$

$\Rightarrow \boldsymbol{v - v_0 = at}(1)$

So $\vec{\boldsymbol{v}} = (\boldsymbol{at + v_0})\vec{\boldsymbol{\imath}}$

2- The position vector$\overrightarrow{OM}$ knowing that x(t=0)=x_0 .

$$v = \frac{dx}{dt} = a\,t + v_0 \quad \Rightarrow \int_{x_0}^{x} dx = \int_0^t (a\,t + v_0)d\,t = a\int_0^t td\,t \; + v_0\int_0^t d\,t$$

$$\Rightarrow x - x_0 = \left[a\frac{t^2}{2} + v_0 t\right]_0^t$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \text{and} \int \frac{dx}{x} = lnx$$

$\boldsymbol{x = \frac{1}{2}at^2 + v_0 t + x_0}(2)$

$$\Rightarrow \overrightarrow{\boldsymbol{OM}} = \left(\frac{\boldsymbol{1}}{\boldsymbol{2}}\,\boldsymbol{a\,t^2 + v_0 t + x_0}\right)\vec{\boldsymbol{\imath}}$$

3. Show that $v^2 - v_0^2 = 2a(x - x_0)$

(1)$\Rightarrow t = \frac{v - v_0}{a}$ in (2) $x - x_0 = \frac{1}{2}a\left(\frac{v-v_0}{a}\right)^2 + v_0\left(\frac{v-v_0}{a}\right) = \frac{v^2+v_0^2-2vv_0}{2a} + \frac{vv_0-v_0^2}{a}$

$$\Rightarrow x - x_0 = \frac{v^2 + v_0^2 - 2vv_0}{2a} + \frac{2vv_0 - 2v_0^2}{2a}$$

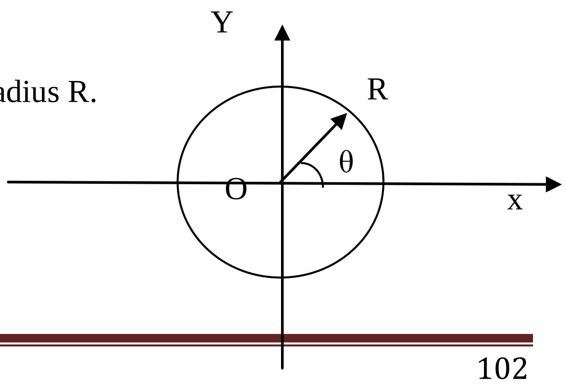
$$\Rightarrow x - x_0 = \frac{v^2 - v_0^2}{2a}$$

so $2a(x - x_0) = v^2 - v_0^2$

4- For motion to be uniformly accelerated, $\vec{a}.\vec{v}$ must be positive.

For motion to be uniformly retarded, $\vec{a}.\vec{v}$ must be negative.

Exercise 18



a- Point M describes a circle with center O and radius R.

The coordinates of point M are:

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases}$$

b- The velocity at point M

$$\begin{cases} v_x = \dfrac{dx}{dt} = -R\dfrac{d\theta}{dt}\sin\theta = -R\theta^{\cdot}\sin\theta \\ v_y = R\dfrac{d\theta}{dt}\cos\theta = R\theta^{\cdot}\cos\theta \end{cases}$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{R^2\theta^{\cdot 2}(-sin^2)\theta + R^2\theta^{\cdot 2}(cos^2)\theta}$$

$$\Rightarrow v = R\theta^{\cdot}$$

c- The components of acceleration

$$\begin{cases} a_x = \dfrac{dv_x}{dt} = -R\theta^{\cdot\cdot}\sin\theta - R\theta^{\cdot 2}\cos\theta \\ a_y = \dfrac{dv_y}{dt} = R\theta^{\cdot\cdot}\cos\theta - R\theta^{\cdot 2}\sin\theta \end{cases}$$

$$a = \sqrt{R^2\theta^{\cdot\cdot 2} + R^2\theta^{\cdot 4}}$$

The tangential acceleration:

$$a_T = \frac{dv}{dt} = \frac{dR\theta^{\cdot}}{dt} \Rightarrow a_T = R\theta^{\cdot}$$

The normal acceleration:

$$a_N = \frac{v^2}{R} = R\theta^{\cdot 2}$$

d- We have $\alpha = \frac{d\omega}{dt} \Rightarrow d\omega = \alpha dt \ \ so \ \ \omega - \omega_0 = \alpha t \Rightarrow \boldsymbol{\omega = \omega_0 + \alpha t}$ **(*)**

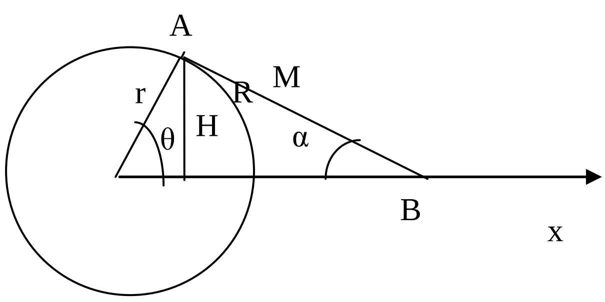
At (t=0), θ_0=0 and $\omega=\omega_0$ $with\ \omega = \frac{d\theta}{dt} = \omega_0 + \alpha t \Rightarrow d\theta = \omega_0 dt + \alpha t dt$

$So\ \int_0^{\theta} d\theta = \int_0^t \omega_0 dt + \int_0^t \alpha t dt$ Then $\theta = \alpha\frac{t^2}{2} + \omega_0 t$

(*)$\Rightarrow t = \frac{\omega-\omega_0}{\alpha} \Rightarrow \ \theta = \frac{\alpha}{2}\left(\frac{\omega-\omega_0}{\alpha}\right)^2 + \omega_0\left(\frac{\omega-\omega_0}{\alpha}\right) = \frac{\omega^2-\omega_0^2}{2\alpha}$

$\Rightarrow\ \boldsymbol{\theta = \frac{\omega^2-\omega_0^2}{2\alpha}}$

Exercise 19



a- The time equations of point M (the middle of AB).

With OA=r and OA=R

Along the Ox axis :

$$x_M = r\,cos\theta + \frac{R}{2}\,cos\alpha$$

$$AH = r\,sin\,\alpha = R\,sin\,\theta \Rightarrow \sin\alpha = \frac{r\,sin\theta}{R}$$

$$cos^2\alpha + sin^2\alpha = 1 \Rightarrow cos^2\alpha = 1 - sin^2\alpha = 1 - \frac{r^2}{R^2}sin^2\theta$$

$$So\quad cos\alpha = \sqrt{1 - \frac{r^2}{R^2}sin^2\theta}$$

$$\theta = \omega t\ \ then\ \ x_M = r\,cos\omega t + \frac{R}{2}\sqrt{1 - \frac{r^2}{R^2}sin^2\omega t}$$

$$y_M = \frac{R}{2}sin\alpha = \frac{R}{2}\frac{r\,sin\theta}{R} = \frac{r}{2}sin\omega t \qquad \text{so} \quad y_M = \frac{r}{2}sin\omega t$$

$$\begin{cases} x_M = r\,cos\omega t + \dfrac{R}{2}\sqrt{1 - \dfrac{r^2}{R^2}sin^2\omega t} \\ y_M = \dfrac{r}{2}sin\omega t \end{cases}$$

b- The abscissa of B:

$$x_B = r\,cos\theta + R\,cos\alpha = r\,cos\omega t + R\sqrt{1 - \frac{r^2}{R^2}sin^2\omega t}$$

According to the equation of motion, point B does not have sinusoidal motion

c- The velocity at point M

$$v_M = \frac{d\overrightarrow{oM}}{dt} \Rightarrow \begin{cases} v_x = \dfrac{dx}{dt} = -r(\omega sin\omega t) - R\left(\dfrac{r^2\omega sin\omega t cos\omega t}{\sqrt{R^2 - r^2 sin^2\omega t}}\right) \\ v_y = \dfrac{dy}{dt} = \dfrac{r\omega}{2}cos\omega t \end{cases}$$

d- If r=R Showing that the movement of B is sinusoidal

$$\begin{cases} x_B = rcos\omega t + r\sqrt{1 - \dfrac{r^2}{r^2}sin^2\omega t} = 2rcos\omega t \\ y_B = 0 \end{cases}$$

$$\Rightarrow x_B = 2rcos\omega t$$

So the movement of B in this case is sinusoidal.

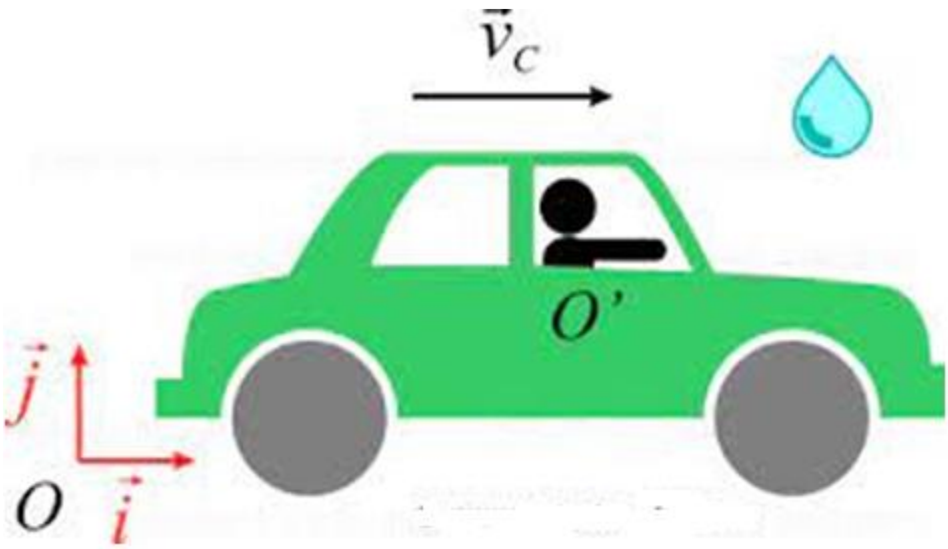
University of Tlemcen
Faculty of Sciences
Department of Mathematics

U.Y: 2023/2024

COURSE OF MECHANICS

OF THE MATERIAL POINT

Chapter IV: Relative motion



Glossary

In English	In French	In Arabic
Relative motion	Mouvement relative	حركة نسبية
Absolute reference frame	Un référentiel absolu	مرجع مطلق
Relative reference frame	Un référentiel relatif	مرجع نسبي
Absolute motion	Mouvement absolu	حركة مطلقة
Entrainment motion	Mouvement d'entrainement	حركة المعلم المتحرك بالنسبة للمعلم الثابت
Velocity composition	Composition de la vitesse	تركيب السرعة
Fixed frame of reference	Un référentiel fixe	المعلم الثابت
Moving frame of reference	Un référentiel mobile	المعلم المتحرك
Absolute velocity	Vitesse absolue	السرعة المطلقة
Relative velocity	Vitesse relative	السرعة النسبية
Entrainment velocity	Vitesse d'entrainement	سرعة المعلم المتحرك بالنسبة للمعلم الثابت
Composition of acceleration	Composition des accelerations	تركب التسارع
Absolute acceleration	Acceleration absolue	التسارع المطلق
Relative acceleration	Acceleration relative	التسارع النسبي
Entrainment acceleration	Acceleration d'entrainement	التسارع المعلم المتحرك بالنسبة للمعلم الثابت
Coriolis acceleration	Acceleration de Coriolis	التسارع Coriolis

1. Introduction

State of motion or state of rest are two essentially relative notions, meaning that each of the two states depends on the position of the moving body relative to the body taken as reference frame. All the motions we have studied so far have been in a Galilean frame of reference, i.e. at rest or in uniform rectilinear motion. When two observers linked to two different reference frames are in motion relative to each other, the position, trajectory, velocity and acceleration of the same moving body vary according to the reference frame chosen by the observer.

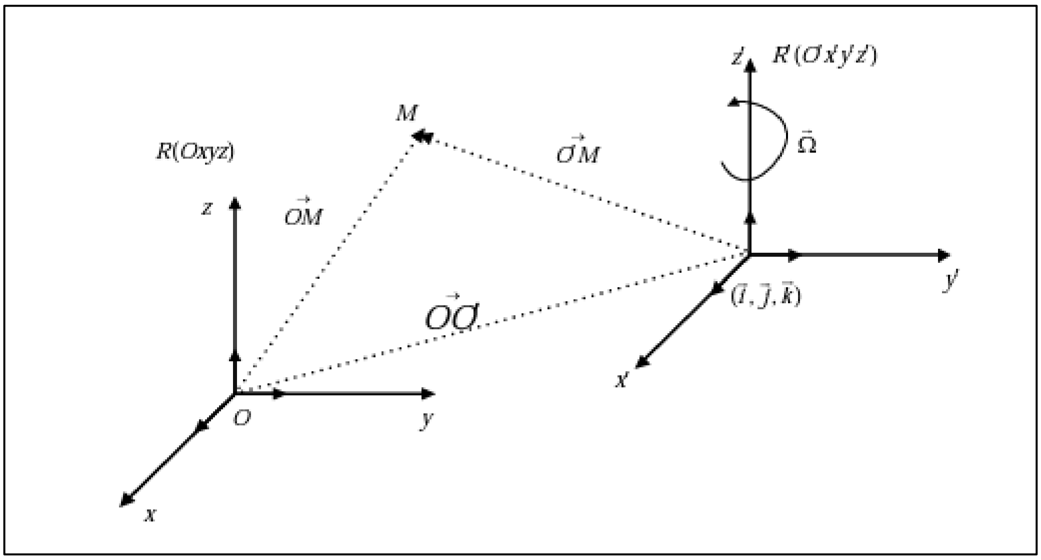
A bus passenger, for example, is in motion relative to an observer seated on the side of the road, whereas he is at rest relative to another observer (a passenger lending the same bus). Clearly, then, the notion of motion or rest is intimately linked to the position of the observer.

To say observer is to say to choose a frame of reference to determine the position, velocity and acceleration of a moving object at each instant.

2. Composition of movements مركبات الحركة

Let R (O, x, y, z) be a fixed or absolute (Galilean) frame of reference and R' (O', x', y', z') a moving or relative (non-Galilean) frame of reference relative to (R).

It's always useful to know how to determine the position, velocity and acceleration of a material point M in a fixed reference frame if they are known in the other relative reference frame and vice-versa.



Let's associate to the reference frame R (called **absolute reference frame**) the reference frame R($O,\vec{i}, \vec{j},\vec{k}$) and to the reference frame R' (called **relative reference frame**) the reference frame R' ($O',\vec{i'}, \vec{j'},\vec{k'}$).

If M is a movable point in space, defined by coordinates (x, y, z) in the fixed reference frame (R) and by (x', y', z') in the movable reference frame (R').

We will call:

Relative motion الحركة النسبية : the motion of M relative to (R').

Absolute motion الحركة المطلقة : the motion of M relative to (R).

Entrainment motion: the motion of the moving frame of reference (R') relative to the fixed frame of reference (R).

2.1. Velocity composition قانون تركيب السرعات

If we know the relative motion, i.e. the motion of the material point considered in the moving frame of reference (relative frame of reference المعلم المتحرك) and that of the moving frame of reference relative to the fixed frame of reference (fixed frame of reference المعلم الثابت).

We have: $\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M} = \overrightarrow{OO'} + \left(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}\right)$

With: $\overrightarrow{OM} = \left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right)$in the fixed reference frame R.

$and \quad \overrightarrow{O'M} = \left(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}\right)$in the moving reference frame R'.

The velocity is then: $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$

$$\frac{d\overrightarrow{OM}}{dt} = \frac{d\overrightarrow{OO'}}{dt} + \frac{d\overrightarrow{O'M}}{dt}$$

$$= \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'} + \frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt} = \vec{v_a} = \vec{v_r} + \vec{v_e}$$

$$\text{with} \begin{cases} \vec{v_a} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k} = \overrightarrow{v(M)}/(R) \\ \vec{v_r} = \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'} = \frac{d\overrightarrow{O'M}}{dt}/(R') = \overrightarrow{v(M)}/(R') \\ \vec{v_e} = \frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt} = \overrightarrow{v(R')}/(R) \end{cases}$$

$\boldsymbol{\vec{v_a}}$ represents absolute velocityالسرعة المطلقة , the derivative of $\overrightarrow{OM}$ with respect to time in the fixed reference frame.

$\boldsymbol{\vec{v_r}}$ is the relative velocity السرعة النسبية, i.e. the derivative of $\overrightarrow{O'M}$ with respect to time in the moving reference frame.

$\overrightarrow{\boldsymbol{v_e}}$ is entrainment velocity سرعة المعلم المتحرك بالنسبة للمعلم الثابت, this is the derivative of $\overrightarrow{OM}$ with respect to time in the fixed reference frame, considering the moving point M fixed in the moving reference frame (x',y' and z' are constant).

It also represents the velocity of the moving frame of reference relative to the fixed frame.

The law of velocity composition is given by:

$$\overrightarrow{\boldsymbol{v_a}} = \overrightarrow{\boldsymbol{v_r}} + \overrightarrow{\boldsymbol{v_e}}$$

<u>Note</u> :

1. When the entrainment motion is translational, the vectors $(\vec{\imath'}, \vec{\jmath'} and \vec{k'})$ remain parallel to the unit vectors $(\vec{\imath}, \vec{\jmath}\ and\ \vec{k})$ of the fixed reference frame.

So $\frac{d\vec{\imath'}}{dt} = \frac{d\vec{\jmath'}}{dt} = \frac{d\vec{k'}}{dt} = \vec{0}$ and $\vec{v_e} = \left(\frac{d\overrightarrow{OO'}}{dt}\right)/R$

2. If the moving frame of reference (R') rotates relative to the fixed frame of reference (R), the entrainment velocity can also be written as:

$$\vec{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$$

With $\vec{\omega}$ represents the rotational velocity السرعة الزاوية of R'/R.

3. In the case of translational motion $\vec{\omega} = \vec{0}$ then $\vec{v_e} = \frac{d\overrightarrow{OO'}}{dt}$.

2.2. Composition of acceleration قانون تركيب التسارعات

Acceleration is the derivative of velocity with respect to time :

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

$$\vec{v} = \frac{d\overrightarrow{OO'}}{dt} + \frac{\boldsymbol{dx'}}{\boldsymbol{dt}}\vec{\boldsymbol{\imath'}} + \frac{\boldsymbol{dy'}}{\boldsymbol{dt}}\vec{\boldsymbol{\jmath'}} + \frac{\boldsymbol{dz'}}{\boldsymbol{dt}}\vec{\boldsymbol{k'}} + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}$$

$$\frac{d}{dt}\left(\frac{dx'}{dt}\vec{\imath'}\right) = \frac{d^2x'}{dt^2}\vec{\imath'} + \frac{dx'}{dt}\frac{d\vec{\imath'}}{dt}$$

$$\frac{d}{dt}\left(\frac{dy'}{dt}\vec{\jmath'}\right) = \frac{d^2y'}{dt^2}\vec{\jmath'} + \frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt}$$

$$\frac{d}{dt}\left(\frac{dz'}{dt}\vec{k'}\right) = \frac{d^2z'}{dt^2}\vec{k'} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}$$

and

$$\frac{d}{dt}\left(x'\frac{d\vec{\imath'}}{dt}\right) = \frac{dx'}{dt}\frac{d\vec{\imath'}}{dt} + x'\frac{d^2\vec{\imath'}}{dt^2}$$

$$\frac{d}{dt}\left(y'\frac{d\overrightarrow{j'}}{dt}\right)=\frac{dy'}{dt}\frac{d\overrightarrow{j'}}{dt}+y'\frac{d^2\overrightarrow{j'}}{dt^2}$$

$$\frac{d}{dt}\left(z'\frac{d\overrightarrow{k'}}{dt}\right)=\frac{dz'}{dt}\frac{d\overrightarrow{k'}}{dt}+z'\frac{d^2\overrightarrow{k'}}{dt^2}$$

$$\vec{a}=\frac{d^2x'}{dt^2}\overrightarrow{i'}+\frac{d^2y'}{dt^2}\overrightarrow{j'}+\frac{d^2z'}{dt^2}\overrightarrow{k'}+\frac{d^2\overrightarrow{OO'}}{dt^2}+x'\frac{d^2\overrightarrow{i'}}{dt^2}+y'\frac{d^2\overrightarrow{j'}}{dt^2}+z'\frac{d^2\overrightarrow{k'}}{dt^2}+2(\frac{dx'}{dt}\frac{d\overrightarrow{i'}}{dt}+\frac{dy'}{dt}\frac{d\overrightarrow{j'}}{dt}+\frac{dz'}{dt}\frac{d\overrightarrow{k'}}{dt})$$

with

$$\overrightarrow{a_a}=\frac{d^2\overrightarrow{OM}}{dt^2}=\frac{d^2x}{dt^2}\vec{i}+\frac{d^2y}{dt^2}\vec{j}+\frac{d^2z}{dt^2}\vec{k}$$

$$\overrightarrow{a_r}=\frac{d^2\overrightarrow{O'M}}{dt^2}=\frac{d^2x'}{dt^2}\overrightarrow{i'}+\frac{d^2y'}{dt^2}\overrightarrow{j'}+\frac{d^2z'}{dt^2}\overrightarrow{k'}$$

$$\overrightarrow{a_e}=\frac{d^2\overrightarrow{OO'}}{dt^2}+x'\frac{d^2\overrightarrow{i'}}{dt^2}+y'\frac{d^2\overrightarrow{j'}}{dt^2}+z'\frac{d^2\overrightarrow{k'}}{dt^2}$$

$$\Rightarrow\overrightarrow{a_e}=\frac{d^2\overrightarrow{OO'}}{dt^2}+\vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right)+\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$$

$$\overrightarrow{a_c}=\ 2\left(\frac{dx'}{dt}\frac{d\overrightarrow{i'}}{dt}+\frac{dy'}{dt}\frac{d\overrightarrow{j'}}{dt}+\frac{dz'}{dt}\frac{d\overrightarrow{k'}}{dt}\right)$$

$$\Rightarrow\overrightarrow{a_c}=2(\vec{\omega}\Lambda\overrightarrow{v_r})/\mathrm{R}'$$

Then the law of composition of the acceleration will be :

$$\boldsymbol{\overrightarrow{a_a}=\overrightarrow{a_r}+\overrightarrow{a_c}+\overrightarrow{a_e}}$$

$\boldsymbol{\overrightarrow{a_a}}$ is the absolute acceleration التسلرع المطلق representing the second derivative of $\overrightarrow{OM}$ with respect to time in the fixed reference frame. This is the acceleration of M in the fixed frame of reference.

$\boldsymbol{\overrightarrow{a_r}}$ is the relative acceleration التسارع النسبي representing the second derivative of $\overrightarrow{O'M}$ with respect to time in the moving frame of reference. This is the acceleration of M in the moving frame of reference.

$\boldsymbol{\overrightarrow{a_e}}$ is the entrainment acceleration تسارع المعلم المتحرك بالنسبة للمعلم الثابت, which represents the acceleration of the motion of the moving frame relative to the fixed frame. This is the acceleration of the moving frame R' relative to the fixed frame R.

$\boldsymbol{\overrightarrow{a_c}}$ is the Coriolis or complementary acceleration (it has no physical meaning).

Note:

If the moving frame of reference (R') rotates relative to the fixed frame of reference (R), the drive speed can also be written as:

$\overrightarrow{v_e} = \left(\frac{d\overrightarrow{OO'}}{dt}\right)/R + (\vec{\omega}\Lambda\overrightarrow{O'M})/R'$,

Entrainment acceleration by:

$\overrightarrow{a_e} = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)/R + (\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M})/R' + \left(\vec{\omega}\Lambda\vec{\omega}\Lambda\overrightarrow{O'M}\right)/\text{R'}$

and the Coriolis acceleration by:

$$\overrightarrow{a_c} = 2.(\vec{\omega}\Lambda\overrightarrow{v_r})/R'$$

With $\overrightarrow{\omega}$ represents the rotational velocity of R'/R.

Particular cases:

1. When the training motion is translational, the vectors $\overrightarrow{\imath'}, \overrightarrow{\jmath'} and\ \overrightarrow{k'}$ remain parallel to the unit vectors ($\vec{\imath}, \vec{\jmath}$ and $\vec{k}$) of the fixed reference frame.

So $\frac{d\overrightarrow{\imath'}}{dt} = \frac{d\overrightarrow{\jmath'}}{dt} = \frac{d\overrightarrow{k'}}{dt} = \vec{0}$ and $\boldsymbol{\vec{\omega}} = \boldsymbol{\vec{0}}$

Then $\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt}, \overrightarrow{a_e} = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)$ and $\overrightarrow{a_c} = \vec{0}$.

2. If R' has a uniform rectilinear motion then $\boldsymbol{\vec{\omega}} = \boldsymbol{\vec{0}},\ \overrightarrow{a_c} = \vec{0}$, $\frac{d\overrightarrow{OO'}}{dt} = cst$ and

$\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$ So $\overrightarrow{a_e} = \vec{0}$ because $\overrightarrow{v_e} = cst$.

3. If R' has a pure rotation about R (R' and R have the same origin), so we have:

$\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$ and $\frac{d\overrightarrow{OO'}}{dt} = \vec{0}$ then $\overrightarrow{v_e} = (\vec{\omega}\Lambda\overrightarrow{O'M})/R'$

And $\overrightarrow{a_e} = (\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M})/R' + \left(\vec{\omega}\Lambda\vec{\omega}\Lambda\overrightarrow{O'M}\right)/\text{R'}$

Proposed exercises about chapter IV

Exercise 1

A moving train of a speed v passes through a station. At the moment t=0 a M bulb is detached from the ceiling of one of its compartments. The movement of M is then observed by a passenger of the train and by the station chief motionless on the platform.

Describe the movement of M for each observer.

Exercise 2

A man mounting a horse galloping at a constant speed v, launches an arrow into the air with speed v_0 relative to the horse.

At what angle θ should v_0 make with the vertical for the arrow to fall back onto the man?

(air resistance will be neglected).

Exercise 3

The coordinates of a moving particle in the reference frame (R) provided with the reference frame $\left(O,\vec{\imath},\vec{\jmath},\vec{k}\right)$ are given as a function of time by:

$$x = 2t^3 + 1, \qquad y = 4t^2 + t - 1, \qquad z = t^2$$

In a second frame of reference (R') with the reference frame$\left(O',\vec{\imath'},\vec{\jmath'},\vec{k'}\right)$ with $\vec{\imath} = \vec{\imath'}, \vec{\jmath} = \vec{\jmath'}, \vec{k} = \vec{k'}$ the coordinates of a moving particle are given by:

$$x' = 2t^3, \qquad y' = 4t^2 - 3t + 2, \qquad z' = t^2 - 5$$

1- Calculate the express the velocity v of m in (R) as a function of its velocity v' in (R'), and proceed in the same way for the accelerations.
2- Define the entrainment motion of (R') relative to (R).

Exercise 4

In the (Oxy) plane, consider a system of moving axes (OXY) with the **same origin O**, rotating with a constant angular velocity ω around (OZ). A moving point M moves along axis

(OX) with **constant acceleration γ** and no initial velocity. We call relative motion of M its motion with respect to (OXY), and absolute motion with respect to (Oxy).

At time t=0, axes (Ox) and (OX) are coincident and M is in OA.

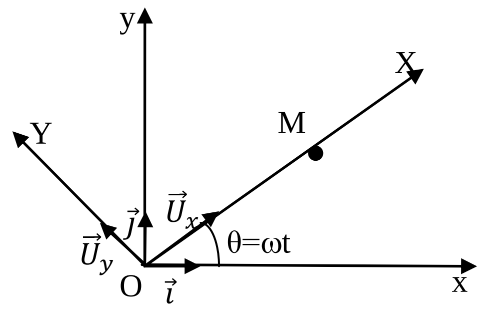
Calculate in the moving reference frame :

1- The velocity and relative acceleration of M.

2- Entrainment velocity and acceleration.

3- Coriolis acceleration.

4- Deduce its absolute velocity and acceleration.



Exercise 5

A point M moves with a constant **velocity *v_0* on the axis (OX)** of a coordinate system (OXYZ) which rotates with a constant angular velocity ω around (Oz) in the plane (Oxy).

1- What is the expression of $\overrightarrow{OM}$ in the fixed frame (Oxy)? Calculate the absolute velocity and absolute acceleration.

2- Calculate the relative velocity $\overrightarrow{v_r}$ and the entrainment velocity $\overrightarrow{v_e}$, verify that $\overrightarrow{v_a}=\overrightarrow{v_r}+\overrightarrow{v_e}$.

3- Calculate the relative acceleration $\overrightarrow{a_r}$, the entrainment acceleration $\overrightarrow{a_e}$, and the coriolis acceleration $\overrightarrow{a_c}$, verify that $\overrightarrow{a_a}=\overrightarrow{a_r}+\overrightarrow{a_e}+\overrightarrow{a_c}$.

Exercise 6

In the (**Oxy)** plane, we consider a system of moving axes **(OXY)** with the same **origin o** and such that (OX) makes a variable angle θ with (Ox). A point M moving **along axis (OX)** is marked by **OM=r**. We call relative motion of M, its motion with respect to (OXY), and absolute motion with respect to (Oxy).

Calculate in the moving frame of reference (**polar coordinates**):

1- Relative velocity and acceleration of M.

2- The velocity and entrainment acceleration of M.

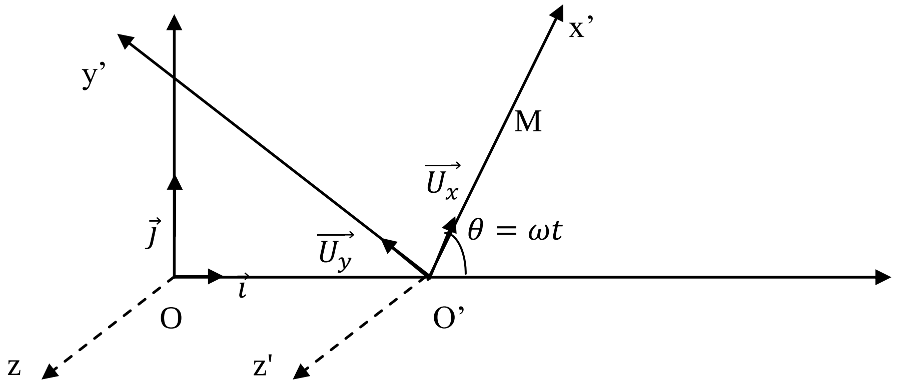
3- Coriolis acceleration.

4- Deduce its absolute velocity and acceleration.

Exercise 7

Consider the reference frame R(Oxyz) where point O' moves **along the axis (Ox)** with **constant velocity v**. O' is linked to the reference frame (O'x'y'z'), which rotates **around (oz)** with **constant angular velocity ω**. A moving point M **moves along the axis O'x'** such that **|O'M|=t²**.

At time t=0, the axes (Ox) and (O'x') are coincident and M is at O.



1. Calculate the relative velocity $\overrightarrow{v_r}$ and the entrainment velocity $\overrightarrow{v_e}$, deduce the absolute velocity $\overrightarrow{v_a}$.

2. Calculate the relative acceleration $\overrightarrow{a_r}$, the entrainment acceleration$\overrightarrow{a_e}$ and the Coriolis acceleration$\overrightarrow{a_c}$, deduce the absolute acceleration $\overrightarrow{a_a}$.

Exercise 8

Consider the reference frame R(Oxyz) where point **O' moves along axis (Oy)** with constant **acceleration γ**. We link to O' the reference frame (O'XYZ) which **rotates around (Oz)** with a **constant angular velocity ω**. The coordinates of a moving body M in the moving frame of reference **are x'=t²** and **y'=t**.

At time t=0, the axis (O'x) coincides with (Ox).

Calculate in the moving frame of reference:
1- Velocity $\overrightarrow{v_r}$ and $\overrightarrow{v_e}$, deduce the absolute velocity $\overrightarrow{v_a}$.
2- Relative acceleration $\overrightarrow{a_r}$, entrainment acceleration$\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$, deduce the absolute acceleration $\overrightarrow{a_a}$.

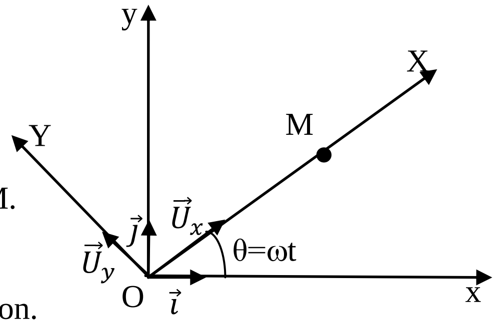
Exercise 9

In a plane (Oxy), consider a system of moving axes (OXY), of the **same origin O**, rotates around (Oz) with a constant angular velocity ω. A moving point on axis (OX) is marked by: $\left|\overrightarrow{OM}\right| = r = r_0(1 + \sin\omega t)$

Where r_0 is a positive constant:

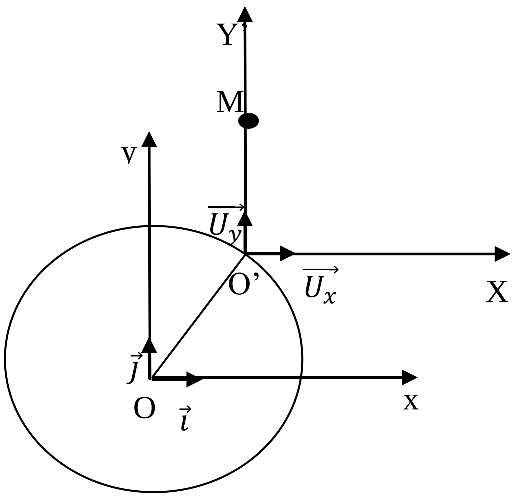
Calculate in the moving reference frame:

1) Relative velocity and acceleration of M.
2) Entrainment velocity and acceleration of M.
3) Coriolis acceleration.
4) Deduce its absolute velocity and acceleration.



Exercise 10

In the plane (Oxy) of a reference frame (Oxyz), a point O', to which the reference frame (O'XYZ) is linked, describes a circle of **center O and radius R**, and rotates **with a constant angular velocity ω**. A point M moves **along the axis (O'Y) parallel to Oy** with constant **acceleration γ** (at time t=0, M is merged with $M_0(R,0,0)$ and its initial velocity is positive).



1- Calculate in the (Oxyz) reference frame the position vector $\overrightarrow{OM}$, the absolute velocity $\overrightarrow{v_a}$. And the absolute acceleration $\overrightarrow{a_a}$.

2- Knowing that O'X// O x, O'Y// Oy and O'Z// Oz, calculate:

a- Relative speed and drive velocity, check that $\overrightarrow{v_a}=\overrightarrow{v_r}+\overrightarrow{v_e}+\overrightarrow{v_c}$.

b- A relative acceleration $\overrightarrow{a_r}$, entrainment acceleration $\overrightarrow{a_e}$, and the Coriolis acceleration $\overrightarrow{a_c}$, check that $\overrightarrow{a_a}=\overrightarrow{a_r}+\overrightarrow{a_e}+\overrightarrow{a_c}$.

Exercise 11

Consider a fixed reference frame (Oxyz) and a moving reference frame (Ox'y'z') which rotates **around (Oz)** with a **constant angular velocity ω**.

A moving point M **(OM=r)** moves **along the axis (Ox')** according to the law:

$\boldsymbol{r = r_0\,(\cos\omega t + \sin\omega t)}$ with r_0= constant.

Determine in the **moving reference frame** (Ox'y'z'):

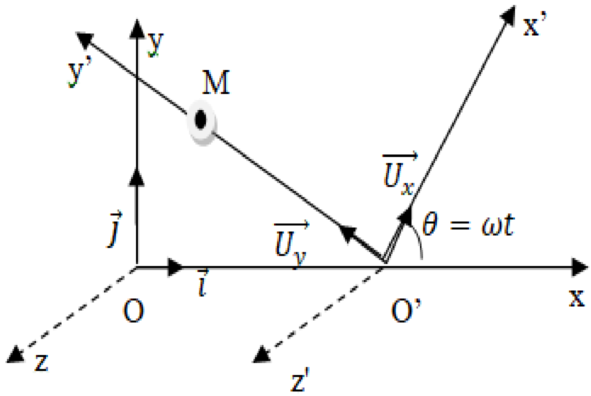
1- The velocity $\overrightarrow{v_r}$ and the entrainment velocity $\overrightarrow{v_e}$, deduce the absolute velocity$\overrightarrow{v_a}$.

2- Relative acceleration $\overrightarrow{a_r}$, entrainment acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$, deduce absolute acceleration $\overrightarrow{a_a}$.

Exercise 12

Consider the fixed reference frame R(Oxyz) where point O' moves **along axis (Ox)** with **constant velocity v_0**. Linked to O' is the moving reference frame (O'x'y'z') which rotates **around (Oz)** with **constant** angular velocity ω. A moving point M moves **along the (O'y')** axis with constant **acceleration γ**.

At time t=0, the axes (Ox) and (O'x') are coincident and M is at O.



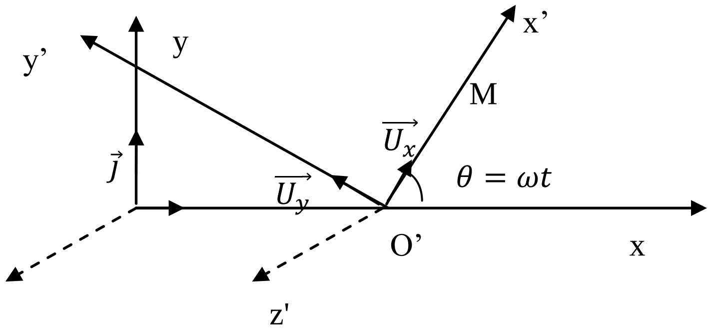
Calculate in the moving frame:

1- The relative velocity $\overrightarrow{v_r}$ and the entrainment velocity $\overrightarrow{v_e}$, deduce the absolute velocity $\overrightarrow{v_a}$.

2- The relative acceleration $\overrightarrow{a_r}$, the entrainment acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$, deduce the absolute acceleration $\overrightarrow{a_a}$.

Exercise 13

In the (Oxy) plane, a point O' (the origin of the moving reference frame) moves along the **(Ox) axis** such that **|OO'|=t**. The reference frame (O'X'Y') rotates **around (Oz)** with a constant angular velocity ω. A moving point M (O'M=r) moves along the **axis (O'X')** according to the law: $\boldsymbol{r = r_0\,(\cos\omega t + \sin\omega t)}$ With; r_0= constant.



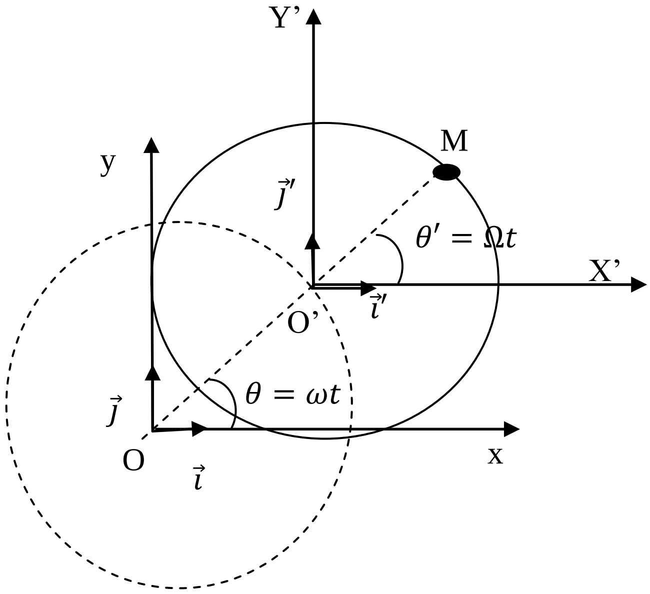
Determine at time t as a function of r_0 and ω:

1- The relative velocity $\overrightarrow{v_r}$ and the entrainment velocity $\overrightarrow{v_e}$, deduce the absolute velocity $\overrightarrow{v_a}$ in the moving frame of reference.

2- The relative acceleration $\overrightarrow{a_r}$, the entrainment acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$ in the moving frame of reference, deduce the absolute acceleration $\overrightarrow{a_a}$ in this frame of reference.

Exercise 14

In the frame (Oxyz), a point O' to which the frame (O'XYZ) is attached, describes a circle with center O and radius **R**; it rotates with a constant angular velocity **ω'** in the plane (Oxy). A point M describes a circle with center O' and radius **d** in the plane (O'XY); it rotates with a constant angular velocity **Ω**.



Calculate in the fixed frame (Oxyz) knowing that (O'X')∥(OX):

1- The absolute velocity $\overrightarrow{v_a}$, relative velocity $\overrightarrow{v_r}$, and entrainment velocity $\overrightarrow{v_e}$. Verify that $\overrightarrow{v_a}=\overrightarrow{v_r}+\overrightarrow{v_e}$.

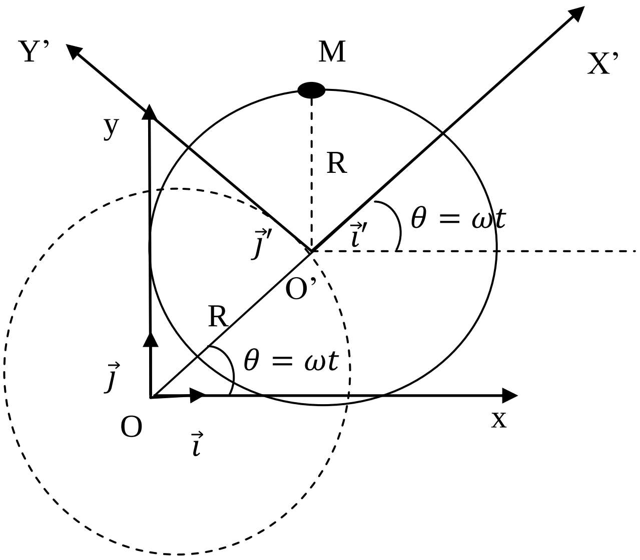
2- The absolute acceleration $\overrightarrow{a_a}$, relative acceleration $\overrightarrow{a_r}$, entrainment acceleration $\overrightarrow{a_e}$, and Coriolis acceleration $\overrightarrow{a_c}$. Verify that $\overrightarrow{a_a}=\overrightarrow{a_r}+\overrightarrow{a_e}+\overrightarrow{a_c}$.

Exercise 15

In the plane (Oxy), a point O' to which the frame (O'x'y'z') is attached, describes a circle with diameter **R** rotates at a constant angular velocity **ω** around point O. A point M initially at O' moves along the circumference in the positive direction with the same angular velocity **ω** and the same radius **R**.

At time t=0, M is on the O'x' axis parallel to O'x':

1. Provide the expression for $\overrightarrow{OM}$ in the fixed reference frame R(Oxy).
2. Calculate the components of the velocity and acceleration vectors of M in the reference frame R(Oxy).
3. Calculate the components of the velocity and acceleration vectors of M in the reference frame R'(O'x'y').
4. Calculate the entrainment velocity $\overrightarrow{v_e}$, the entrainment acceleration $\overrightarrow{a_e}$, and the Coriolis acceleration $\overrightarrow{a_c}$.

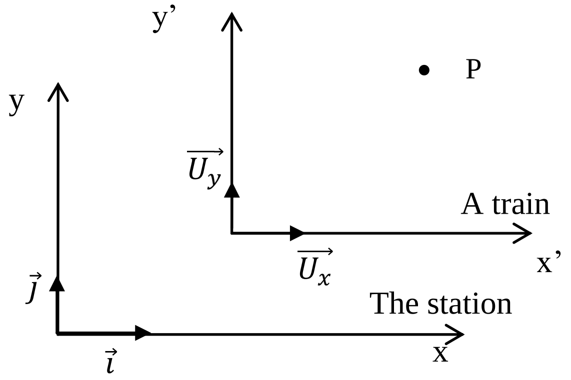


Correction of exercises about chapter IV

Exercise 1

We consider the station as the fixed reference frame and the train as the moving reference frame. The motion of “P” observed by a passenger (inside the train)

$\overrightarrow{v_p} = -gt\vec{j} = \overrightarrow{v_r}$



The speed of the train relative to the station $\vec{v} = \overrightarrow{v_e} = v\vec{i}\, with\, \overrightarrow{U_x} = \vec{i}\, and\, \overrightarrow{U_y} = \vec{j}$

The reference frames are parallel because the train undergoes a translational motion relative to the station.

$v = \frac{dx}{dt} \Rightarrow dx = vdt$ so $x = vt$ (at : t=0, x=0)

$$x = vt \Rightarrow t = \frac{x}{v}$$

The motion of “P” observed by the stationary stationmaster on the platform:

$$\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} = -gt\vec{j} + v\vec{i} \Rightarrow \overrightarrow{v_a} = -g\left(\frac{x}{v}\right)\vec{j} + v\vec{i}$$

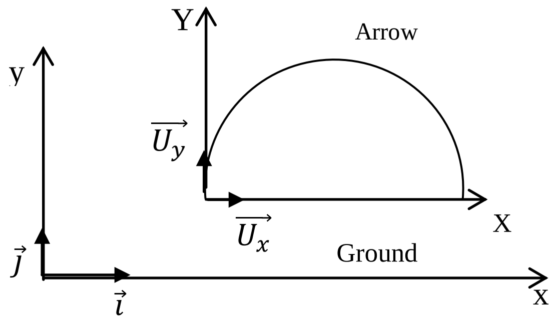
The position :

$$\begin{cases} x = vt \\ y = \dfrac{1}{2}gt^2 + y_0 \end{cases}$$

Or $y = \frac{1}{2}g\frac{x^2}{v^2} + Y_0$

Exercise 2

In this case, the ground is the fixed reference frame, and the horse is the moving reference frame. The horse has a translational motion relative to the ground, so the unit vectors of the two frames are equal.

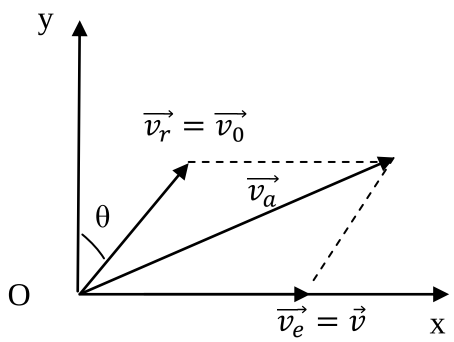


We study the motion of the arrow in both frames.

$\overrightarrow{v_{\text{Horse}}}/ground = \overrightarrow{v_e} = \vec{v}$ and $\overrightarrow{v_{Arrow}}/horse = \overrightarrow{v_r} = \overrightarrow{v_0}$

$$\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} = v_{ax}\vec{i} + v_{ay}\vec{j} \Rightarrow \begin{cases} v_{ax} = v + v_0 \operatorname{Sin}\theta \\ v_{ay} = v_0 \operatorname{Cos}\theta \end{cases} \quad (*)$$

The entrainment velocity is along the ox axis (translation motion), whereas the relative velocity makes an angle θ with the vertical oy.



$$(*) \Rightarrow \begin{cases} x_{Arrow}/ground = \int_0^r v_{ax}dt = \int_0^r (v + v_0 sin\,\theta)dt = (v + v_0 sin\,\theta)t \\ x'_{Horse}/ground = \int_0^r v_e dt = \int_0^r vdt = vt \end{cases}$$

For the arrow to fall back on the man, it's necessary that $(v + v_0 sin\,\theta)t = vt \Rightarrow sin\theta = 0$.

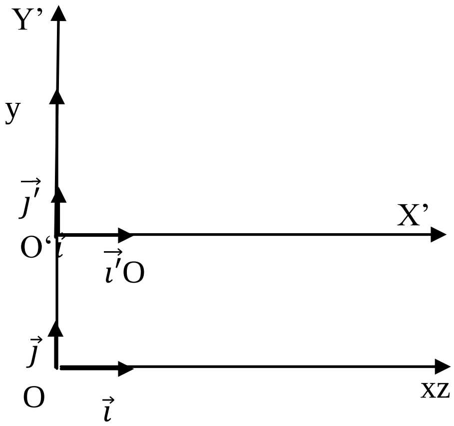
Thus, it's required that θ=0, meaning v_0 should be vertical.

Exercicse 3

$$\overrightarrow{OM}/(R)\begin{cases} x = 2t^3 + 1 \\ y = 4t^2 + t - 1 \\ z = t^2 \end{cases} \quad and \quad \overrightarrow{O'M}(R')\begin{cases} x' = 2t^3 \\ y' = 4t^2 - 3t + 2 \\ z' = t^2 - 5 \end{cases}$$

1. The speed of point m in the fixed reference frame (R) and the moving reference frame (R').

$$\vec{v} = \overrightarrow{v_a} = \frac{d\overrightarrow{OM}}{dt}\begin{cases} \frac{dx}{dt} = 6t^2 \\ \frac{dy}{dt} = 8t + 1 \\ \frac{dz}{dt} = 2t \end{cases}$$



$$\overrightarrow{v'} = \overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}\begin{cases} \frac{dx'}{dt} = 6t^2 \\ \frac{dy'}{dt} = 8t - 3 \\ \frac{dz'}{dt} = 2t \end{cases}$$

So:

$\vec{v}= 6t^2\vec{\imath} +(8t + 1)\vec{\jmath} + 2t\vec{k}$ And $\overrightarrow{v'} = 6t^2\vec{\imath} +(8t - 3)\vec{\jmath} + 2t\vec{k}$

We have :$\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} \Rightarrow \overrightarrow{v_e} = \overrightarrow{v_a} - \overrightarrow{v_r}$

$\overrightarrow{v_e}$=$(6t^2\vec{\imath} +(8t + 1)\vec{\jmath} + 2t\vec{k}) - (6t^2\vec{\imath} +(8t - 3)\vec{\jmath} + 2t\vec{k})$=$4\vec{\jmath}$

So: $\boldsymbol{\vec{v}}$=$\boldsymbol{\vec{v}' + 4\vec{j}}$

And $\boldsymbol{\vec{\imath} = \overrightarrow{\imath'}, \vec{\jmath} = \overrightarrow{\jmath'}, \vec{k} = \overrightarrow{k'}}$

2. The acceleration of point m in the two reference frames, fixed (R) and moving (R'), is as follows:

$$\vec{a} = \overrightarrow{a_a} = \frac{d\vec{v}}{dt}\begin{cases} \frac{dv_x}{dt} = 12t \\ \frac{dv_y}{dt} = 8 \\ \frac{dv_z}{dt} = 2 \end{cases}$$

And $\overrightarrow{a'} = \overrightarrow{a_r} = \frac{d\overrightarrow{v'}}{dt} \begin{cases} \frac{dv'_x}{dt} = 12t \\ \frac{dv'_y}{dt} = 8 \\ \frac{dv'_z}{dt} = 2 \end{cases}$

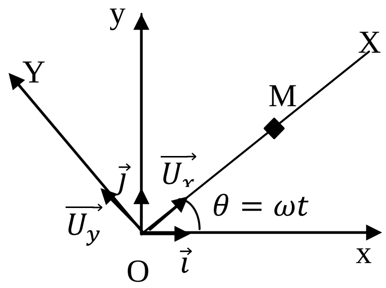
So: $\vec{a} = \overrightarrow{a'}$ Or $\overrightarrow{a_a} = \overrightarrow{a_r}$

Conclusion:

The motion of frame of reference (R') relative to the fixed frame of reference (R) is a uniform translational motion along axis (Oy) with a constant speed of 4m/s.

Exercise 4

The fixed frame of reference and the moving frame of reference have the same origin, so O' and O are the same.



Then, $\overrightarrow{OO'} = \vec{0}$ and $\overrightarrow{OM} = \overrightarrow{O'M} = \frac{1}{2}\gamma t^2 \overrightarrow{U_x}$

with $\overrightarrow{U_x} = \cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}$

and $\overrightarrow{U_y} = (-\sin\omega t\,\vec{i} + \cos\omega t\,\vec{j})$

we have also $\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$ and $\frac{d\omega}{dt} = 0$

1. **Absolute velocity**

$\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}/(OXY)$ with $\overrightarrow{O'M} = \overrightarrow{OM} = \frac{1}{2}\gamma t^2\, \overrightarrow{U_x}$ so $\overrightarrow{v_r} = \gamma t \overrightarrow{U_x}$

Absolute acceleration :

$$\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}/(O'XY) \quad \text{with } \overrightarrow{v_r} = \gamma t\, \overrightarrow{U_x}$$

So $\overrightarrow{a_r} = \frac{d^2\overrightarrow{O'M}}{dt^2} = \gamma\, \overrightarrow{U_x}$

2. **Entrainment velocity :**

$\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$ with $\overrightarrow{OO'} = \vec{0}$ so $\overrightarrow{v_e} = \vec{\omega}\Lambda\overrightarrow{O'M}$

$$\vec{\omega}\Lambda\overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ \frac{1}{2}\gamma t^2 & 0 & 0 \end{vmatrix} = \omega \frac{1}{2}\gamma t^2\, \overrightarrow{U_y} \text{ so } \overrightarrow{v_e} = \omega\, \frac{1}{2}\gamma t^2\, \overrightarrow{U_y}$$

Entrainment acceleration

$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) + \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$; $\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0}$ because ω constant and $\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$

And $\vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) = \vec{\omega}\Lambda\left(\omega\frac{1}{2}\gamma t^2\ \overrightarrow{U_y}\right) = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ 0 & \omega\frac{1}{2}\gamma t^2 & 0 \end{vmatrix} = -\omega^2\frac{1}{2}\gamma t^2\,\overrightarrow{U_x}$

So $\overrightarrow{a_e} = -\omega^2\frac{1}{2}\gamma t^2\,\overrightarrow{U_x}$

3. **Coriolis acceleration**

$\overrightarrow{a_c} = 2\vec{\omega}\Lambda\overrightarrow{v_r} = 2\begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ \gamma t & 0 & 0 \end{vmatrix} = 2\omega\gamma t\overrightarrow{U_y}$ so $\overrightarrow{a_c} = 2\omega\gamma t\overrightarrow{U_y}$

4. **Absolute velocity :**

$\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} = \gamma t\overrightarrow{U_x} + \omega\ \frac{1}{2}\gamma t^2\,\overrightarrow{U_y}$

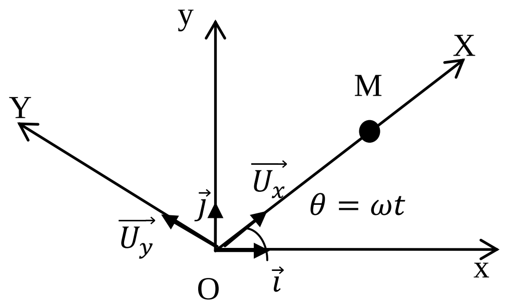
$= \gamma t(\cos\omega t\,\vec{\imath} + \sin\omega t)\vec{\jmath} + \omega\ \frac{1}{2}\gamma t^2\,(-\sin\omega t\,\vec{\imath} + \cos\omega t\,\vec{\jmath})$

Acceleration absolute

$\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e} = (\gamma - \omega^2\frac{1}{2}\gamma t^2)\,\overrightarrow{U_x} + 2\omega\gamma t\overrightarrow{U_y}$

$\overrightarrow{a_a} = (\gamma - \omega^2\frac{1}{2}\gamma t^2)\ (\cos\omega t\,\vec{\imath} + \sin\omega t) + 2\omega\gamma t\ (-\sin\omega t\,\vec{\imath} + \cos\omega t\,\vec{\jmath})$

Exercise 5



In the farme (OXY) $\overrightarrow{OM} = v_0 t\overrightarrow{U_x}$

With $\overrightarrow{U_X} = \mathrm{Cos}\,\omega t\,\vec{I} + \mathrm{Sin}\,\omega t\,\vec{J}$ And $\overrightarrow{U_Y} = -\sin\omega t\,\vec{I} + \mathrm{Cos}\,\omega t\,\vec{J}$

$\overrightarrow{OM}/(R) = v_0 t\,\left(\mathrm{Cos}\,\omega t\,\vec{I} + \mathrm{Sin}\,\omega t\,\vec{J}\right)$ In the farme (Oxy)

Absolute velocity

$$\overrightarrow{v_a} = \frac{d\overrightarrow{OM}}{dt}/(R) = v_0\,(\text{Cos}\,\omega t\,\vec{\text{I}} + \text{Sin}\,\omega t\,\vec{\text{J}}) - v_0\omega t\,\text{Sin}\,\omega t\,\vec{\text{I}} + v_0\omega t\,\text{Cos}\,\omega t\,\vec{\text{J}}$$

$=v_0\,(\cos\omega t\,\vec{\text{I}} + v_0\,\text{Sin}\,\omega t\,\vec{\text{J}}) + v_0\omega t\,(-\sin\omega t\,\vec{\text{I}} + \text{Cos}\,\omega t\,\vec{\text{J}})$ $=v_0\overrightarrow{\text{U}_\text{X}} + v_0\omega t\overrightarrow{\text{U}_\text{Y}}$

(or) $\overrightarrow{v_a} = \frac{d\overrightarrow{OM}}{dt}/(R) = \frac{d(v_0 t\overrightarrow{U_x})}{dt} = v_0\overrightarrow{U_x} + v_0 t\frac{d\overrightarrow{U_x}}{dt}$,

$$with\ \frac{d\overrightarrow{U_x}}{dt} = \frac{d\overrightarrow{U_x}}{d\theta}\frac{d\theta}{dt}\ \ and\ \ \theta = \omega t$$

$$\Rightarrow \frac{d\overrightarrow{U_x}}{dt} = \frac{d\overrightarrow{U_x}}{d\theta}\frac{d\omega t}{dt} = \omega\overrightarrow{U_y}\ because\ \frac{d\overrightarrow{U_x}}{d\theta} = \overrightarrow{U_y}$$

So : $\overrightarrow{v_a} = v_0\overrightarrow{\text{U}_\text{X}} + v_0\omega t\overrightarrow{\text{U}_\text{Y}}$

Absolute acceleration

$$\overrightarrow{a_a} = \frac{d\overrightarrow{v_a}}{dt}/(R)$$

$$= -v_0\omega\,\text{Sin}\,\omega t\,\vec{\text{I}} + v_0\,\omega\,\cos\omega t\,\vec{\text{J}} - v_0\omega\,\text{Sin}\,\omega t\,\vec{\text{I}} + v_0\,\omega\cos\omega t\,\vec{\text{J}}$$
$$- v_0\omega^2\text{T}\,\cos\omega t\,\vec{\text{I}} + v_0\omega^2\text{T}\,\text{Sin}\,\omega t\,\vec{\text{J}}$$

$$\Rightarrow \overrightarrow{a_a} = \frac{d\overrightarrow{v_a}}{dt}/(R) = 2v_0\omega\left(-\sin\omega t\,\vec{\text{I}} + \text{Cos}\,\omega t\,\vec{\text{J}}\right) - v_0\omega^2\text{T}\left(\text{Cos}\,\omega t\,\vec{\text{I}} + \text{Sin}\,\omega t\,\vec{\text{J}}\right)$$

$\Rightarrow \overrightarrow{a_a} = 2v_0\omega\overrightarrow{U_y} - v_0\omega^2\text{T}\overrightarrow{U_x}$

Relative velocity

$\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}/(OXY)\ \ with\ \ \overrightarrow{O'M} = \overrightarrow{OM} = v_0 t\,\overrightarrow{U_x}$ so $\overrightarrow{v_r} = v_0\overrightarrow{U_x}$

Entrainment velocity

The fixed frame and the mobile frame have the same origin so O' is confused with O, then $\overrightarrow{OO'} = \vec{0}$.

$\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + X\frac{d\overrightarrow{U_x}}{dt} = \vec{0} + v_0 t\frac{d\overrightarrow{U_x}}{dt}$ With $\frac{d\overrightarrow{U_x}}{dt} = \omega\overrightarrow{U_y}$

So ; $\overrightarrow{v_e} = \omega\, v_0 t\, \overrightarrow{U_y}$

Or

$$\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$$

$\overrightarrow{OO'} = \vec{0}$ so $\overrightarrow{v_e} = \vec{\omega}\Lambda\overrightarrow{O'M}$

$$\vec{\omega}\Lambda\overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ v_0 t & 0 & 0 \end{vmatrix} = \omega v_0 t\, \overrightarrow{U_y} \quad \text{So} \quad \overrightarrow{v_e} = \omega\, v_0 t\, \overrightarrow{U_y}$$

With $\overrightarrow{\mathrm{U_X}} = \mathrm{Cos}\,\omega t\,\vec{\mathrm{I}} + \mathrm{Sin}\,\omega t\,\vec{\mathrm{J}}$ And $\overrightarrow{\mathrm{U_Y}} = -\sin\omega t\,\vec{\mathrm{I}} + \mathrm{Cos}\,\omega t\,\vec{\mathrm{J}}$

So $\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} = v_0\left(\mathrm{Cos}\,\omega t\,\vec{\mathrm{I}} + \mathrm{Sin}\,\omega t\,\vec{\mathrm{J}}\right) + \omega\, v_0 t\, \left(-\sin\omega t\,\vec{\mathrm{I}} + \mathrm{Cos}\,\omega t\,\vec{\mathrm{J}}\right)$

$\Rightarrow \overrightarrow{v_a} = v_0\overrightarrow{U_x} + \omega\, v_0 t\, \overrightarrow{U_y}$

So $\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e}$ $\quad is\ verified$

Relative acceleration

$\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}$ $with\ \overrightarrow{v_r} = v_0\,\overrightarrow{\mathrm{U_X}}$ Then $\overrightarrow{a_r} = \vec{0}$

Entrainment acceleration

$$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + X\frac{d^2\overrightarrow{U_x}}{dt^2} = v_0 t\frac{d}{dt}\left(\frac{d\overrightarrow{U_x}}{dt}\right) \Rightarrow \overrightarrow{a_e} = v_0 t\frac{d}{dt}\left(\omega\ \overrightarrow{U_y}\right)$$

$\Rightarrow \overrightarrow{a_e} = v_0 t\omega\left(\frac{d\overrightarrow{U_y}}{dt}\right)$ With $\frac{d\overrightarrow{U_y}}{dt} = \frac{d\overrightarrow{U_y}}{d\theta}\frac{d\omega t}{dt} = -\omega\overrightarrow{U_x}$

$$\Rightarrow \overrightarrow{a_e} = v_0 t\omega\left(+\ \left(-\omega\overrightarrow{U_x}\right)\right)$$

$$\Rightarrow \overrightarrow{a_e} = -v_0\omega^2\, t\, \overrightarrow{U_x}$$

<u>Or</u>

$$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) + \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$$

$\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0}$ because ω constant and $\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$

And $\vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) = \vec{\omega}\Lambda\left(\omega\, v_0 t\,\overrightarrow{U_y}\right) = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ 0 & \omega\, v_0 t & 0 \end{vmatrix} = -\omega^2 v_0 t\,\overrightarrow{U_x}$

So ; $\overrightarrow{a_e} = -\omega^2 v_0 t\overrightarrow{U_x}$

Coriolis acceleration

$\overrightarrow{a_c} = 2\frac{dX}{dt}\frac{d\overrightarrow{U_x}}{dt}\ with = v_0 t\ \ et\frac{d\overrightarrow{U_x}}{dt} = \omega\overrightarrow{U_y}$

So $\overrightarrow{a_c} = 2v_0\omega\overrightarrow{U_y}$

<u>Or</u>

$\overrightarrow{a_c} = 2\vec{\omega}\Lambda\overrightarrow{v_r} = 2\begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ v_0 & 0 & 0 \end{vmatrix} = 2\omega v_0\overrightarrow{U_y}$

So $\overrightarrow{a_c} = 2\omega v_0\overrightarrow{U_y}$

Absolute acceleration

$\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e} = -\omega^2 v_0 t\overrightarrow{U_x} + 2\omega v_0\overrightarrow{U_y}$

$\Rightarrow \overrightarrow{a_a} = -\omega^2 v_0 t\left(\mathrm{Cos}\omega\,\mathrm{t}\,\vec{\mathrm{I}} + \mathrm{Sin}\,\omega\mathrm{t}\,\vec{\mathrm{J}}\right) + 2\omega v_0\left(-\sin\omega\mathrm{t}\,\vec{\mathrm{I}} + \mathrm{Cos}\,\omega\mathrm{t}\,\vec{\mathrm{J}}\right)$

So

$\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e}$ Is verified

Exercise 6

In the mobile frame (OXY): $\overrightarrow{OM} = r\overrightarrow{U_x}$

In the mobile frame (polar coordinates) we have:

Relative velocity

$\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}/(OXY) = \frac{dr}{dt}\overrightarrow{U_x}\overrightarrow{v_r} = r^{\cdot}\overrightarrow{U_x}$

Relative acceleration

$$\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}/(OXY) \qquad with\ \overrightarrow{v_r} = r^{\cdot}\ \overrightarrow{\mathrm{U_X}}$$

So : $\overrightarrow{a_r} = \frac{d^2r}{dt^2}\overrightarrow{\mathrm{U_X}} = r^{\cdot\cdot}\ \overrightarrow{\mathrm{U_X}}$

Entrainment velocity

$\overrightarrow{OO'} = \vec{0}$ "because both reference frames have the same origin."

$$\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + X\frac{d\overrightarrow{U_x}}{dt} = \vec{0} + r\frac{d\overrightarrow{U_x}}{dt}$$

With $\frac{d\overrightarrow{U_x}}{dt} = \frac{d\overrightarrow{U_x}}{d\theta}\frac{d\theta}{dt}$ $\ and\ \ \theta = \omega t$

So $\frac{d\overrightarrow{U_x}}{dt} = \frac{d\overrightarrow{U_x}}{d\theta}\frac{d\omega t}{dt} = \omega\overrightarrow{U_y}\ car\ \frac{d\overrightarrow{U_x}}{d\theta} = \overrightarrow{U_y}$

Then $\overrightarrow{v_e} = \omega r\ \overrightarrow{U_y}$

<u>Or</u> $\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$

$\overrightarrow{OO'} = \vec{0}$ so $\overrightarrow{v_e} = \vec{\omega}\Lambda\overrightarrow{O'M}$

$\vec{\omega}\Lambda\overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ r & 0 & 0 \end{vmatrix} = \omega r\,\overrightarrow{U_y}$ So $\overrightarrow{v_e} = \omega\ r\ \overrightarrow{U_y}$

Entrainment acceleration

$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + X\frac{d^2\overrightarrow{U_x}}{dt^2} = r\frac{d}{dt}\left(\frac{d\overrightarrow{U_x}}{dt}\right) \Rightarrow r\frac{d}{dt}\left(\omega\ \ \overrightarrow{U_y}\right)$

$\Rightarrow \overrightarrow{a_e} = r\ \ \omega\left(\frac{d\overrightarrow{U_y}}{dt}\right)$ With $\frac{d\overrightarrow{U_y}}{dt} = \frac{d\overrightarrow{U_y}}{d\theta}\frac{d\omega t}{dt} = -\omega\overrightarrow{U_x}$

$\Rightarrow \overrightarrow{a_e} = \ \ r\omega\left(+\ \left(-\omega\overrightarrow{U_x}\right)\right)$

$\Rightarrow \overrightarrow{a_e} = -\ \ r\omega^2\overrightarrow{U_x}$

<u>Or</u>

$$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) + \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$$

$$\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0} \text{ because } \omega \text{ constant and } \frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$$

$$\vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) = \vec{\omega}\Lambda\left(\omega\, r\,\overrightarrow{U_y}\right) = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ 0 & \omega\, r & 0 \end{vmatrix} = -\omega^2\, r\,\overrightarrow{U_x}$$

So $\overrightarrow{a_e} = -\omega^2 r\overrightarrow{U_x}$

Coriolis acceleration

$$\overrightarrow{a_c} = 2\frac{dX}{dt}\frac{d\overrightarrow{U_x}}{dt} \;\; with \;\; X = r \;\; and \;\; \frac{d\overrightarrow{U_x}}{dt} = \omega\overrightarrow{U_y}$$

So $\overrightarrow{a_c} = 2r^{\cdot}\omega\overrightarrow{U_y}$

__**Or**__

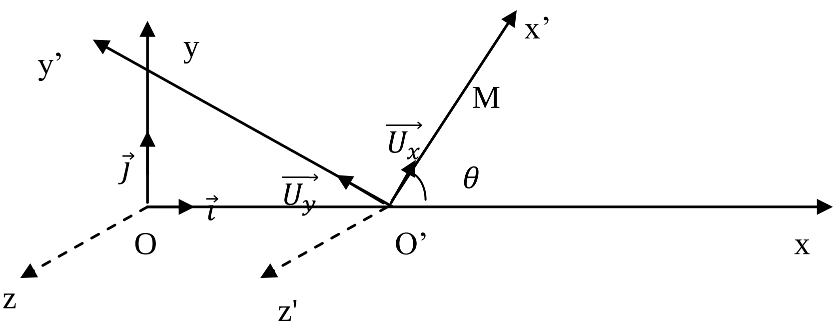
$$\overrightarrow{a_c} = 2\vec{\omega}\Lambda\overrightarrow{v_r} = 2\begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ r^{\cdot} & 0 & 0 \end{vmatrix} = 2\omega r^{\cdot}\overrightarrow{U_y}$$

So $\overrightarrow{a_c} = 2\omega r^{\cdot}\overrightarrow{U_y}$

Absolute velocity $\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} = r^{\cdot}\overrightarrow{U_x} + \omega\, r\,\overrightarrow{U_y}$

Absolute acceleration $\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e} = (r^{\cdot\cdot} - \omega^2 r)\overrightarrow{U_x} + 2\omega r^{\cdot}\overrightarrow{U_y}$

Exercise 7



Relative velocity

$\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}/(R')$ with $\overrightarrow{O'M} = t^2\overrightarrow{U_x}$ so $\overrightarrow{v_r} = 2t\overrightarrow{U_x}$

Entrainment velocity:

$$\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$$

We are looking for the vector $\overrightarrow{OO'}$.

The point O' moves along the axis (Ox) with a velocity v, thus $\overrightarrow{v_{O'}} = \frac{dOO'}{dt}\vec{\imath} = v\vec{\imath}$

At t=0, x=0 so $\frac{dOO'}{dt} = v \Rightarrow OO' = vt \quad so \; \overrightarrow{OO'} = vt\vec{\imath}$

$$\vec{\omega}\Lambda\overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ t^2 & 0 & 0 \end{vmatrix} = \omega t^2\overrightarrow{U_y} \quad and \quad \frac{d\overrightarrow{OO'}}{dt} = v\vec{\imath}$$

So $\overrightarrow{v_e} = \omega t^2\overrightarrow{U_y} + v\vec{\imath}$

"we need to express $\overrightarrow{v_e}$ In the same coordinate system. To do this, we will express $\vec{\imath}$ In terms of $\overrightarrow{U_x}$ and $\overrightarrow{U_y}$.

We have $\begin{cases} \overrightarrow{U_x} = cos\,\theta\vec{\imath} + \text{Sin}\,\theta\,\vec{\jmath} \\ \overrightarrow{U_y} = -\text{sin}\theta\vec{\imath} + cos\;\theta\,\vec{\jmath} \end{cases} \Rightarrow \vec{\imath} = cos\,\theta\overrightarrow{U_x} - \text{Sin}\,\theta\,\overrightarrow{U_y}$

So $\overrightarrow{v_e} = \omega t^2\overrightarrow{U_y} + v\left(cos\,\theta\overrightarrow{U_x} - \text{Sin}\,\theta\,\overrightarrow{U_y}\right) = v\,cos\,\theta\overrightarrow{U_x} + (\omega t^2 - v\,sin\,\theta)\overrightarrow{U_y}$

Absolute velocity

$$\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} = 2t\overrightarrow{U_x} + \omega t^2\overrightarrow{U_y} + v\left(cos\,\theta\overrightarrow{U_x} - \text{Sin}\,\theta\,\overrightarrow{U_y}\right)$$

$$\Rightarrow \overrightarrow{v_a} = (2t + v\,cos\,\theta)\overrightarrow{U_x} + (\omega t^2 - v\,sin\,\theta)\overrightarrow{U_y}$$

Relative acceleration

$\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}/(R')$ with $\overrightarrow{v_r} = 2t\overrightarrow{U_x}$ so $\overrightarrow{a_r} = 2\overrightarrow{U_x}$

Entrainement acceleration

$$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) + \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$$

$\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}\,, \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0}$ $\quad \Omega$ constant

And $\vec{\omega}\Lambda(\vec{\omega}\Lambda\overrightarrow{O'M}) = \vec{\omega}\Lambda\omega t^2\overrightarrow{U_y} = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ 0 & \omega t^2 & 0 \end{vmatrix} = -\omega^2 t^2\overrightarrow{U_x}$

So $\overrightarrow{a_e} = -\omega^2 t^2\overrightarrow{U_x}$

Coriolis acceleration

$$\overrightarrow{a_c} = 2\vec{\omega}\Lambda\overrightarrow{v_r} = 2\begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ 2t & 0 & 0 \end{vmatrix} = 4t\omega\overrightarrow{U_y}$$

Absolute acceleration

$$\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e} = 2\overrightarrow{U_x} - \omega^2 t^2\overrightarrow{U_x} + 4t\omega\overrightarrow{U_y}$$

$So\ \ \overrightarrow{a_a}=(2-\omega^2t^2)\overrightarrow{U_x} + 4t\,\omega\overrightarrow{U_y}$

Exercise 8

The coordinates of point m in the moving reference frame M(t^2, t)/(R').

So $\overrightarrow{O'M}$ is written: $\overrightarrow{O'M} = t^2\overrightarrow{U_x} + t\overrightarrow{U_y}$

Or O' moves on the axis (Oy) with a constant acceleration γ.

at instant t=0, the axis (O'x) is confused with (Ox). So v_0=0 and y_0=0

then; $the\ acceleration\ of O' is: \gamma = \frac{dv}{dt} \Rightarrow dv = \gamma dt$

After integration ***v=γ.t***

And $\frac{dy}{dt} = \gamma t \Rightarrow dy = \gamma t dt \quad so\ \ y = \frac{1}{2}\gamma t^2$ And $\overrightarrow{OO'} = \frac{1}{2}\gamma t^2\vec{j}$

Relative velocity: $\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}/(R')$ with $\overrightarrow{O'M} = t^2\overrightarrow{U_x} + t\overrightarrow{U_y}$ So $\overrightarrow{v_r} = 2t\overrightarrow{U_x} + \overrightarrow{U_y}$

Entrainment velocity: $\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$ With $\overrightarrow{OO'} = \frac{1}{2}\gamma t^2\vec{j} \Rightarrow \frac{d\overrightarrow{OO'}}{dt} = \gamma t\vec{j}$

$$\vec{\omega}\Lambda\overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ t^2 & t & 0 \end{vmatrix} = -\omega t\overrightarrow{U_x} + \omega t^2\overrightarrow{U_y} \quad so\ \ \overrightarrow{v_e} = \gamma t\vec{j} - \omega t\overrightarrow{U_x} + \omega t^2\overrightarrow{U_y}$$

We need to write $\vec{v_e}$ in the same coordinate system, so we'll write $\vec{j}$ as a function of $\overrightarrow{U_x}$ and $\overrightarrow{U_y}$:

We have: $\begin{cases} \overrightarrow{U_x} = cos\theta\vec{\imath} + \text{Sin}\,\theta\vec{j} \\ \overrightarrow{U_y} = -\text{sin}\theta\vec{\imath} + cos\,\theta\vec{j} \end{cases} \Rightarrow \vec{j} = sin\theta\overrightarrow{U_x} + cos\,\theta\overrightarrow{U_y}$

So $\vec{v_e} = \omega t^2\overrightarrow{U_y} - \omega t\overrightarrow{U_x} + \gamma t\left(sin\theta\overrightarrow{U_x} + cos\,\theta\overrightarrow{U_y}\right)$

$$\Rightarrow \vec{v_e} = (\gamma t\,sin\,\theta - \omega t)\overrightarrow{U_x} + (\omega t^2 + \gamma t cos\,\theta)\overrightarrow{U_y}$$

Absolute velocity :

$$\vec{v_a} = \vec{v_r} + \vec{v_e} = 2t\overrightarrow{U_x} + \overrightarrow{U_y} + (\gamma t\,sin\,\theta - \omega t)\overrightarrow{U_x} + (\omega t^2 + \gamma t cos\,\theta)\overrightarrow{U_y}$$

$$\Rightarrow \vec{v_a} = (\gamma t\,sin\,\theta - \omega t + 2t)\overrightarrow{U_x} + (\omega t^2 + \gamma t cos\,\theta + 1)\overrightarrow{U_y}$$

Relative acceleration :

$\vec{a_r} = \frac{d\vec{v_r}}{dt}/(R')with\ \vec{v_r} = 2t\overrightarrow{U_x} + \overrightarrow{U_y}$ So $\vec{a_r} = 2\overrightarrow{U_x}$

Entrainement acceleration :

$$\vec{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) + \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$$

$\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0}$ Because ω constant and $\frac{d^2\overrightarrow{OO'}}{dt^2} = \gamma\vec{j}$

$$and\quad \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) = \vec{\omega}\Lambda\left(-\omega t\overrightarrow{U_x} + \omega t^2\overrightarrow{U_y}\right) = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ -\omega t & \omega t^2 & 0 \end{vmatrix} = -\omega^2 t^2\overrightarrow{U_x} - \omega^2 t\overrightarrow{U_y}$$

Then $\vec{a_e} = \vec{j} - \omega^2 t^2\overrightarrow{U_x} - \omega^2 t\overrightarrow{U_y} = \gamma\left(\text{Sin}\,\theta\overrightarrow{U_x} + cos\,\theta\overrightarrow{U_y}\right) - \omega^2 t^2\overrightarrow{U_x} - \omega^2 t\overrightarrow{U_y}$

$$\vec{a_e} = (\gamma\,\text{Sin}\,\theta - \omega^2 t^2)\overrightarrow{U_x} + (\gamma\,cos\,\theta - \omega^2 t)\overrightarrow{U_y}$$

Coriolis acceleration :

$$\vec{a_c} = 2\vec{\omega}\Lambda\vec{v_r} = 2\begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ 2t & 1 & 0 \end{vmatrix} = 4t\omega\overrightarrow{U_y} - 2\omega\overrightarrow{U_x}$$

Absolute acceleration :

$$\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e}$$

$\Rightarrow \overrightarrow{a_a} = 2\overrightarrow{U_x} + (\gamma \operatorname{Sin}\theta - \omega^2 t^2)\overrightarrow{U_x} + (\gamma \cos\theta \;\; - \omega^2 t)\overrightarrow{U_y} + 4t\,\omega\overrightarrow{U_y} - 2\omega\overrightarrow{U_x}$

So $\overrightarrow{a_a}$=$(2 - 2\omega + \gamma \operatorname{Sin}\theta - \omega^2 t^2)\overrightarrow{U_x} + (\gamma \cos\theta - \omega^2 t + 4t\omega\,)\overrightarrow{U_y}$

Exercise 9

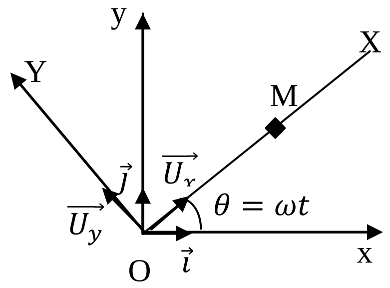
The fixed frame of reference and the moving frame of reference have the same origin, so O' and O are the same.

Then $\overrightarrow{OO'} = \vec{0}$ and $\overrightarrow{OM} = \overrightarrow{O'M} = |\mathrm{OM}|\,\overrightarrow{U_x} =$ r $= \mathrm{r}_0(1 + \sin\omega t)\overrightarrow{U_x}$

with $\overrightarrow{\mathrm{U_x}} = \cos\omega t\,\vec{\mathrm{i}} + \sin\omega t\,\vec{\mathrm{j}}$

and $\overrightarrow{\mathrm{U_y}} = (-\sin\omega t\,\vec{\mathrm{i}} + \cos\omega t\,\vec{\mathrm{j}})$

we have also ; $\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$ and $\frac{d\omega}{dt} = 0$



Relative velocity

$\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}/(OXY)$ so $\overrightarrow{v_r} = \mathrm{r}_0(\omega\cos\omega t)\overrightarrow{U_x}$

Relative acceleration

$\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}/(O'XY)$ so $\overrightarrow{a_r} = \frac{d^2\overrightarrow{O'M}}{dt^2} = -\mathrm{r}_0(\omega^2 \sin\omega t)\,\overrightarrow{\mathrm{U_x}}$

Entrainment velocity

$\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$ with $\overrightarrow{OO'} = \vec{0}$ so $\overrightarrow{v_e} = \vec{\omega}\Lambda\overrightarrow{O'M}$

$$\vec{\omega}\Lambda\overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ \mathrm{r}_0(1 + \sin\omega t) & 0 & 0 \end{vmatrix} = \omega \mathrm{r}_0(1 + \sin\omega t)\,\overrightarrow{U_y}$$

so $\overrightarrow{v_e} = \omega\,\mathrm{r}_0(1 + \sin\omega t)\,\overrightarrow{U_y}$

Entraiment acceleration

$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda(\vec{\omega}\Lambda\overrightarrow{O'M}) + \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$; $\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0}$ because ω constant and $\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$

And $\vec{\omega}\Lambda(\vec{\omega}\Lambda\overrightarrow{O'M}) = \vec{\omega}\Lambda\left(\omega\frac{1}{2}\gamma t^2\;\overrightarrow{U_y}\right) = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ 0 & \omega \mathrm{r}_0(1 + \sin\omega t) & 0 \end{vmatrix}$

$= -\omega^2 r_0(1 + \sin\omega t)\,\overrightarrow{U_x}$

so $\overrightarrow{a_e} = -\omega^2 r_0(1 + \sin\omega t)\,\overrightarrow{U_x}$

Coriolis acceleration

$$\overrightarrow{a_c} = 2\vec{\omega}\Lambda\overrightarrow{v_r} = 2\begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ r_0(\omega\cos\omega t) & 0 & 0 \end{vmatrix} = 2\omega r_0(\omega\cos\omega t)\overrightarrow{U_y}$$

so $\overrightarrow{a_c} = 2\omega r_0(\omega\cos\omega t)\overrightarrow{U_y}$

Absolute velocity

$\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} = r_0(\omega\cos\omega t)\overrightarrow{U_x} + \omega\, r_0(1 + \sin\omega t)\,\overrightarrow{U_y}$

In the Cartesian coordinate base:

$\overrightarrow{v_a} = \gamma t(\cos\omega t\,\vec{\imath} + \sin\omega t)\vec{\jmath} + \omega\,\frac{1}{2}\gamma t^2\,(-\sin\omega t\,\vec{\imath} + \cos\omega t\,\vec{\jmath})$

Absolute acceleration

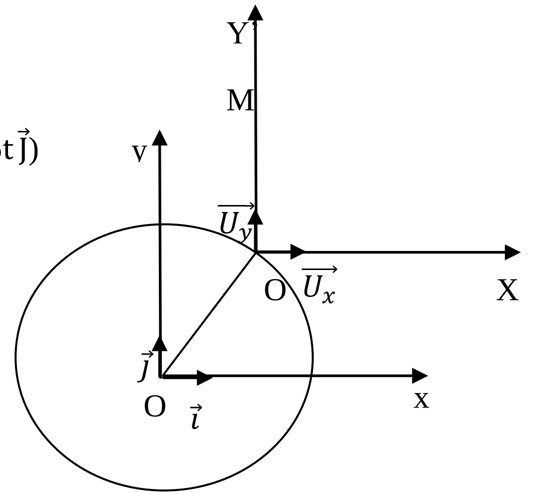
$$\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e}$$

$\overrightarrow{a_a} = 2\omega r_0(\omega\cos\omega t)\overrightarrow{U_y} - \omega^2 r_0(1 + \sin\omega t)\overrightarrow{U_x} - r_0(\omega^2\sin\omega t)\,\overrightarrow{U_x}$

In the Cartesian coordinate base:

$$\overrightarrow{a_a} = (-r_0(\omega^2\sin\omega t) - \omega^2 r_0(1 + \sin\omega t))\,(\cos\omega t\,\vec{\imath} + \sin\omega t) + 2\omega r_0(\omega\cos\omega t)\,(-\sin\omega t\,\vec{\imath} + \cos\omega t\,\vec{\jmath})$$

Exercise 10



At t=0, y'=0 and v=v_0

$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$ $\quad with\ \overrightarrow{OO'} = R(\cos\ \omega t\,\vec{\imath} + \mathrm{Sin}\,\omega t\,\vec{\jmath})$

M moves along the axis (O'y) parallel to Oy.

With constant acceleration γ.

$\overrightarrow{OM} = y\overrightarrow{U_y},\quad \gamma = \frac{dv}{dt} \Rightarrow dv = \gamma dt$

After integration $v = \gamma t + v_0$

$\frac{dy}{dt} = \gamma t + v_0 \Rightarrow dy = \gamma t dt + v_0 dt \quad So\ \ y = \frac{1}{2}\gamma t^2 + v_0 t$

$\overrightarrow{O'M} = \left(\frac{1}{2}\gamma t^2 + v_0 t\right)\overrightarrow{U_y}$

Since, (O'y)//(oy) then $\overrightarrow{U_y} = \vec{j}$ So $\overrightarrow{O'M} = \left(\frac{1}{2}\gamma t^2 + v_0 t\right)\overrightarrow{U_y} = \left(\frac{1}{2}\gamma t^2 + v_0 t\right)\vec{j}$

Finally $\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M} = \mathrm{R}(\cos\ \omega t\,\vec{\imath} + \mathrm{Sin}\,\omega t\,\vec{\jmath}) + \left(\frac{1}{2}\gamma t^2 + v_0 t\right)\vec{j}$

$$\Rightarrow \overrightarrow{OM} = \mathrm{R}\cos\ \omega t\,\vec{\imath} + \left(r\,\mathrm{Sin}\,\omega t + \frac{1}{2}\gamma t^2 + v_0 t\right)\vec{j}$$

Absolute velocity :

$$\overrightarrow{v_a} = \frac{d\overrightarrow{OM}}{dt}/(R) = -\mathrm{r}\,\omega\sin\ \omega t\,\vec{\imath} + (r\omega\,\mathrm{Cos}\,\omega t + \gamma t + v_0)\vec{j}$$

Absolute accelertion :

$$\overrightarrow{a_a} = \frac{d\overrightarrow{v_a}}{dt}/(R) = -\mathrm{r}\,\omega^2\mathrm{Cos}\ \omega t\,\vec{\imath} + (-r\omega^2\,\mathrm{Sin}\,\omega t + \gamma\,)\vec{j}$$

Relative velocity :

$$\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}/(R') = (\gamma t + v_0)\vec{j}$$

Entrainement velocity :

$$\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$$

$\vec{\omega}\Lambda\overrightarrow{O'M} = \vec{0}$ Because the unit vectors of the two marks are parallel, so there is no rotational movement.

There is a translational movement

$\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} = -\mathrm{r}\,\omega\sin\ \omega t\,\vec{\imath} + r\omega\,\mathrm{Cos}\,\omega t\,\vec{j}$

Let's check that : $\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e}$

$\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} = (\gamma t + v_0)\vec{j} - \mathrm{R}\,\omega\sin\ \Omega t\,\vec{\imath} + r\omega\,\mathrm{Cos}\,\omega t\,\vec{j}$

$= -\mathrm{r}\,\omega\sin\ \omega t\,\vec{\imath} + (r\omega\,\mathrm{Cos}\,\omega t + \gamma t + v_0)\vec{j}$

So $\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e}$ Is verified.

Relative acceleration :

$\overrightarrow{a_r} = \left(\frac{d\overrightarrow{v_r}}{dt}\right) / R' \; with \; \overrightarrow{v_r} = (\gamma t + v_0)\vec{j}$ So $\overrightarrow{a_r} = \gamma\vec{j}.$

Entrainement acceleration :

$$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) + \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$$

$\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0} \;\; and \;\; \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) = \vec{0}$

Because there is a translational movement between the reference marks.

$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} = -\mathrm{r}\,\omega^2\mathrm{Cos}\;\;\omega\mathrm{t}\,\vec{\imath} + -r\omega^2\,\mathrm{Sin}\,\omega\mathrm{t}\,\vec{\jmath}$ So $\overrightarrow{a_e} = -\omega^2 r_0(\mathrm{Cos}\;\;\omega\mathrm{t}\vec{\imath} + \mathrm{Sin}\,\omega\mathrm{t}\,\vec{\jmath})$

Coriolis acceleration :

$$\overrightarrow{a_c} = \; 2\vec{\omega}\Lambda\overrightarrow{v_r} = \vec{0}$$

Let's check that : $\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e}$

$$\overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e} = \;\gamma\vec{\jmath} + -\mathrm{r}\,\omega^2\mathrm{Cos}\;\;\omega\mathrm{t}\,\vec{\imath} \pm r\omega^2\,\mathrm{Sin}\,\omega\mathrm{t}\,\vec{\jmath}$$

$= -\mathrm{r}\,\omega^2\mathrm{Cos}\;\;\omega\mathrm{t}\,\vec{\imath} + (-r\omega^2\,\mathrm{Sin}\,\omega\mathrm{t} + \gamma)\vec{\jmath}$

So $\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e}$ Is verified

Exercise 11

$$r = r_0(\mathrm{Cos}\;\;\omega\mathrm{t} + \mathrm{Sin}\,\omega\mathrm{t})\,\overrightarrow{\mathrm{U_X}}$$

Relative velocity: $\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}/(R')$

O' is confused with O, then : $\overrightarrow{OM} = \overrightarrow{O'M} \Rightarrow \overrightarrow{v_r} = \frac{d\overrightarrow{OM}}{dt}/(R')$

$$\overrightarrow{v_r} = r_0\omega(-sin\omega\mathrm{t} + \mathrm{Cos}\,\omega\mathrm{t})\,\overrightarrow{\mathrm{U_X}}$$

Entrainment velocity : $\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$ With $\overrightarrow{OO'} = \vec{0} \Rightarrow \frac{d\overrightarrow{OO'}}{dt} = \vec{0}$

$$\vec{\omega}\Lambda\overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ r & 0 & 0 \end{vmatrix} = \omega\, r\,\overrightarrow{U_y} \;\; so\; \overrightarrow{v_e} = \omega\, r\,\overrightarrow{U_y} = \omega\, r_0(\mathrm{Cos}\;\;\omega\mathrm{t} + \mathrm{Sin}\,\omega\mathrm{t})\,\overrightarrow{\mathrm{U_Y}}$$

Absolute velocity : $\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} = r_0\omega[(-sin\omega t + \text{Cos }\omega t)\overrightarrow{U_x} + (\text{Cos }\ \Omega t + \text{Sin }\omega t)\overrightarrow{U_y}]$

$$\Rightarrow \lceil\overrightarrow{v_a}\rceil = r_0\omega\sqrt{(-sin\omega t + \text{Cos }\Omega t)^2 + (\text{Cos }\ \omega t + \text{Sin }\omega t)^2}$$

Then $\lceil\overrightarrow{v_a}\rceil = r_0\omega\sqrt{2}$ So $\lceil\overrightarrow{v_a}\rceil$ Is constant

Relative acceleration : $\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}/(R')\ \overrightarrow{v_r} = r_0\omega(-sin\omega t + \text{Cos }\omega t)\ \overrightarrow{\text{U}_\text{X}}$

So $\overrightarrow{a_r} = r_0\omega^2(-\text{Cos }\omega t - sin\omega t)\ \overrightarrow{\text{U}_\text{X}}$

Entrainment acceleration :

$$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) + \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$$

With $\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0}$ Becauseω constantand $\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$

And $\vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) = \vec{\omega}\Lambda\left(\omega r\overrightarrow{U_y}\right) = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ 0 & \omega r & 0 \end{vmatrix} = -\omega^2 r\overrightarrow{U_x}$

Then$\overrightarrow{a_e} = -\omega^2 r_0(\text{Cos }\ \omega t + \text{Sin }\omega t)\overrightarrow{U_x}$

Coriolis acceleration: $\overrightarrow{a_c} = 2\vec{\omega}\Lambda\overrightarrow{v_r} = 2\begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ v_r & 0 & 0 \end{vmatrix} = 2\omega v_r\overrightarrow{U_y}$

So$\overrightarrow{a_c} = 2\omega v_r\overrightarrow{U_y} = 2r_0\omega^2(-sin\omega t + \text{Cos }\omega t)\overrightarrow{U_y}$

Absolueacceleration :$\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e}$

$$\overrightarrow{a_a} = -r_0\omega^2(\text{Cos }\omega t + sin\omega t)\ \overrightarrow{\text{U}_\text{X}} - \omega^2 r_0(\text{Cos }\ \omega t + \text{Sin }\omega t)\overrightarrow{U_x} + 2r_0\omega^2(-sin\omega t + \text{Cos }\omega t)\overrightarrow{U_y}$$

$$\Rightarrow \overrightarrow{a_a} = -2r_0\omega^2(\text{Cos }\ \omega t + \text{Sin }\omega t)\overrightarrow{U_x} + 2r_0\omega^2(-sin\omega t + \text{Cos }\omega t)\overrightarrow{U_y}$$

$$\lceil\overrightarrow{a_a}\rceil = 2r_0\omega^2\sqrt{(-(\text{Cos }\ \omega t + \text{Sin }\omega t))^2 + (-sin\omega t + \text{Cos }\omega t)^2}$$

Then, $\lceil\overrightarrow{a_a}\rceil = 2\ r_0\omega^2\sqrt{2}$ so $\lceil\overrightarrow{a_a}\rceil$ Is constant.

Exercise 12

1- Speeds:

M moves along the (Oy') axis with constant acceleration, so: $\overrightarrow{O'M} = Y\,\overrightarrow{u_y}$ with $\gamma = \frac{dv}{dt}$ and at t=0 the point M is at O' :

$\gamma = \frac{dv}{dt} \Rightarrow \int_0^v dv = \gamma \int_0^t dt$ $so\ v = \gamma t$ (at t=0, v_0 (M)=0)

$v = \gamma t = \frac{dY}{dt} \Rightarrow \int_0^Y dY = \gamma \int_0^t tdt$ so $Y = \frac{1}{2}\gamma t^2$ (at t=0, Y_0 (M)=0)

$\overrightarrow{O'M} = \frac{1}{2}\gamma t^2\,\overrightarrow{u_y}$

O' moves on Ox with a constant speed v_0 so $\overrightarrow{OO'} = x\vec{\imath}\ and\ v_0 = \frac{dx}{dt}$ and à t=0, axis (O'x') is confused with (Ox).

$v_0 = \frac{dx}{dt} \Rightarrow \int_0^x dx = v_0 \int_0^t dt$ so $x = v_0 t$ (at t=0, x_0 (O')=0) then $\overrightarrow{OO'} = v_0 t\vec{\imath}$

$\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt} = \gamma\, t\,\overrightarrow{\mathbf{u_y}}$

$\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega} \wedge \overrightarrow{O'M}$ with $\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$

$\overrightarrow{u_x} = cos\theta\vec{\imath} + sin\theta\vec{\jmath}\ and\ \overrightarrow{u_y} = -sin\theta\vec{\imath} + cos\theta\vec{\jmath}$

Using the passage table :

So $\vec{\imath} = \cos\theta\,\overrightarrow{u_x} - \sin\theta\,\overrightarrow{u_y}$

	$\overrightarrow{u_x}$	$\overrightarrow{u_y}$
$\vec{\imath}$	Cosθ	-sin θ
$\vec{\jmath}$	Sinθ	cosθ

$\frac{d\overrightarrow{OO'}}{dt} = v_0\,\vec{\imath} = v_0\left(\cos\omega t\,\overrightarrow{u_x} - \sin\omega t\,\overrightarrow{u_y}\right)$

$$\vec{\omega} \wedge \overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{u_x} & \overrightarrow{u_y} & \overrightarrow{u_z} \\ 0 & 0 & \omega \\ 0 & \frac{1}{2}\gamma t^2 & 0 \end{vmatrix} = -\frac{1}{2}\gamma t^2\omega\,\overrightarrow{u_x}$$

$\overrightarrow{\mathbf{v_e}} = \left(-\frac{1}{2}\gamma t^2\omega + v_0\cos\omega t\right)\overrightarrow{\mathbf{u_x}} + (-v_0\sin\omega t)\,\overrightarrow{\mathbf{u_y}}$

$\overrightarrow{\mathbf{v_a}} = \overrightarrow{\mathbf{v_r}} + \overrightarrow{\mathbf{v_e}} = \left(-\frac{1}{2}\gamma t^2\omega + v_0\cos\omega t\right)\overrightarrow{\mathbf{u_x}} + (\gamma t - v_0\sin\omega t)\,\overrightarrow{\mathbf{u_y}}$

2- The accelerations :

$\overrightarrow{\mathbf{a_r}} = \frac{\overrightarrow{\mathbf{dv_r}}}{\mathbf{dt}} = \gamma\overrightarrow{\mathbf{u_y}}$

$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M} + \vec{\omega} \wedge \vec{\omega} \wedge \overrightarrow{O'M}$ with $\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$

$$\vec{\omega} \therefore \vec{\omega} \therefore \overrightarrow{O'M} = \begin{vmatrix} \vec{u_x} & \vec{u_y} & \vec{u_z} \\ 0 & 0 & \omega \\ -\frac{1}{2}\gamma t^2\omega & 0 & 0 \end{vmatrix} = -\frac{1}{2}\gamma t^2\omega^2\,\vec{u_y}$$

$$\vec{a_e} = -\frac{1}{2}\gamma t^2\omega^2\,\vec{u_y}$$

$$\vec{\mathbf{a_c}} = \mathbf{2}\vec{\omega} \therefore \vec{\mathbf{v_r}} = \begin{vmatrix} \vec{u_x} & \vec{u_y} & \vec{u_z} \\ 0 & 0 & 2\omega \\ 0 & \gamma t & 0 \end{vmatrix} = -2\gamma t\,\boldsymbol{\omega}\,\vec{\mathbf{u_x}}$$

$$\vec{a_a} = \vec{a_r} + \vec{a_e} + \vec{a_c}$$

So $\vec{\mathbf{a_a}} = (-2\gamma t\,\omega)\,\vec{u_x} + \left(\gamma - \frac{1}{2}\gamma t^2\omega^2\right)\vec{u_y}$

Exercise 13

$$r = r_0(\cos\ \omega t + \sin\omega t)\,\overrightarrow{U_x}$$

Relative velocity

$$\vec{v_r} = \frac{d\overrightarrow{O'M}}{dt}/(R') \text{ with } \overrightarrow{O'M} = \overrightarrow{O'M} \Rightarrow \vec{v_r} = \frac{d\overrightarrow{OM}}{dt}/(R')$$

$$\vec{v_r} = r_0\omega(-sin\omega t + \cos\omega t)\,\overrightarrow{U_x}$$

Entrainment velocity

$$\vec{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M} \quad \text{with} \quad \overrightarrow{OO'} = t\vec{\imath} \Rightarrow \frac{d\overrightarrow{OO'}}{dt} = \vec{\imath}$$

$$\vec{\omega}\Lambda\overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ r & 0 & 0 \end{vmatrix} = \omega\, r\,\overrightarrow{U_y}$$

So $\vec{v_e} = \vec{\imath} + \omega\, r\,\overrightarrow{U_y} = \left(\cos\theta\overrightarrow{U_x} - sin\,\theta\,\overrightarrow{U_y}\right) - \omega\, r_0(\cos\ \omega t + \sin\omega t)\,\overrightarrow{U_y}$

$$\vec{v_e} = \cos\theta\overrightarrow{U_x} - (sin\theta + \omega\, r_0(\cos\ \omega t + \sin\omega t)\,\overrightarrow{U_y}$$

Absolute velocity

$$\vec{v_a} = \vec{v_r} + \vec{v_e} = [\left(\cos\omega t + r_0\omega(-sin\omega t + \cos\omega t)\right)\overrightarrow{U_x} - (sin\theta + r_0\omega(\cos\ \omega t + \sin\omega t)\overrightarrow{U_y}]$$

Absolute acceleration

$$\vec{a_r} = \frac{d\vec{v_r}}{dt}/(R') \quad \text{with} \quad \vec{v_r} = r_0\omega(-sin\omega t + \cos\omega t)\,\overrightarrow{U_x}$$

So $\vec{a_r} = r_0\omega^2(-\cos\omega t - sin\omega t)\,\overrightarrow{U_x}$

Entrainment acceleration

$$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) + \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$$

$$\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0} \text{ because } \omega \text{ constant and } \frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$$

And $\vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) = \vec{\omega}\Lambda\left(\omega\, r\, \overrightarrow{U_y}\right) = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ 0 & \omega\, r & 0 \end{vmatrix} = -\omega^2\, r\overrightarrow{U_x}$

so $\overrightarrow{a_e} = -\omega^2 r_0(\cos\ \omega t + \sin \omega t)\overrightarrow{U_x}$

Coriolis acceleration

$\overrightarrow{a_c} = 2\vec{\omega}\Lambda\overrightarrow{v_r} = 2\begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ v_r & 0 & 0 \end{vmatrix} = 2\,\omega v_r\,\overrightarrow{U_y}$

so $\overrightarrow{a_c} = 2\,\omega v_r\,\overrightarrow{U_y} = 2r_0\omega^2(-sin\omega t + \cos \omega t)\overrightarrow{U_y}$

Absolute acceleration

$$\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e}$$

$$\overrightarrow{a_a} = r_0\omega^2(-\cos \omega t - sin\omega t)\,\overrightarrow{\mathrm{U_x}} - \omega^2 r_0(\cos\ \omega t + \sin \omega t)\overrightarrow{U_x} + 2r_0\omega^2(-sin\omega t + \cos \omega t)\overrightarrow{U_y}$$

$\Rightarrow \overrightarrow{a_a} = -2r_0\omega^2(\cos\ \omega t + \sin \omega t)\overrightarrow{U_x} + 2r_0\omega^2(-sin\omega t + \cos \omega t)\overrightarrow{U_y}$

Exercise 14

$\overrightarrow{OO'} = R(\mathrm{Cos}\,\omega' t\,\vec{\imath} + \mathrm{Sin}\,\omega' t\,\vec{\jmath})$ *and* $\overrightarrow{O'M} = d(\mathrm{Cos}\,\Omega t\,\vec{\imath} + \mathrm{Sin}\,\Omega t\,\vec{\jmath})$

$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M} = R(\mathrm{Cos}\,\omega' t\,\vec{\imath} + \mathrm{Sin}\,\omega' t\,\vec{\jmath}) + d(\mathrm{Cos}\,\Omega t\,\vec{\imath} + \mathrm{Sin}\,\Omega t\,\vec{\jmath})$

Absoltue velocity

$\overrightarrow{v_a} = \frac{d\overrightarrow{OM}}{dt}/(R) = \frac{d}{dt}(\overrightarrow{OO'} + \overrightarrow{O'M})$

The axes (O'X), (O'Y) (moving reference) are parallel with the axes (Ox), (Oy) (fixed reference), therefore:

$$\vec{\imath} = \vec{\imath}'\, and\ \vec{\jmath} = \vec{\jmath}'$$

$\overrightarrow{v_a} = \mathrm{R}\,\omega' \left(-\sin \omega' t\,\vec{\imath} + \mathrm{Cos}\,\omega' t\,\vec{\jmath}\right) + d.\Omega\ \left(-\sin \Omega t\,\vec{\imath} + \mathrm{Cos}\,\Omega t\,\vec{\jmath}\right)$

$\Rightarrow \overrightarrow{v_a} = -(\mathrm{R}\,\omega' \sin \omega' t + d.\Omega \sin \Omega t)\vec{\imath} + (\mathrm{R}\,\omega' \cos \omega' t + d.\Omega \cos \Omega t)\vec{\jmath}$

Relative velocity

$$\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}/(R') = d.\Omega\ \left(-\sin \Omega t\,\vec{\imath} + \mathrm{Cos}\,\Omega t\,\vec{\jmath}\right)$$

Entrainment velocity

$\overrightarrow{v_e} = \dfrac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$

$\vec{\omega}\Lambda\overrightarrow{O'M} = \vec{0}$ Because the unit vectors of the two frames of reference are parallel, so there is a translational movement and not a rotational movement of axes of moving reference frame.

$\overrightarrow{v_e} = \dfrac{d\overrightarrow{OO'}}{dt} = -\mathrm{R}\,\omega' \sin\ \omega' t\,\vec{\imath} + R\omega' \,\mathrm{Cos}\,\omega' t\,\vec{\jmath}$

Absolute velocity

$$\begin{aligned}\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} &= d.\Omega\ \left(-\sin \Omega t\,\vec{\imath} + \mathrm{Cos}\,\Omega t\,\vec{\jmath}\right) + \mathrm{R}\,\omega' \left(-\sin \omega' t\,\vec{\imath} + \mathrm{Cos}\,\omega' t\,\overrightarrow{\jmath}\right)\\ &= -(\mathrm{R}\,\omega' \,\mathrm{Sin}\,\omega' t + d\Omega\,\mathrm{Sin}\,\Omega t)\,\vec{\imath} + (\mathrm{R}\,\omega' \,\mathrm{Cos}\,\omega' t + d\Omega \cos \Omega t)\vec{\jmath}\end{aligned}$$

So $\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e}$ Est is verifiesd

Absolute acceleration

$$\overrightarrow{a_a} = \frac{d\overrightarrow{v_a}}{dt}/(R) = \frac{d}{dt}\left[-(\mathrm{R}\,\omega' \,\mathrm{Sin}\,\omega' t + d\Omega\,\mathrm{Sin}\,\Omega t)\,\vec{\imath} + (\mathrm{R}\,\omega' \,\mathrm{Cos}\,\omega' t + d\Omega \cos \Omega t)\vec{\jmath}\right]$$

$$\overrightarrow{a_a} = -(\mathrm{R}\omega'^2 \mathrm{Cos}\,\omega' t + d\Omega^2 \mathrm{Cos}\,\Omega t)\vec{\imath} - (\mathrm{R}\omega'^2 \mathrm{Sin}\,\omega' t + d.\Omega^2 \mathrm{Sin}\,\Omega t)\vec{\jmath}$$

Relative acceleration

$\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}/(R')$ with $\overrightarrow{v_r} = d.\Omega\ \left(-\sin \Omega t\,\vec{\imath} + \mathrm{Cos}\,\Omega t\,\vec{\jmath}\right)$

Donc $\overrightarrow{a_r} = d.\Omega^2\ \left(-\cos \Omega t\,\vec{\imath} - sin\,\Omega t\,\vec{\jmath}\right)$

Entrainment acceleration

$\overrightarrow{a_e} = \dfrac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) + \dfrac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$

$\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0}$ *and* $\vec{\omega}\Lambda(\vec{\omega}\Lambda\overrightarrow{O'M}) = \vec{0}$

Because there is a translation movement between the two references

$$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} = -\mathrm{R}\,\omega'^2\mathrm{Cos}\ \omega'\mathrm{t}\,\vec{\imath} + -R\omega'^2\,\mathrm{Sin}\,\omega'\mathrm{t}\,\vec{\jmath}$$

So $\overrightarrow{a_e} = -\omega'^2 R(\mathrm{Cos}\ \omega'\mathrm{t}\,\vec{\imath} + \mathrm{Sin}\,\omega'\mathrm{t}\,\vec{\jmath})$

Coriolis accélération

$$\overrightarrow{a_c} = 2\vec{\omega}\Lambda\overrightarrow{v_r} = \vec{0}$$

Absolute accélération

$$\overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e} = d.\Omega^2\ (-\mathrm{Cos}\,\Omega\mathrm{t}\ \vec{\imath} - sin\,\Omega\mathrm{t}\,\vec{\jmath}) + -\mathrm{R}\,\omega'^2\mathrm{Cos}\ \omega'\mathrm{t}\,\vec{\imath} \pm R\omega'^2\,\mathrm{Sin}\,\omega'\mathrm{t}\,\vec{\jmath}$$
$$= -(\mathrm{R}\omega'^2\,\mathrm{Cos}\,\omega'\mathrm{t} + d.\Omega^2\mathrm{Cos}\,\Omega\mathrm{t})\vec{\imath} - (\mathrm{R}\omega'^2\,\mathrm{Sin}\,\omega'\mathrm{t} + d.\Omega^2\mathrm{Sin}\,\Omega\mathrm{t})\vec{\jmath}$$

So $\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_c} + \overrightarrow{a_e}$ is vérified

Exercise 15

The axes (Ox) and (O’x’) are not parallel, and the rotation is along the (Oz) axis.

The expression of (OM) vector in the fixed frame (Oxy).

$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$ with R=O’M=OO’

$$\overrightarrow{OO'} = R(\mathrm{Cos}\,\omega t\,\vec{\imath} + \mathrm{Sin}\,\omega t\,\vec{\jmath})\ \ and\ \ \overrightarrow{O'M} = R\left(\mathrm{Cos}\,\omega t\,\vec{\imath'} + \mathrm{Sin}\,\omega t\,\vec{\jmath'}\right)$$

with $\vec{\imath'} = \mathrm{Cos}\,\omega t\,\vec{\imath} + \mathrm{Sin}\,\omega t\,\vec{\jmath}$ *and* $\vec{\jmath'} = -\mathrm{sin}\,\omega t\,\vec{\imath} + \mathrm{Cos}\,\omega t\,\vec{\jmath}$

So $\overrightarrow{O'M} = R\big(\mathrm{Cos}\,\omega t(\mathrm{Cos}\,\omega t\,\vec{\imath} + \mathrm{Sin}\,\omega t\,\vec{\jmath}) + sin\,\omega t(-\mathrm{sin}\,\omega t\,\vec{\imath} + \mathrm{Cos}\,\omega t\,\vec{\jmath})\big)$

$$\Rightarrow \overrightarrow{O'M} = R(\mathrm{Cos}\,2\omega t\vec{\imath} + sin\,2\omega t\vec{\jmath})$$

So $\overrightarrow{OM} = R[(\mathrm{Cos}\,\omega t\,\vec{\imath} + \mathrm{Sin}\,\omega t\,\vec{\jmath}) + (\mathrm{Cos}\,2\omega t\vec{\imath} + sin\,2\omega t\vec{\jmath})]$

$$\Rightarrow \overrightarrow{O'M} = R[(\mathrm{Cos}\,2\omega t + \mathrm{Cos}\,\omega t)\vec{\imath} + (sin2\omega t + \mathrm{sin}\,\omega t)\,\vec{\jmath})]$$

"The expression of absolute velocity and absolute acceleration in the fixed frame."

Absolute velocity

$$\overrightarrow{v_a} = \frac{d\overrightarrow{OM}}{dt}/(R) \ = \ \frac{d}{dt}(\overrightarrow{OO'} + \overrightarrow{O'M})$$

$$\Rightarrow \overrightarrow{v_a} = R[(-\ \omega sin\omega t - 2\omega \operatorname{Sin} 2\omega t)\vec{\imath} + (\omega \operatorname{Cos} \omega t + 2\omega cos\omega t)\ \vec{\jmath}]$$

Absolute accélération

$$\overrightarrow{a_a} = \frac{d\overrightarrow{v_a}}{dt}/(R) \ = \ \frac{d}{dt}(R[(-\ \omega sin\omega t - 2\omega \operatorname{Sin} 2\omega t)\vec{\imath} + (\omega \operatorname{Cos} \omega t + 2\omega cos\omega t)\ \vec{\jmath}])$$

$$\Rightarrow \overrightarrow{a_a} = R[(-\ \omega^2 cos\omega t - 2\omega^2 \operatorname{Cos} 2\omega t)\vec{\imath} - (\omega^2 \operatorname{Sin} \omega t + 2\omega^2 sin\omega t)\ \vec{\jmath}]$$

"The expression of relative velocity and relative acceleration in the fixed frame."

Relative velocity

$$\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}/R' = \ R\,\omega\ (-\sin \omega \mathrm{t}\,\overrightarrow{\imath'} \ + \operatorname{Cos} \omega \mathrm{t}\,\overrightarrow{\jmath'})$$

Relative acceleration

$$\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}/R' = \ -R\,\omega^2\ (\cos \omega \mathrm{t}\,\overrightarrow{\imath'} \ + \operatorname{Sin} \omega \mathrm{t}\,\overrightarrow{\jmath'})$$

Entrainment velocity

$$\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M} \quad \text{with } \frac{d\overrightarrow{OO'}}{dt} = \ R\,\omega\ (-\sin \omega \mathrm{t}\,\vec{\imath} \ + \operatorname{Cos} \omega \mathrm{t}\,\vec{\jmath})$$

$$\vec{\omega}\Lambda\overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{\imath'} & \overrightarrow{\jmath'} & \overrightarrow{k'} \\ 0 & 0 & \omega \\ Rcos\omega t & Rsin\omega t & 0 \end{vmatrix} = \ R\,\omega\ (-\sin \omega \mathrm{t}\,\overrightarrow{\imath'} \ + \operatorname{Cos} \omega \mathrm{t}\,\overrightarrow{\jmath'})$$

$$\Rightarrow \overrightarrow{v_e} = \ R\,\omega\ (-\sin \omega \mathrm{t}\,\vec{\imath} \ + \operatorname{Cos} \omega \mathrm{t}\,\vec{\jmath}) + R\,\omega\ (-\sin \omega \mathrm{t}\,\overrightarrow{\imath'} \ + \operatorname{Cos} \omega \mathrm{t}\,\overrightarrow{\jmath'})$$

$$\Rightarrow \overrightarrow{v_e} = \ R\,\omega\ (-\sin \omega \mathrm{t}\,\vec{\imath} \ + \operatorname{Cos} \omega \mathrm{t}\,\vec{\jmath}) + R\,\omega\ (-\sin \omega \mathrm{t}\,(\operatorname{Cos} \omega t\,\vec{\imath} + \operatorname{Sin} \omega t\,\vec{\jmath}) + \operatorname{Cos} \omega \mathrm{t}\ \ (-\sin \omega \mathrm{t}\,\vec{\imath} \ + \operatorname{Cos} \omega \mathrm{t}\,\vec{\jmath}))$$

$$\Rightarrow \overrightarrow{v_e} = \ R\,\omega\ [(-\sin \omega \mathrm{t}\,\vec{\imath} \ + \operatorname{Cos} \omega \mathrm{t}\,\vec{\jmath}) + (-\sin \omega \mathrm{t}\cos\omega \mathrm{t} - \operatorname{Cos}^2\omega \mathrm{t})\,\vec{\imath} + (-\operatorname{Sin}^2\omega \mathrm{t} + \sin\omega \mathrm{t}\,\cos\omega \mathrm{t})\vec{\jmath}]$$

Entrainment acceleration

$$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) + \frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$$

$$\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M} = \vec{0}$$

With $\frac{d^2\overrightarrow{OO'}}{dt^2} = -R\,\omega^2\,(\cos\omega\mathrm{t}\,\vec{\imath} + \mathrm{Sin}\,\omega\mathrm{t}\,\vec{\jmath})$

$\vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right) = \vec{\omega}\Lambda R\,\omega\,(-\sin\omega\mathrm{t}\,\vec{\imath'} + \mathrm{Cos}\,\omega\mathrm{t}\,\vec{\jmath'})$

$$= \begin{vmatrix} \vec{\imath'} & \vec{\jmath'} & \vec{k'} \\ 0 & 0 & \omega \\ -R\omega sin\omega t & R\omega cos\omega t & 0 \end{vmatrix}$$

$= -R\,\omega^2\,(\cos\omega\mathrm{t}\,\vec{\imath'} + \mathrm{Sin}\,\omega\mathrm{t}\,\vec{\jmath'})$

$\Rightarrow \overrightarrow{a_e} = -R\,\omega^2\,(\cos\omega\mathrm{t}\,\vec{\imath} + \mathrm{Sin}\,\omega\mathrm{t}\,\vec{\jmath}) - R\,\omega^2\,(\cos\omega\mathrm{t}\,\vec{\imath'} + \mathrm{Sin}\,\omega\mathrm{t}\,\vec{\jmath'})$

$$\Rightarrow \overrightarrow{a_e} = -R\,\omega^2\,(\cos\omega\mathrm{t}\,\vec{\imath} + \mathrm{Sin}\,\omega\mathrm{t}\,\vec{\jmath}) - R\,\omega^2\,(\cos\omega\mathrm{t}\,(\mathrm{Cos}\,\omega t\,\vec{\imath} + \mathrm{Sin}\,\omega t\,\vec{\jmath}) \\ + \mathrm{Sin}\,\omega\mathrm{t}\;(-\sin\omega\mathrm{t}\,\vec{\imath} + \mathrm{Cos}\,\omega\mathrm{t}\,\vec{\jmath}))$$

$\Rightarrow \overrightarrow{a_e} = -R\,\omega^2\,[(\cos\omega\mathrm{t}\,\vec{\imath} + \mathrm{Sin}\,\omega\mathrm{t}\,\vec{\jmath}) - (\mathrm{Cos}\,2\omega t\,\vec{\imath} + \mathrm{Sin}\,2\omega t\,\vec{\jmath})\,]$

Coriolis acceleration

$\overrightarrow{a_c} = 2\vec{\omega}\Lambda\overrightarrow{v_r}$

$$\Rightarrow \overrightarrow{a_c} = \begin{vmatrix} \vec{\imath'} & \vec{\jmath'} & \vec{k'} \\ 0 & 0 & \omega \\ -R\omega sin\omega t & R\omega cos\omega t & 0 \end{vmatrix} = -R\,\omega^2\,(\cos\omega\mathrm{t}\,\vec{\imath'} + \mathrm{Sin}\,\omega\mathrm{t}\,\vec{\jmath'})$$

University of Tlemcen
Faculty of Sciences
Department of Mathematics

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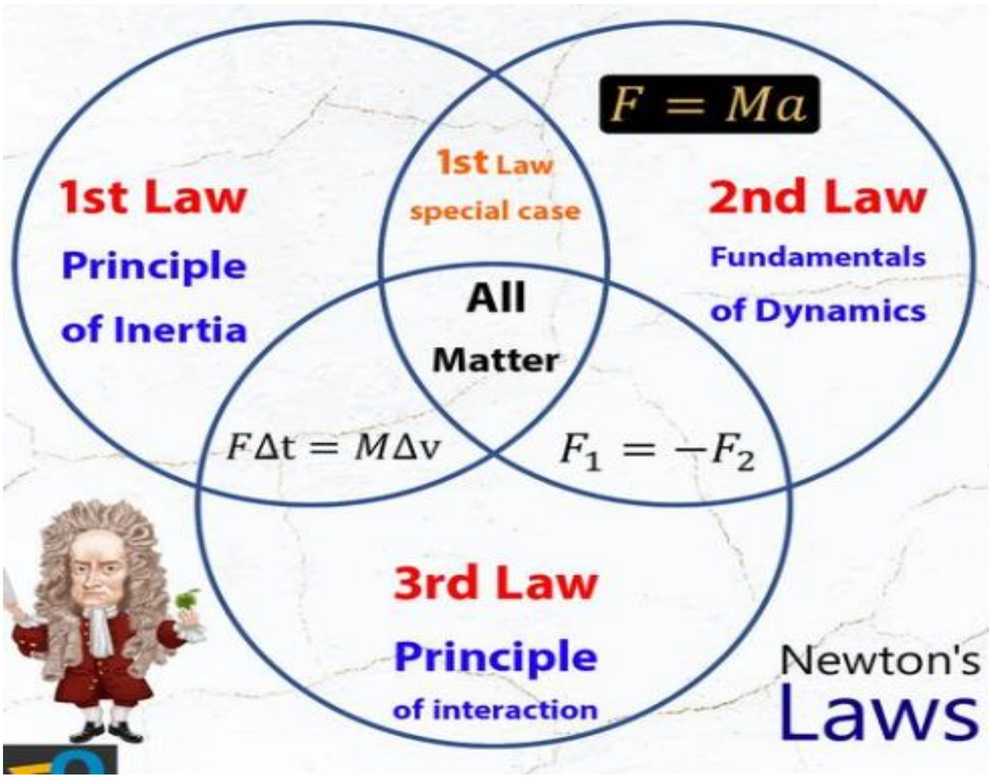
COURSE OF MECHANICS

OF THE MATERIAL POINT

Chapter V: Dynamics of a particle



Glossary

In English	In French	In Arabic
The momentum	La quantité du mouvement	كمية الحركة
Isolated system	Un système isolé	جملة معزولة
The principle of inertia	Principe d'inertie	مبدا العطالة
A free and isolated particle	Une particule libre et isolée	جسم معزول او حر
Fundamental Principle of Dynamics	Principe fondamental de la dynamique	المبدا الأساسي للتحريك
Newtonianmechanics	La mécanique newtonienne	ميكانيك نيوتن
Principle of action and reaction	Principe d'action et de réaction	مبدا الفغل و رد الفعل
Force of gravity-weight	La force de gravitation	الثقل او قوة الجادبية
Force ata distance	La force à distance	القوة عن بعد
Force électrique	La force électrique	القوة الكهربائية
Binding or contact forces	La force de réaction	قوة رد الفعل
Equilibrium	Equilibre	حالة التوازن
Friction forces	La force de frottement	قوة الاحتكاك
Static friction force	La force de frottement statique	قوة الاحتكاك في حالة السكون
The coefficient of static friction	Le coefficient de frottement statique	معامل الاحتكاك في حالة السكون
The coefficient of dynamic friction	Le coefficient de frottement dynamique	معامل الاحتكاك في حالة الحركة
Elastic forces	La force élastique	قوة الارجاع او قوة المرونية
The spring stiffness constant	La constante de raideur d'un ressort	ثابت المرونة لنابض

1. Introduction

In physics, dynamics is the science that studies the relationship between a body in motion and the causes of that motion. It also predicts the motion of a body located in a given environment. Dynamics, more precisely, is the analysis of the relationship between applied force and changes in body motion.

2. Newton's laws of motion

2.1. The momentum كمية الحركة

The momentum of a particle is the product of its mass and its instantaneous velocity vector.

$$\vec{P} = m\vec{v}$$

Experiments have shown that the momentum of a system composed of two particles, subject only to their mutual influences, remains constant.

Theorem:

"In an isolated system of two particles, the variation in the momentum of one particle over time is equal to and opposite in direction to the variation in the momentum of the other particle over the same time".

2.2. Newton's three laws

2.2.1. Galilean principle of inertia مبدا العطالة (Newton's first laws)

Newton's first law, also known as the law of inertia, states that any object continues to move at a constant speed in a straight line, or remains at rest, unless an external force is applied to it. In other words, if the material body is not subjected to any force, it is either in uniform rectilinear motion, or at rest, if it was initially at rest.

For a particle the principle of inertia thus states: "A free and isolated particle moves in rectilinear motion with constant velocity".

Note: A free particle always moves with a constant momentum (principle of inertia).

2.2.2. Newton's second law (Fundamental Principle of Dynamics) المبدا الأساسي للتحريك

In an abstract sense, force represents the effort required to modify a body's state of motion, in particular to modify its speed. Different bodies have different inertia, i.e. different resistance to a change in their state of inertia, and therefore different resistance to a change in their state of motion. This property must therefore be taken into account in the definition of force. To this end, we introduce a new physical quantity called the momentum of a body.

$\vec{P} = m\vec{v}$. Consequently, force can be defined by the derivative of momentum P. This means that the resultant of the forces applied to a particle is:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

This equation is called the "equation of motion»

$$\vec{F} = \frac{dm\vec{v}}{dt} \Rightarrow \vec{F} = m\frac{d\vec{v}}{dt} + \vec{v}\overset{=0}{\frac{dm}{dt}} = m\frac{d\vec{v}}{dt}$$

So, $\vec{F} = m\vec{a}$

This is because the mass m of the moving particle is constant (as is often the case in Newtonian mechanics).

In general, Newton's second law for a moving particle can be written as:

$$\sum \overrightarrow{F_{ext}} = m\vec{a}$$

In the S.I. system, the unit of force is: ***1Newton=1N=1kg.m.s^{-2}***.

Statement of the Fundamental Principle of Dynamics (2ndNewton law) :

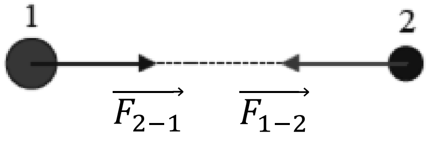
In a Galilean frame of reference, the sum of the external forces applied to a system is equal to the derivative of the momentum vector of the system's center of inertia.

2.2.3. Newton's third law or principle of action and reactionمبدا الفعل و رد الفعل

Newton's third law, often referred to as the law of action and reaction, states that for every action, there is an equal and opposite reaction. In other words, when two particles are under mutual influence, the force applied by the first particle on the second is equal to, and opposite in sign to, the force applied by the second particle on the first.

This is shown in the following figure, which allows us to write:

$$\overrightarrow{F_{1-2}} = -\overrightarrow{F_{2-1}}$$



$$\left|\overrightarrow{F_{1-2}}\right| = \left|\overrightarrow{F_{2-1}}\right|$$

3. Notion of force and law of force

The definition of force by the equation $\vec{F} = m\vec{a}$ allows us to express the force corresponding to the effect studied as a function of physical factors such as distance, mass, electric charge of the bodies....We will ultimately arrive at deriving "the law of force".

This law clearly shows the expression of the force (the resultant) applied to a material point in a well-defined situation.

3.1. Force of gravity "weight $\vec{p}$" الثقل او قوة الجادبية

It's gravitation that makes all the bodies in the universe attract each other. It's an attractive, long-range, low-amplitude force. The gravitational phenomenon is created by the interaction between two bodies. The force of gravity acting on a human being when on Earth is the result of the interaction between the earth and the human body. As the Earth is more imposing, the gravitational force pulls the human body towards the center of the Earth. This is gravity.

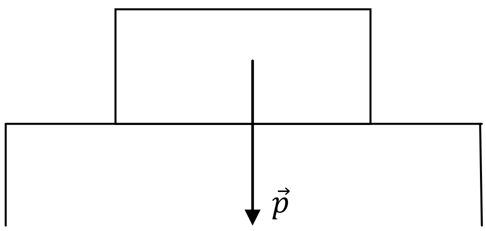
Mass (m) is the total amount of matter that makes up an object, while weight (p) is the result of the force of gravity (g) on mass. The mathematical formula is as follows :

$$p = m \times g.$$

The gravity field is represented at any point on the globe by the vector: $\vec{\boldsymbol{p}} = \boldsymbol{m}\vec{\boldsymbol{g}}$.

With $\vec{\boldsymbol{g}}$ is the gravity acceleration vector, it depends on the altitude and latitude at which the body is located. It is generally considered to be constant, and the value adopted, at mean sea level, is 9.81 $m.s^{-2}$.

Representation of the force of weight: $\vec{\boldsymbol{p}}$ is always vertical, and directed downwards.



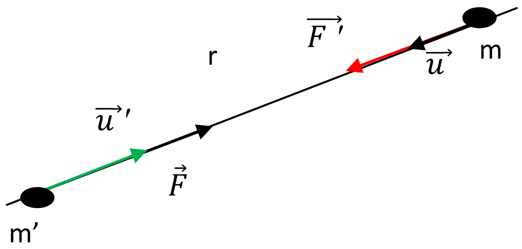
3.2. Force at a distance

Assume two bodies separated by a distance r, of mass m and m' respectively.

The attractive force exerted by m on m' is : $\vec{F} = \overrightarrow{F m_{/m'}} = -G\frac{mm'}{r^2}\vec{u}$

The attractive force exerted by m' on m is : $\vec{F'} = \overrightarrow{F m'_{/m}} = G\frac{mm'}{r^2}\vec{u}$

Then: $\vec{F} = -\vec{F}\,'$.



where G is a constant, the value of which is experimentally determined to be:

G = 6.673 × 10 $^{-11}$ N.m^2.kg^{-2}.

3.3. Force électrique القوة الكهربائية

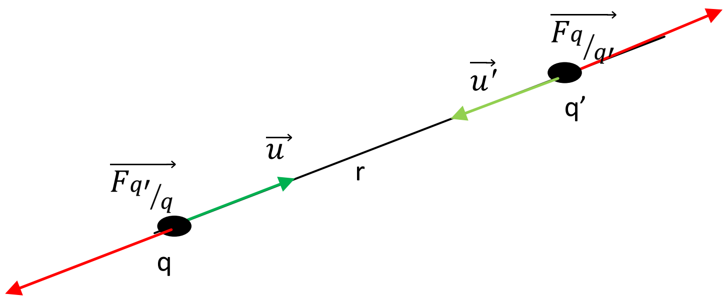
Consider two electric charges q and q' separated by a distance r. The electric force exerted by q on q' is given by:

$$\overrightarrow{F_{q/_{q'}}} = k\frac{qq'}{r^2}\overrightarrow{u}$$

With : k a constant

The electric force exerted by q' on q is given by :

$$\overrightarrow{F_{q'/_{q}}} = k\frac{qq'}{r^2}\overrightarrow{u'}$$



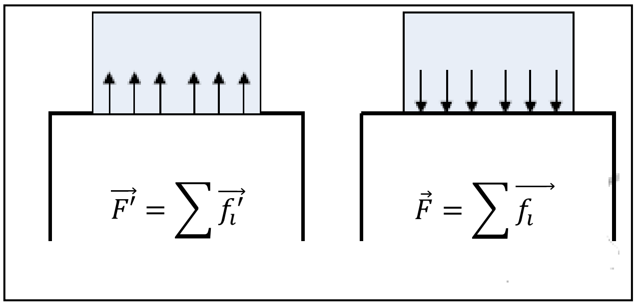
3.4. Binding or contact forces قوة رد الفعل

These are the forces acting mutually between bodies in contact.

Consider a solid body placed on a table. The body is in equilibrium on the table, i.e. the acceleration is zero ($\vec{a} = \vec{0}$).

Faced with the force $\vec{F}$, representing the resultant of all the interactions of the molecules making up the body, and applied to the table, the latter in turn applies the force F $^{\rightarrow}$'which is the resultant of all the interactions of the molecules making up the surface of the table that is in contact with the body. The two forces $\vec{F}$ and $\vec{F}'$are called contact or binding forces because of the contact between the two bodies.

With $\vec{F} = -\overrightarrow{F'}$and $|\vec{F}| = |\overrightarrow{F'}|$.



3.5. Friction forces القوة الاحتكاك

Whenever there is contact between the rough surfaces of two solid bodies, a resistance arises that opposes the relative movement of the two bodies. This resistance is called frictional force.

Friction is influenced by a number of factors. Consider the type of surface in contact. Smooth surfaces generally offer less friction than rough ones. Friction between solid bodies can be both static and dynamic.

a- Static friction force القوة الاحتكاك في حالة السكون

Static friction is the force that keeps a body at rest even in the presence of an external force.

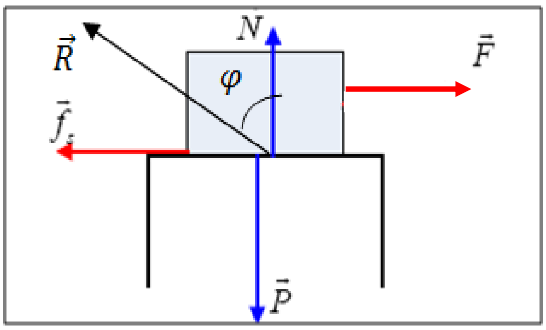
Example:

A body resting on a horizontal plane:

Consider the body shown in the figure below. It is subjected to four forces.

Let f_s , be the static friction force and $\vec{P}$ and $\vec{N}$ be the weight and normal reaction force of the support respectively.

For the body on the table to move, a minimum force $\vec{F}$ must be applied.



The mass remains stationary as long as F<fs, there is resistance to movement.

In this case the reaction of the support is the resultant force given by: $\vec{R} = \vec{N} + \vec{f_s}$

At equilibrium:

$$\sum \overrightarrow{F_{ext}} = \vec{0} \;\; \Rightarrow \vec{N} + \vec{f_s} + \vec{P} + \vec{F} = \vec{0}$$

$$\Rightarrow \vec{R} + \vec{P} + \vec{F} = \vec{0}$$

By projecting onto the two axes Ox and Oy:

On (Oy) : P=N and (Ox) : F=f_s

The mass starts moving when F>f_s.

Experience shows that the ratio (f_s/N) is constant.

$$tg\varphi = \frac{f}{N} = k = \mu$$

μ: is the coefficient of friction and φ is the angle of friction.

The coefficient of friction is called static when the body is stationary. The coefficient of static friction is a ratio between the static frictional force of an object and the normal force, and is written as follows :

$$tg\varphi = \frac{f_s}{N} = \mu_s$$

b- Dynamic friction force القوة الاحتكاك في حالة الحركة

Kinetic or dynamic friction is the frictional force present when an object is in motion on another object.

The dynamic friction coefficient is a ratio between the dynamic friction force of an object and the normal force.

Mass starts moving when F>f_d.

The coefficient of dynamic friction is written as :

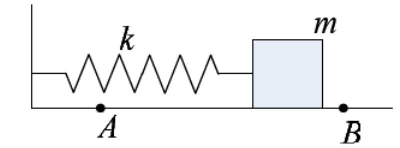
$$tg\varphi = \frac{f_d}{N} = \mu_d$$

Note :

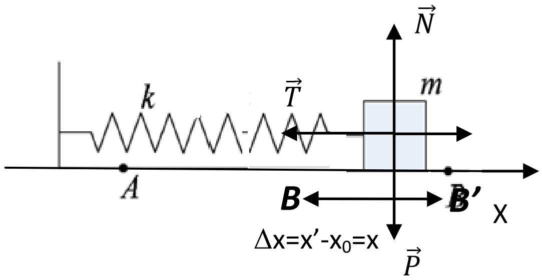
Experience has shown that the coefficient of static friction is greater than the coefficient of dynamic friction $\mu_s > \mu_d$.

3.6. Elastic forces قوة الارجاع او قوة المرونية

Elastic force is the force applied to an object that tends to return to its shape after being deformed. Elastic forces cause periodic movements. The most common is sinusoidal motion, as in the case of a spring.



(At equilibrium)



(Motion)

We have: FPD: $\sum\overrightarrow{F_{ext}}=m\vec{a}$

$\vec{N}+\vec{P}+\vec{T}=m\vec{a}$

M is the mass of the body

N is the reaction force of the support **قوة رد الفعل**

P is the force of the body weight **قوة الثقل**

T is the spring return force **قوة الارجاع**

By projection:

On (Ox) we have: $T=-kx=ma$

On (Oy) we have: N-P=0 So N=P=mg

Where k is the spring stiffness constant ثابت المرونة .

Proposed exercises about chapter V

Exercise 1

Consider a small block of mass m abandoned without initial velocity at point **A** of an inclined plane at an angle α=30° to the horizontal. Point **A** is at height **h**.

1- What is the value of the coefficient of static friction μ_s that keeps the mass in equilibrium at point **A**.

Exercise 2

A man pushes a 20 kg lawnmower with a force of 80N directed parallel to the handle, which is inclined at 30° to the horizontal.

1. If moving at constant speed, what is the modulus of the friction force due to the ground?
2. What force parallel to the handle would produce an acceleration of $1m/s$, given that the friction force is that found in question 1?

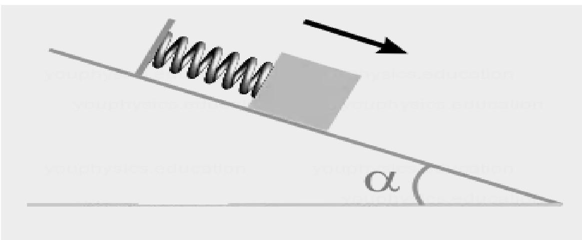
Exercise 3

A block of mass m ascends along a plane inclined by an angle α, with respect to the horizontal, with initial velocity v_0, and coefficient of friction μ_d.

1. Determine how far the block travels before coming to rest.
2. What is the maximum value that the static friction coefficient μ_s can take for the body to remain stationary.
3. For a value of the dynamic friction coefficient μ_d lower than the maximum value found in the second question, what is the velocity v_1 of the body when it returns to its starting position.

Exercise 4

A mass m = 15 kg suspended from a spring of stiffness K = 100N / m descends along an inclined plane which makes an angle α = 30º with the horizontal.



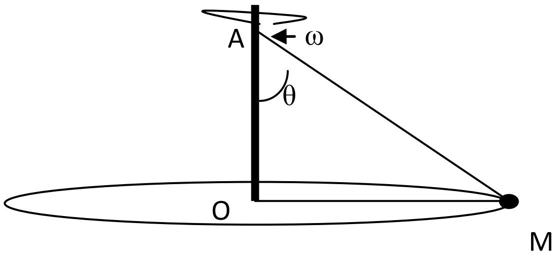
Assuming there is no friction, determine the normal reaction of the support and the acceleration of the mass when the spring is stretched by a length x = 0.02m.

Exercise 5

A ball of mass m is attached by two wires (Am and Om) to a vertical pole. The whole system rotates with a constant angular velocity ω around the axis of the post (we know g the acceleration of gravity, θ and $L = |\overrightarrow{OM}|$)

1. Assuming ω is large enough to keep both wires taut, find the force (wire tension) each wire exerts on the ball.

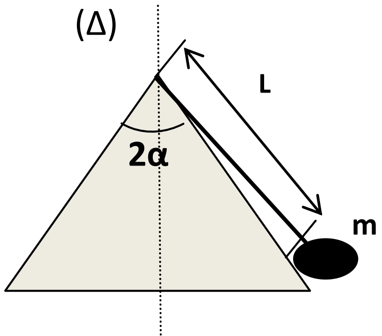
2. What is the minimum angular velocity ω_{min} for which the bottom wire remains taut?



Exercise 6

A body of mass (m=1kg) is attached by a wire of length L=30cm to the top of a cone, of axis (Δ) and angle at the top 2α=60°. This body rotates without friction on the surface of the cone with a rotational speed ω=10 rpm.(10.2π/60s)

1. Calculate the body's linear velocity.
2. Using the fundamental principle of dynamics, determine the reaction (R_N) of the cone surface on the body and the thread tension (T).



Exercise 7

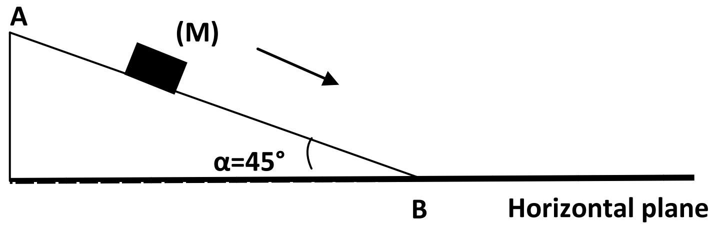
A block (M) of mass m is thrown from the top of an inclined plane AB=1m at an angle α=45° to the horizontal, with initial velocity v_A=1m/s.

1- Knowing that the coefficient of friction μ=0.5 on AB.

- Demonstrate, what is the nature of the motion on AB?
- Calculate the speed of (M) when it reaches point B.

2- Friction forces are considered negligible on the horizontal plane:

- Demonstrate the nature of the motion on the horizontal plane.
- Will the block (M) stop? Justify your answer.

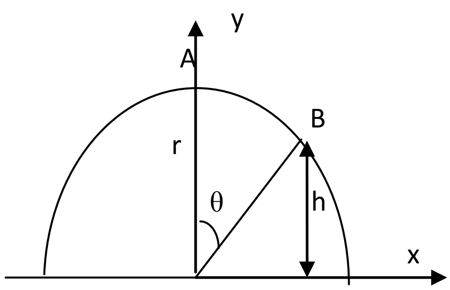


Exercise 8

A piece of ice M of mass m slides frictionlessly over the outer surface of an igloo, which is a half-sphere of radius r with a horizontal base.

At t=0, it is released from point A without any initial velocity.

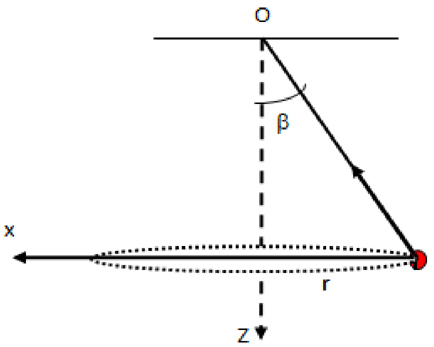
1- Find the expression for the velocity at point B, as a function of g, r and θ.
2- Using the fundamental relation of dynamics, determine the expression of $|\vec{N}|$ the reaction of the igloo on M at point B as a function of velocity v_B.
3- At what height does M leave the sphere?
4- At what speed does M arrive at the axis (Ox)?



Exercise 9

A material point, of mass m, is suspended at a fixed point O by a wire of length l inextensible and negligible mass. By rotating it around axis Oz, it acquires a constant angular velocity ω. It describes a horizontal circle of radius r.

1. Find the expression for the wire tension.
2. Find the expression for the inclination β of the wire with respect to the vertical.

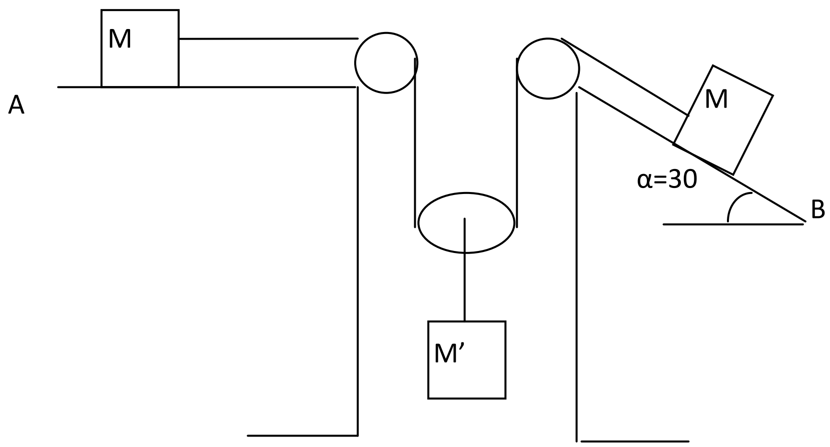


Exercise 10

Two carriages A and B of the same mass M are linked by a wire carrying a pulley of negligible mass. The axis of the pulley carries a mass M'.

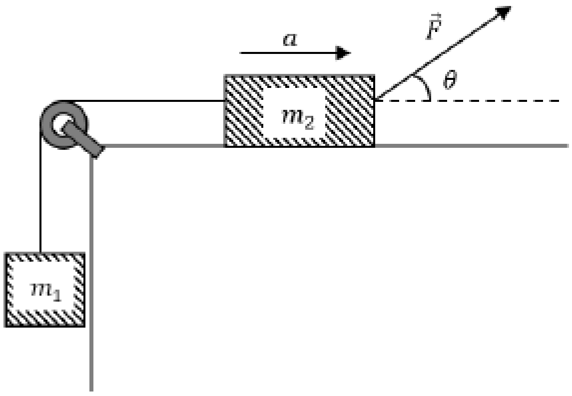
1- Neglecting all friction, calculate the ratio M'/M so that cart B remains stationary.

2- If M'=2M, calculate the accelerations to which the masses are subjected.



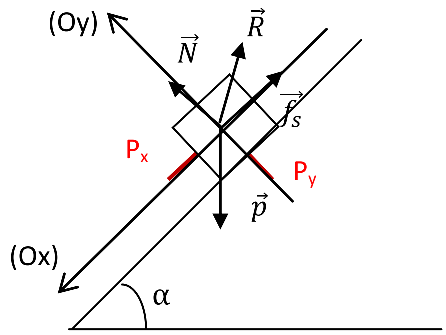
Exercise 11

A block of mass m_1 assimilated to a material point can slide on a horizontal surface with a coefficient of kinetic friction μ_d one of its ends is connected by an inextensible wire of negligible mass passing through a pulley of negligible mass connected to a second mass m_2. A force of modulus F is applied to m_2 at an angle θ to the horizontal. Find the accelerations of the two masses.



Exercises correction of about chapter V

Exercise 1



What is the value of the coefficient of static friction μs that keeps the mass in equilibrium at point A?

-At equilibrium:

$$\sum \overrightarrow{F_{ext}} = \vec{0} \;\; \Rightarrow \vec{N} + \vec{f_s} + \vec{P} = \vec{0} \Rightarrow \vec{R} + \vec{P} = \vec{0}$$

Following (Ox): $-f_s + p_x = 0 \Rightarrow f_s = m\,g\,\sin\alpha$

Following (Oy): $N - p_y = 0 \Rightarrow N = m\,g\,\cos\alpha$

In order for the body to remain stationary on the plane, the following conditions must be met $f_s > p_x$.

$$\text{We have } tg\varphi = \frac{f_s}{N} = \mu_s = \frac{\mathrm{m\,g\,sin\alpha}}{\mathrm{m\,g\,cos\alpha}} = tg\alpha$$

The maximum value that the coefficient of static friction μ_s can take is tgα.

Note: experience shows that: $\mu_s \geq \mu_d$

Exercise 2

Modulus of friction force:

M=20kg, $\alpha = 30°$ and F=80N.

1- FPD: $\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \overrightarrow{R_N} + \vec{F} + \vec{f} = m\vec{a}$

with v=cst so a=0.

Following (Ox): $F\cos\alpha - f = 0$

Following (Oy): $R_N - P = 0 \Rightarrow R_N = m\,g$

Then: f=80.cos.30=69.28N

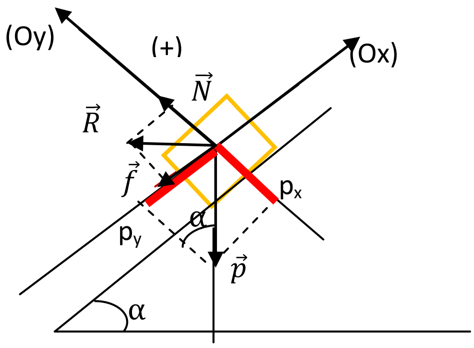
2- Force F for a=1m/s^2

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \overrightarrow{R_N} + \overrightarrow{F'} + \vec{f} = m\vec{a}$$

Following (Ox): -f+ F'cosα - f = m.a

$$F' = \frac{m.a + f}{cos\alpha} = 103.1N$$

Exercise 3



At: t=0 ,v=v_0 and μ= μ_d

1- Let's find out how far the block can travel before it stops.

According to the fundamental principle of dynamics:

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \vec{R} = \vec{p} + \vec{N} + \vec{f} = m\vec{a}$$

Initial velocity v_i= v_0 and final velocity v_f=0 (the body will stop)

We have: $\boldsymbol{v_f^2 - v_i^2 = 2al}$

(*l* being the distance covered by the body)

So ; $a = \frac{v_f^2 - v_i^2}{2l}$

The reference frame must be chosen so that the axis (Ox) follows the axis of motion, so it is parallel to $\vec{f}$and (Oy) is perpendicular to (Ox), so it is parallel to $\vec{N}$.

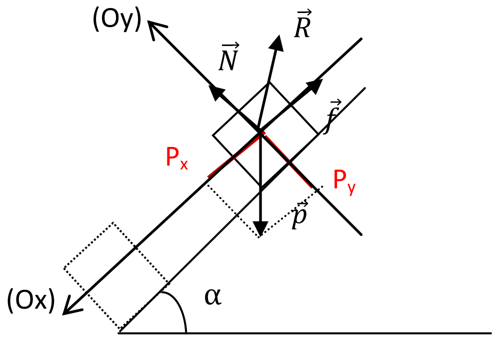
Following (Ox): – f- p_x= -f-m g sinα=ma

Following (Oy): N- p_y=0 ⇒ N=m g cosα

$\mu_d = tg\varphi = f/N \Rightarrow f = N\ tg\ \varphi = N.\ \mu_d$ so $f = \mu_d\, m\, g\, \cos\alpha$

$- \mu_d\, m\, g\, \cos\alpha - m\, g\, \sin\alpha = ma \Rightarrow - \mu_d\ g\, \cos\alpha - g\, \sin\alpha = \frac{v_f^2 - v_i^2}{2l}$

Then, $l = \frac{-v_i^2}{2(-\mu_d\ g\cos\alpha - g\sin\alpha)} = \frac{v_0^2}{2g(\mu_d\ \cos\alpha + \sin\alpha)}$



2- The maximum value that the static friction coefficient fs can take for the body to sink,

- At equilibrium

Following (Ox): $-f + p_x = 0 \Rightarrow f = m\, g\, \sin\alpha$

Following (Oy): $N - p_y = 0 \Rightarrow N = m\, g\, \cos\alpha$

For the body to be able to descend, it must: $p_x > f$

$p_x \geq f \ \Rightarrow m\, g\, \sin\alpha \geq N\, \mu_s$ (*) ($\mu_s = f/N$)

with $f = N\, \mu_s$ and μ_s is the coefficient of static friction at which the body begins its motion (with $\mu_s = f/N \ \Rightarrow f = N\, tg\varphi$ so $f = \mu_s m\, g\, \cos\alpha$).

(*)$\Rightarrow m\, g\, \sin\alpha \geq m\, g\, \cos\alpha\ \mathbf{f_s}$ so $\mu_s \leq tg\ \alpha$

The maximum value that can be : μ_s is $tg\alpha$

3- The velocity v_1 of the body as it returns to its initial position;

$x = l$, $v_i = 0$ and we look for v_f.

$$\boldsymbol{v_f^2 - v_i^2 = 2al}$$

Where ''l'' is the distance covered by the body.

So, $a = \frac{v_f^2 - v_i^2}{2l}$

The reference frame must be chosen so that the (Ox) axis follows the axis of motion, i.e. it is parallel to and follows p_x, and the (Oy) axis is perpendicular to (Ox), i.e. it is parallel to $\vec{N}$.

Following (Ox): $- f + p_x = -f + m\ g \sin\alpha = m.a$

Following (Oy): $N - p_y = 0 \Rightarrow N = m\ g \cos\alpha$

$\mu_d = tg\varphi = f/N \Rightarrow f = N\ tg\ \varphi$ so $f = \mu_d m\ g \cos\alpha$

Hence; $-\mu_d\ m\ g \cos\alpha + m\ g \sin\alpha = m.a$

$\Rightarrow -\mu_d\ g \cos\alpha + g \sin\alpha = \frac{v_f^2 - v_i^2}{2l}$

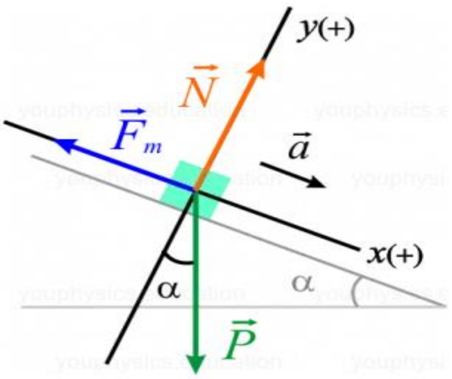
$$v_f^2 = 2gl(\sin\alpha - \mu_d \cos\alpha)$$

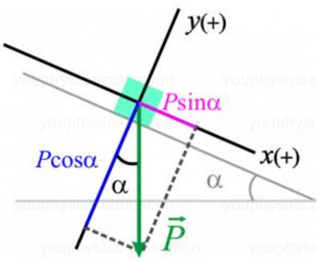
(where l is the same distance found in the question 1)

Exercise 4

To solve this problem, we'll use Newton's second law. The forces acting on the mass are weight (if it's close to the Earth's surface), the normal (because it's resting on the plane) and the force of the spring, which is given by Hooke's law.

Below is a diagram of the forces acting on the mass and the Cartesian axes we'll use to make the projections.





The spring's restoring force acts in the opposite direction to the spring's elongation (and therefore to the direction of mass displacement):

The acceleration of the mass is also shown in the figure. It runs in the positive direction of the x axis. In the diagram below, we've plotted the projections of the weight vector on the axes we've chosen.

Newton's second law applied to mass motion gives:

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \vec{N} + \vec{F_r} = m\vec{a}$$

By projecting onto the Cartesian axes we obtain:

Following (Ox): $- F_r + p_x = - F_r + m\ g\ \sin\alpha = m.a$ (1)

Following (Oy): $N - p_y = 0 \Rightarrow N = m\ g\ \cos\alpha$ (2)

We obtain the norm of the support reaction from equation (2), and as you can see, it's not equal to the weight.

$$N = m\ g\ \cos\alpha$$

On the other hand, the norm of the spring return force is given by: $F_r = k.x$

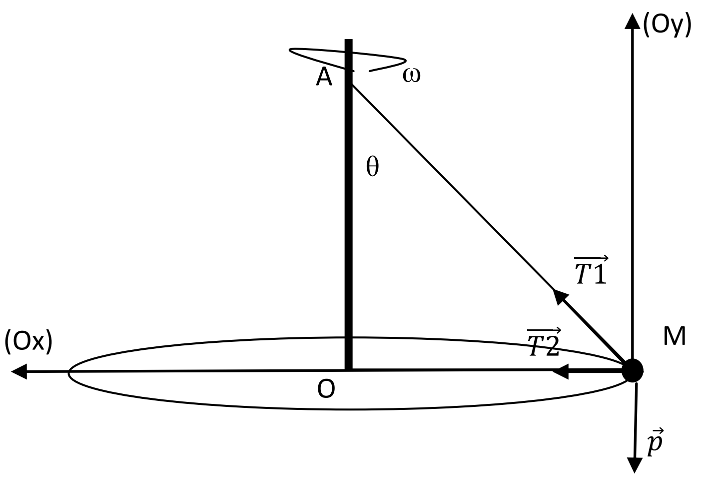
Finally, solving equation (1) yields the acceleration:

$a = (- k.x + m\ g\ \sin\alpha)/m = 4.8\ m/s^2$

by taking : $g = 10\ m/s^2$.

Exercise 5

Calculating the tension T on the wire:



1- Let's find the force (thread tension) that each thread exerts on the ball.

According to the fundamental principle of dynamics FPD:

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \overrightarrow{T_1} + \overrightarrow{T_2} = m\vec{a}$$

The ball's motion is circular, so the acceleration in this case is the normal acceleration a_N, which is directed towards the center of the circle. (with $a_N = v^2/R$)

We choose the reference frame such that :

(Ox) follows the normal acceleration and is directed towards the center of the circle.

By projection onto the axes (Oy) and (Ox) we have :

On (Ox) : $T_2 + T_1 \sin\theta = m.a_N \Rightarrow T_2 + T_1 \sin\theta = m\frac{v^2}{R}$

On (Oy) : $p - T_1 \cos\theta = 0 \Rightarrow mg = T_1 \cos\theta$

$$\text{So } T_1 = \frac{mg}{\cos\theta}$$

$$T_2 = m\frac{v^2}{R} - T_1 \sin\theta = m\frac{v^2}{R} - \frac{mg}{\cos\theta}\sin\theta$$

$$\text{So } T_2 = m\frac{v^2}{R} - m\,g\,\mathrm{tg}\,\theta$$

2- The minimum angular speed ω_{min} at which the bottom wire remains taut.

In order for the lower wire to remain taut, the following conditions must be met **T_2>0.**

$$T_2 = m\frac{v^2}{R} - m\,g\,\mathrm{tg}\,\theta \geq 0 \Rightarrow \frac{v^2}{R} \geq g\,\mathrm{tg}\,\theta$$

With, $v = \omega.R \Rightarrow \frac{\omega^2 R^2}{R} \geq g\,\mathrm{tg}\,\theta$ and $R = OM = L$

So, $\omega^2 L \geq g\,\mathrm{tg}\,\theta \Rightarrow \omega^2 \geq \frac{g\,\mathrm{tg}\,\theta}{L}$

And, $\omega \geq \sqrt{\frac{g\,\mathrm{tg}\,\theta}{L}}$ Then $\omega_{min} = \sqrt{\frac{g\,\mathrm{tg}\,\theta}{L}}$

Exercise 6

1- The linear velocity of the body.

L=30cm, 2α=60°. (α=30°) and ω=10 tr/mn.

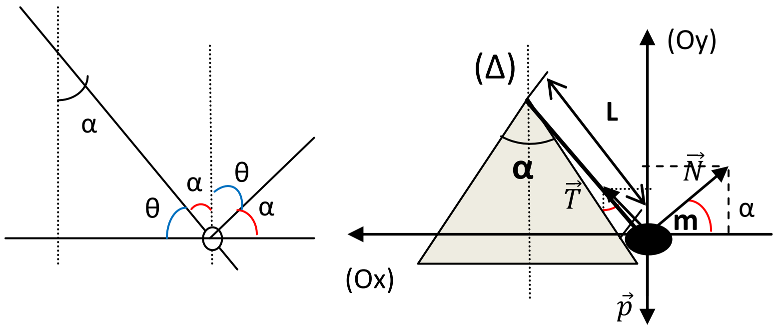
$$\begin{cases} 10x2\pi \longrightarrow 60\ s \\ \omega \longrightarrow 1s \end{cases} \Rightarrow \omega = \frac{10x2\pi}{60} = \frac{\pi}{3} rd/s$$

$v = \omega R$ and $R = l \sin\alpha$

Hence $v = \omega l \sin\alpha = \frac{\pi}{3} . 0,3. sin\ 30 = 0,157 m/s$

Let's determine the reaction (N) of the surface of the cone on the body and the tension of the wire (T).

According to the fundamental principle of dynamics FPD.



$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \vec{N} + \vec{T} = m\overrightarrow{a_N}$$

We choose a reference frame such that (Ox) follows the normal acceleration and is directed towards the center of the cone, and axis (Oy) is perpendicular to (N).

Following (Ox): T_x- N_x =m a_N (1)

Following (Oy): T_y+ N_y - p=m a_T = 0 (a_T =0, because the speed is constant)

$$\Rightarrow T \cos\alpha + N \sin\alpha - p = 0 \quad (2)$$

(1) $\Rightarrow T \sin\alpha - N \cos\alpha = m\frac{v^2}{R} = m\frac{\omega^2 R^2}{R}$

$$\text{So} ; T = \frac{m\omega^2 R}{\sin\alpha} + N\frac{\cos\alpha}{\sin\alpha}$$

We replace it in the second equation:

$$\left(\frac{m\omega^2 R}{\sin\alpha} + N\frac{\cos\alpha}{\sin\alpha}\right)\cos\alpha + N\sin\alpha - p = 0$$

$$\left(\frac{m\omega^2 R}{\sin\alpha}\cos\alpha + N\frac{\cos^2\alpha}{\sin\alpha}\right) + \ N\sin\alpha - p = 0$$

$$N\left(\frac{\cos^2\alpha}{\sin\alpha} + \sin\alpha\right) = p - \frac{m\omega^2 R}{\sin\alpha}\cos\alpha$$

$$\Rightarrow N\left(\frac{1}{\sin\alpha}\right) = \frac{mg\sin\alpha - m\omega^2 R\cos\alpha}{\sin\alpha}$$

So, N= m.$(g\sin\alpha - \omega^2 R\cos\alpha)$, Replacing R with : l sinα , we'll have:

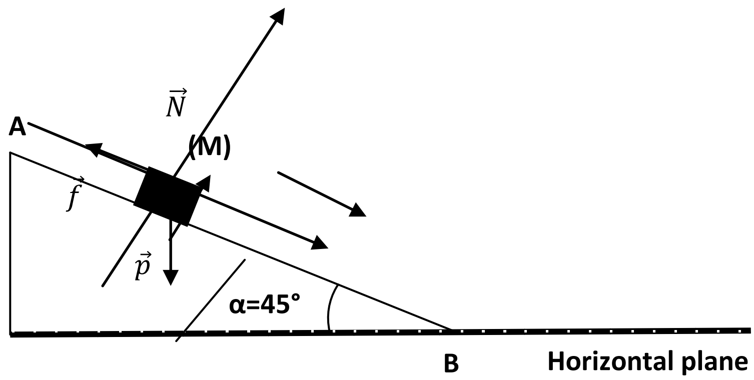
N= m.$(g\sin\alpha - \omega^2 l\sin\alpha\cos\alpha) = 7{,}92$ N

Hence, $T = \frac{m\omega^2 l\sin\alpha}{\sin\alpha} + N\frac{\cos\alpha}{\sin\alpha} = 5{,}88$ N

If we replace N by its expression, we find: T=m g cosα +m. $\omega^2 l\,(1 - \cos^2\alpha)$.

Hence; T= m (g cos α + $\omega^2 l\sin^2\alpha$)

Exercise 7



v_A=1m/s and μ=0,5 on AB.

The nature of movement on AB: FPD:

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \vec{N} + \vec{f} = m\vec{a}$$

We choose the reference frame, such that axis (Ox) is along the axis of motion parallel to $\vec{f}$ and (Oy) is perpendicular to (Ox) therefore along $\vec{N}$.

Following (Ox): $-f + p_x = -f + m\,g\sin\alpha = ma$

Following (Oy): $N - p_y = 0 \Rightarrow N = m\,g\cos\alpha$

$\mu = tg\varphi = f/N \Rightarrow f = N\ tg\ \varphi$ So $f = \mu\ m\ g\ \cos\alpha$

Hence; $-\mu.m\ g.\cos\alpha + m\ g\ \sin\alpha = ma \Rightarrow a = g(\sin\alpha - \mu\cos\alpha)$

$$a = 10\left(\frac{\sqrt{2}}{2} - 0{,}5\ \frac{\sqrt{2}}{2}\right) = 3{,}54\ m/s^2$$

The acceleration a is constant and positive, so the motion is uniformly accelerated.

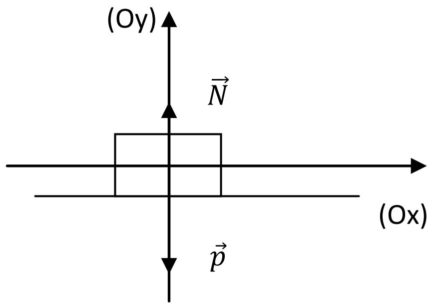
The speed of point M when it reaches point B.

$v_B^2 - v_A^2 = 2al \Rightarrow v_B^2 = v_i^2 + 2al$

With ; l =AB=1

$$v_B = \sqrt{1 + 2a} = 2{,}84\ m/s$$

The nature of movement on the horizontal plane:Friction forces are negligible.



$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \vec{N} = m\vec{a}$$

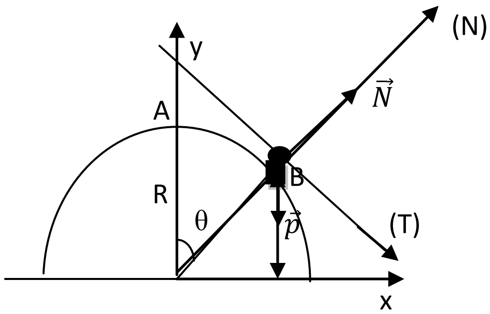
Following (Ox): 0=ma'

Following (Oy): N-p=0 $\Rightarrow N = p = mg$

So a'=0 then the motion is uniformly rectilinear.

Motion is uniform, so speed is constant $v=v_B$ the block will not stop.

Exercise 8



The point at which the point leaves the sphere

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \vec{N} + \vec{f} = m\vec{a}$$

We choose the reference frame, such that axis (T) is tangent to the sphere and axis (N) is perpendicular to (T).

Followig (N): $N - p_N = -m.a_N \Rightarrow N - m\,g\cos\theta = -m\,v^2/R$ (**)

Following (T): $p_T = m.a_T \Rightarrow m\,g\sin\theta = m\,(dv/dt)$ (*)

$v = R\omega \Rightarrow \frac{dv}{dt} = R\frac{d\omega}{dt} = R\frac{d^2\theta}{dt^2}$ with $\omega = \frac{d\theta}{dt} = \dot{\theta}$ and $a_T = R\ddot{\theta} = R\frac{d\dot{\theta}}{dt}$

Equation (*) is multiplied by $\dot{\theta}$:

$$\Rightarrow \dot{\theta}\,g\sin\theta = \dot{\theta}R\frac{d\dot{\theta}}{dt} \quad \text{so} \quad \frac{d\theta}{dt}g\sin\theta = \dot{\theta}R\frac{d\dot{\theta}}{dt}$$

$$\text{then } d\theta\,g\sin\theta = \dot{\theta}Rd\dot{\theta} \Rightarrow g\int_0^{\theta}\sin\theta\,d\theta = R\int_0^{\dot{\theta}}\dot{\theta}\,d\dot{\theta}$$

$g(1-\cos\theta) = R\frac{\dot{\theta}^2}{2} \Rightarrow 2g(1-\cos\theta) = \frac{v^2}{R}$ because $v^2 = R^2\dot{\theta}^2$

$\Rightarrow v^2 = 2(gR - gR\cos\theta)$

The speed at point B is : $v_B = \sqrt{2gR(1-cos\theta)}$

1- The expression for the reaction of the igloo on M

(**)$\Rightarrow N = m\,g\cos\theta - m\,a_N = m\,(g\cos\theta - 2g(1-\cos\theta))$

$$\Rightarrow N = m\,(3\,g\cos\theta - 2g)$$

1- The material point leaves the half-sphere at point p, so N=0

N=0$\Rightarrow$ $mg\,(3\ \cos\theta_0 - 2) = 0$ so $\cos\theta_0 = \frac{2}{3}$

Then, $\theta_0 = 48°$

The angle relative to the horizontal at which the point leaves the half-sphere is 90-48=52

The height h at which the material point leaves the half-sphere is:

$$h_p = R\cos\theta = \frac{2}{3}R$$

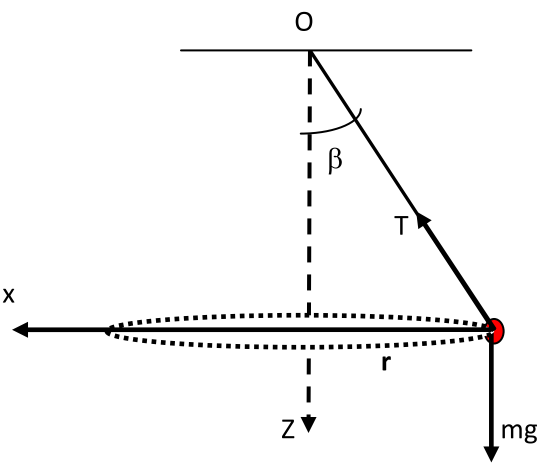
1- The velocity of the material point at this point :

$$2g(1-\cos\theta_0) = \frac{v_p^2}{R} \Rightarrow v_p^2 = 2gR(1-\cos\theta_0)$$

$$then\ \ v_p = \sqrt{2gR(1-\cos\theta_0)} = \sqrt{\frac{2}{3}Rg}$$

(We'll solve the same exercise in the next chapter using the principle of conservation of energy).

Exercise 9



1. Calculating the voltage T on the wire:

The material point is subjected to two mechanical forces; weight mg and tension T.
Expressing Newton's second law, we write:

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \vec{T} = m.\vec{a}$$

In such a rotational movement of m around Oz at the angular velocity ω = constant, the acceleration admits a tangential component and a component normal to the circular trajectory of radius r around OZ. Thus, in the radial direction (ox axis), the projection of Newton's second law gives:

$$T \sin\beta = m.\, a_N = m(v^2/r)$$

$$\Rightarrow m.\, a_N = m.\omega^2 r = m.\omega^2 L\sin\beta$$

$$\text{Since} \quad T = m\omega^2 L$$

2. Calculation of wire inclination to the vertical.

Newton's second law is always written:

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \vec{T} = m\vec{a}.$$

The projection of this relationship along the vertical translates into :

$$- T \cos\beta + mg = 0$$

(because the mass rotates and does not move along Oz).

Taking into account the expression for T, we obtain:

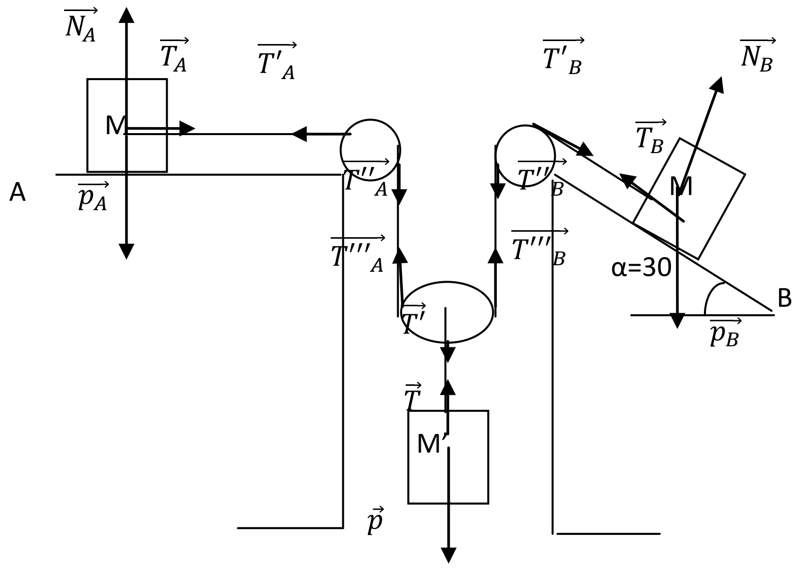
$$cos\beta = \frac{g}{\omega^2 L}$$

Since this angle is inversely proportional to the angular velocity, which must be minimal, and since : **0< β<90°** :

i.e. **$0 < \cos\beta < 1$,** the limit on angular velocity is defined

$$\boldsymbol{\omega \geq \sqrt{g/L}}$$

Exercise 10



1- Neglecting friction, let's calculate the ratio M'/M so that cart B remains stationary?

For system A (carriage of mass M):

$$\Sigma\vec{F} = M\vec{a} \Rightarrow \overrightarrow{p_A} + \overrightarrow{N_A} + \overrightarrow{T_A} = M\overrightarrow{a_A}$$

Projection on the axis of motion

$$T_A = M\, a_A \quad (*)$$

For system B (carriage of mass M):

$\Sigma\vec{F} = M\vec{a} \Rightarrow \overrightarrow{p_B} + \overrightarrow{N_B} + \overrightarrow{T_B} = \vec{0}$ (because carriage B is stationary)

Projection on the axis of motion.

$$-M\, g\, \sin\alpha + T_B = 0 \quad (*')$$

It's the same wire, so $T_B = T_A = T$

M' system (M' mass carriage):

$$\Sigma\vec{F} = M'\overrightarrow{a'} \Rightarrow \overrightarrow{p'} + \overrightarrow{T'} = M'\overrightarrow{a'}$$

Projection on the axis of motion, with $T' = T_A + T_B = 2T$

$$M'g - 2T = M'\, a' \quad (*'')$$

When carriage A moves a distance x_A, carriage M' moves back a distance x' with $x' = x_A/2$.

$x' = \frac{x_A}{2} \Rightarrow v' = \frac{v_A}{2} \quad so \quad a' = \frac{a_A}{2}$

(*'')$\Rightarrow M'g - 2T = M'\frac{a_A}{2}$

(*)$\Rightarrow$ T=M a_A $\Rightarrow a_A{}' = \left(\frac{2gM\prime}{4M+M\prime}\right)$

(*''') $\Rightarrow$ T=M g sin 30=M g/2= M a_A

So $\left(\frac{2gM\prime}{4M+M\prime}\right) = \frac{g}{2} \Rightarrow 4M' = 4M + M'$

Then $\frac{M\prime}{M} = \frac{4}{3}$

1- Let's calculate the accelerations for M'=2M

The cart A : T=M a_A (*)

The cart B : T- Mg sin 30=M $a_B \Rightarrow T - \frac{Mg}{2} = Ma_B$(*')

The mass cart M' (M'=2M): 2 M g – 2 T= 2 M a'

When carriage A moves a distance xA and carriage B moves a distance xB, the mass M' moves down a distance x'=(x_A + x_B)/2

so 2 x'=(x_A + x_B) $\Rightarrow$2 v'=(v_A + v_B) then 2 a'= (a_A +a_B) (*'')

and 2 M g – 2 T= M (a_A +a_B) (*''')

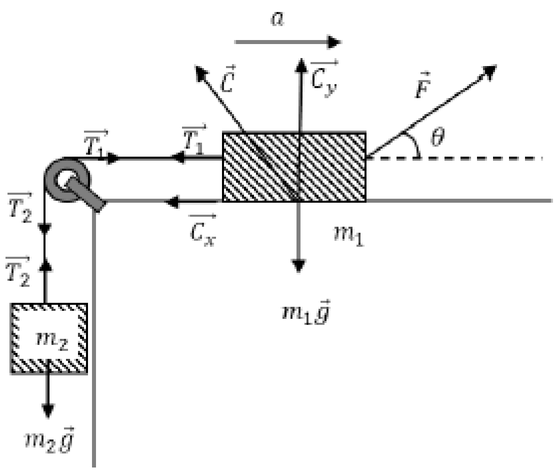
(*) and (*') $\Rightarrow a_A - \frac{g}{2} = a_B$

(*) and (*'') $\Rightarrow$ 2 g – 2 a_A= (a_A +a_B) so 4 a_A=g(5 /2)

$$\text{So} \begin{cases} a_A = g \left(\frac{5}{8}\right) \\ a_B = g \left(\frac{1}{8}\right) \\ a' = g \left(\frac{3}{8}\right) \end{cases}$$

Exercise 11

The fundamental principle of dynamics is applied to the masses m_1 and m_2 :



For system m_1 :

$$\Sigma\vec{F} = m_1\vec{a} \Rightarrow \overrightarrow{p_1} + \overrightarrow{C_1} + \overrightarrow{T_1} + \vec{F} = m_1\vec{a} \ (1)$$

For system m_2 :

$$\Sigma\vec{F} = m_2\vec{a} \Rightarrow \overrightarrow{p_2} + \overrightarrow{T_2} = m_2\vec{a} \ (2)$$

The pulley with negligible mass, so $T_1 = T_2 = T$

Projecting equations (1) and (2) onto the direction of motion gives:

$$-C_x + F\cos\theta - T_1 = m_1.a \quad (3)$$

and

$$T_2 - m_2\, g = m_2.a \quad (4)$$

Projecting equation (1) onto the direction perpendicular to the motion gives:

$C_y = m_1\, g$

And $\mu_d = C_x / C_y = F_f / R_N \Rightarrow C_x = \mu_d\, m_1\, g \ \ (5)$

Summing equations (3) and (4), assuming $T_1 = T_2$ and equation (5), we obtain:

$$a = \frac{F\cos\theta - g(m_2 - \mu_d\, m_1)}{m_1 + m_2}$$

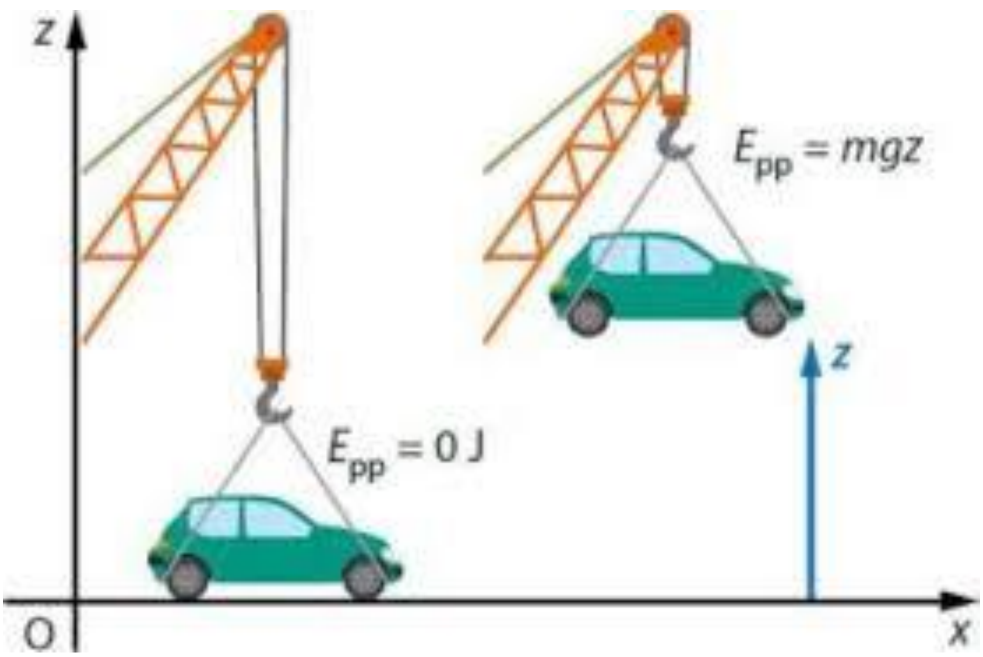
University of Tlemcen
Faculty of Sciences
Department of Mathematics

U.Y: 2023/2024

COURSE OF MECHANICS

OF THE MATERIAL POINT

Chapter VI: Work and Energy



Glossary

In English	In French	In Arabic
The work	Le travail	العمل
External forces	Forces extérieures	القوى الخارجية
The elementary work	Le travail élémentaire	العمل الجزئي
The elementary displacement	Le déplacement élémentaire	التنقل الجزئي
The power	La puissance	الاستطاعة
The average power	La puissance moyenne	الاستطاعة المتوسطة
The instantaneous power	La puissance instantanée	الاستطاعة اللحضية
Energy	L'énergie	الطاقة
Driving work	Le travail moteur	العمل المحرك
Resistive work	Le travail résistant	العمل المقاوم
Kinetic energy	L'énergie cinétique	الطاقة الحركية
Conservatives forces	La force conservative	القوة المنحفضة
Potential energy	L'énergie potentielle	الطاقة الكامنة
Wight force	Force du poids	قوة الثقل
Spring return force	La force de rappel du ressort	قوة الارجاع لنابض
Mechanic energy (TotaleEnergie)	L'énergie mécanique ou l'énergie totale	الطاقة الكلية
Friction force work	Le travail de la force de frottement	عمل قو ة الاحتكاك

1. Introduction

The aim of this chapter is to present the energy tools used in mechanics to solve problems. Indeed, sometimes the fundamental principle of dynamics is not enough to solve a problem. Newton's laws can be used to solve all the problems of classical mechanics. If we know the position and initial velocity of the particles in a system, as well as all the forces acting on them. But in practice, we don't always know all the forces at play, and even if we do, the equations to be solved are too complex. In this case, other concepts such as work and energy must be used. Before describing the different types of energy (kinetic, potential and mechanical) and using them in energy theorems, we'll introduce the notions of power and work of a force.

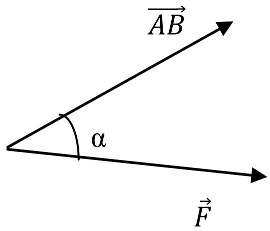
2. The workالعمل

All motion under the action of external forces $\vec{F}$, implies work by these forces. In other words; work supplied by a force moves a body in its own direction and creates motion.

2.1. Work performed by a constant force

Let a particle subjected to a constant force $\vec{F}$ move this body a distance d=AB, the mechanical work W performed by the force $\vec{F}$ is defined as:

$$\boldsymbol{W_{AB} = \vec{F}.\overrightarrow{AB} = |\vec{F}|.|\overrightarrow{AB}|.\cos\alpha}$$



α is the angle between the two vectors $\vec{F} and\ \overrightarrow{AB}$.

- For α=0 $W = |\vec{F}|.|\overrightarrow{AB}|$ because $cos 0 = 1$
- For $\alpha < \frac{\pi}{2}$ with have $W > 0$ It's a driving work.
- For $\alpha = \frac{\pi}{2}$ with have $W = 0$ because $cos\frac{\pi}{2} = 0$.
- For $\frac{\pi}{2} < \alpha < \pi W < 0$ It's a resistive work.

Unity of work in the system MKSA is « **Joule** ».

Note:

Note that work is a scalar quantity, unlike force and displacement, which are vectors.

Example 1:

The muscular effort required to lift an object depends on both its weight (the force of gravity exerted on it), and the height h from which it is lifted.

In this case, the force of the weight is directed downwards, the displacement upwards and θ is 180°.
W = - P. h = -mgh.
The force of the weight is negative, since muscular work must be done against the force of gravity.

Example 2:

To lift a car with a mass of one and a half tons, a force F of 15,000N vertical to the car is required.

Calculate the work done by this force to move the car by a height (AB) of 3 meters.

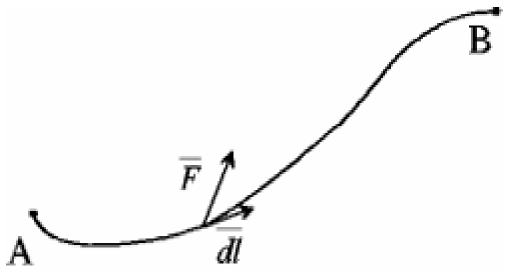
$W_{AB}(\vec{F}) = |\vec{F}|.|\overrightarrow{AB}|.cos\alpha$=F.d.cosα =1.5 10^4 . 3 = 4.5 10^4 J

2.2. The work performed by a variable force

If the force varies in intensity and/or direction during displacement, and if the displacement has any form whatsoever, we need to use integral calculus to generalize the definition of work. Generally speaking, the work of a force depends on the path followed, which is why this elementary work is necessary.

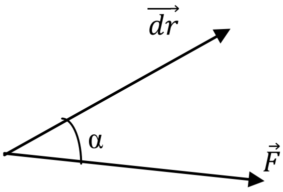
$$\mathbf{dW} = \vec{\mathbf{F}}.\overrightarrow{\mathbf{dr}} = \vec{\mathbf{F}}.\overrightarrow{\mathbf{dl}}$$

where dl is an infinitesimal displacement along the trajectory, tangential to it.



The elementary work dW performed by a force $\vec{F}$ on a point mass m during an elementary displacement dr= dl is given by:

$$\mathbf{dW} = \vec{\mathbf{F}}.\overrightarrow{\mathbf{dr}} = |\vec{\mathbf{F}}|.|\overrightarrow{\mathbf{dr}}| \cos(\vec{\mathbf{F}},\overrightarrow{\mathbf{dr}})$$



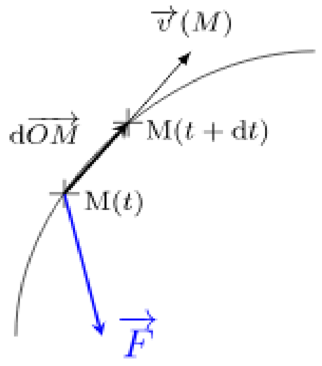
To obtain the work on an AB displacement, we integrate this elementary work:

$$\mathbf{W} = \int \mathbf{dW} = \int_A^B \vec{\mathbf{F}}.\overrightarrow{\mathbf{dr}} = \int \boldsymbol{F.dr.cos\alpha}$$

α is the angle between the two vectors $\vec{F}$ *and* $\overrightarrow{dr}$; $\alpha = (\vec{F},\overrightarrow{dr})$

2.3. The power الاستطاعة

Let a point M move along its trajectory at a velocity $\vec{v}$ (M) relative to the reference frame of study, It experiences a force $\vec{F}$(M) as shown in the figure opposite:



The power of a force $\vec{F}$ is the work per unit time.

We have two types:

- The average power $P_{avr} = \frac{\Delta W}{\Delta t}$
- The instantaneous power $P = \frac{dW}{dt}$

Then the instantaneous power of the $\vec{F}$ is:

$$P(\vec{F}) = \frac{dW}{dt} = \frac{|\vec{F}|.|\overrightarrow{dr}|}{dt} = \vec{F}\,.\vec{v}(M) = \|\vec{F}\|\times\|\vec{v}\,(M)\|\times\cos\alpha$$

Note :

- ✓ The unit of power is the « Watt ».
- ✓ This force can be classified into three types:
  - It is driving, if its power is positive which corresponds to an angle $\alpha<\pi/2$.
  - It is resistive, if its power is negative which corresponds to an angle $\alpha>\pi/2$.
  - Finally, it can be of zero power, in which case $\alpha=\pi/2$.

3. Energyالطاقة

In physics, energy is defined as the capacity of a system to produce work. Energy is not a material substance: it is a physical quantity that characterizes the state of a system; it can be stored and exists in many forms.

3.1. Kinetic energy الطاقة الحركية

In order to accelerate a point mass to a defined speed, work must be done. This work is then stored in the point mass in the form of kinetic energy.

Suppose the object's initial velocity is v_0 and the force F is applied in the direction of v_0, producing a displacement d=dr.

We have: dW=$F.dr$ and $F = ma = m\frac{dv}{dt}$

From this expression we can deduce the following:

$$dW = Fdr = m\frac{dv}{dt}dr$$

$$\Rightarrow\ dW = m\frac{dr}{dt}dv \quad Then \quad dW = mvdv$$

Let's integrate the expression of elementary work, and derive the definition of kinetic energy:

$$W = m\int_A^B vdv \Rightarrow W = \frac{1}{2}m(v_B^2 - v_A^2) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

Where $\boldsymbol{v_A}$ is the velocity of the moving body at point A and $\boldsymbol{v_B}$ its velocity at point B.

The kinetic energy of a material point of mass **m** and instantaneous velocity $\vec{v}$ is given by the expression:

$$Ec = \frac{1}{2}mv^2$$

So: $\boldsymbol{W_{\vec{F}(A\to B)} = E_{C_B} - E_{C_A} = \Delta E_c}$

Note :

- ✓ The unit of energy is the « Joule ».
- ✓ And since *p=mv*, we can also write:

$$Ec = \frac{P^2}{2m}$$

Theorem of the Kinetic Energy Theorem : نضرية الطاقة الحركية

The variation in kinetic energy of a material point subjected to a set of external forces between two positions A and B is equal to the sum of the work of these forces between these two points.

$$\boldsymbol{W_{\vec{F}(A\to B)} = E_{C_B} - E_{C_A} = \Delta E_c \Rightarrow \sum_i W_i = \Delta E_C}$$

3.2. Conservatives forces القوة المنحفضة

A force is said to be conservative, or to derive from a potential, if its work is independent of the path taken, whatever the probable displacement between the starting point and the end point.

Conservative forces include the force of gravity, spring return force and the tension force of a wire.

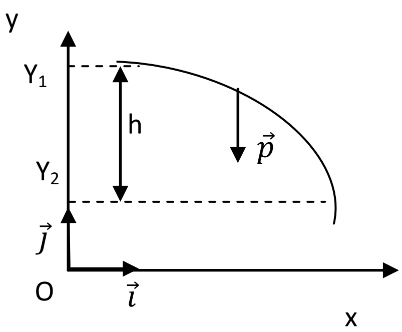
Example:

Let's calculate the work of the force of gravity.

$dW = \vec{p}.\vec{dl}$ with $p = -mg\vec{j}$

$\vec{dl} = dx\vec{i} + dy\vec{j}$

so $dW = -mgdy$



$$W = -mg\int_{y_1}^{y_2} dy = -mg(y_2 - y_1)$$

$$\Rightarrow W = mg(y_1 - y_2) = mgh$$

So the force of gravity $\vec{p}$ is a conservative force because its work does not depend on the path followed, and it is said to derive from a potential.

Spring return force is also a conservative force.

Note:

A force is said to be non-conservative if its work depends on the path followed, as in the case of ***friction force***.

3.3. Potential energy الطاقة الكامنة

Potential energy is a function of coordinates, such as the integration between its two values at start and finish. It represents the work done by the particle to move it from its initial position to its final position.

If the force $\vec{F}$ is a force deriving from a potential (conservative), then:

$$W = \int_A^B \overrightarrow{F_C}.\overrightarrow{dr} = E_{P_A} - E_{P_B} \Rightarrow dW = -dE_p$$

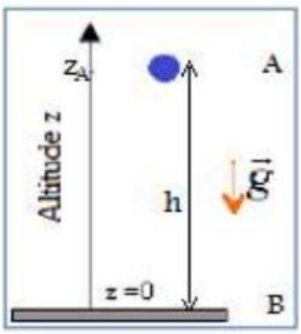
$$\text{Hence}\quad \boldsymbol{W_{A\to B}(\overrightarrow{F_C})) = -\Delta E_p}$$

Potential energy is always calculated relative to a reference frame (Ep=0).

The potential energy function Ep is determined to within one constant.

By identifying the two expressions dE_p and dW, we arrive at the following result: The differential of potential energy is equal to and opposite in direction to the differential of work.

Example 1: Wight force قوة الثقل



The force of weight is a conservative force, hence:

$W_{A\to B}(\overrightarrow{F_C})) = -\Delta E_p$

And $W = mg(y_1 - y_2) = mgh$

So $W_{\vec{p}} = -\Delta E_p = -(E_{pf} - E_{pi}) = Epi = mg(z_A - z_B)$

Because E_{pf} is the reference potential energy.

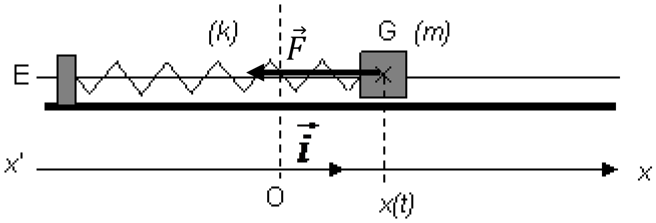
$$\text{So } Ep = mgH$$

Note:

If $Z_A > Z_B$ we have $E_p>0$

If $Z_A < Z_B$ we have $E_p<0$

Example 2 : Spring return force قوة الارجاع لنابض



$\vec{F} = -kx\vec{i}, \overrightarrow{dl} = dx.\vec{i}$et $dW = \vec{F}.\overrightarrow{dl}$

$dW = -dE_p = -kx.dx \Rightarrow dE_p = kxdx$

$$\Rightarrow \int dEp = k\int_{x_i}^{x_f} xdx$$

$$\Rightarrow Ep = \frac{1}{2}k(x_f^2 - x_i^2) = \frac{1}{2}kx^2$$

3.4. Mechanic energy (Totale Energie) الطاقة الكلية

The mechanical energy of a material point at a given instant is equal to the sum of kinetic energy and potential energy:

$$\boldsymbol{E_M = E_C + E_p \Rightarrow E_M = E_C + E_p}$$

- **Principle of conservation of mechanical energy مبدا انحفاظ الطاقة الميكانيكية**

In a conservative (or potential-derived) force field, mechanical energy is conserved over time (**no friction**).

$$\mathbf{E_M = E_C + E_p = Cte}$$

This means that the variation in mechanical energy is zero $\mathbf{\Delta E_M = 0}$, it also means that the variation in kinetic energy is equal to the opposite of the variation in potential energy:

$$\mathbf{\Delta Ec = -\Delta Ep}$$

In other words, if the system is isolated or free, mechanical energy is conserved.

Note:

In the presence of **frictional forces**, the variation in mechanical energy can't be stored, is equal to the sum of the work of the frictional forces. $W(F_{Frott})$:

$$\Delta E_M = E_{Mf} - E_{Mi} = \sum W_{A \to B}(\overrightarrow{F_{NC}}) = W_{A \to B}(\overrightarrow{F_{frot}})$$

- **Friction force work** : عمل قوة الاحتكاك

$$W_{A \to B}\left(\overrightarrow{F_{frot}}\right) = -F_f . AB$$

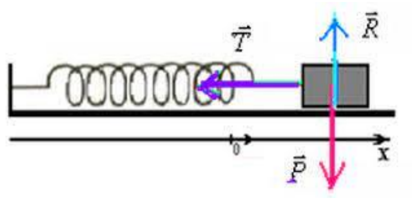
Example:

A mass m is attached to a spring of stiffness k, and the other end of the spring is attached to point C. The mass m can slide on the horizontal surface. Initially, the mass is at rest at point O of equilibrium.

1) Assuming no friction, move mass m from point O to point A, such that OA=a. Determine the work of the Spring return force as m moves from O to A. Then determine the speed of m at point O.

2) Same questions as question 1, but now we assume that friction exists, and give the dynamic friction coefficient μc.

Answers:



1- We have, $\vec{F} = -kx\vec{\imath}$ and $\overrightarrow{dl} = dx\vec{\imath}$

$$\Rightarrow W_{\vec{F}} = \int dW_{\vec{F}} = \int \vec{F} . \overrightarrow{dl} = -k \int_a^0 x dx = \frac{1}{2} k a^2$$

We also have: $\sum_i \boldsymbol{W_i} = \boldsymbol{\Delta E_C} = W_{\vec{p}} + W_{\vec{R}} + W_{\vec{F}}$

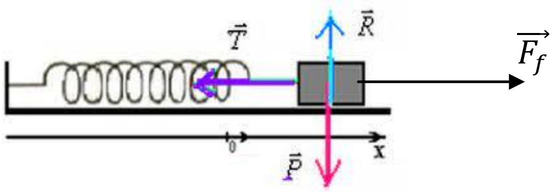
With, $W_{\vec{p}} = W_{\vec{R}} = \vec{0}$ because $\vec{R}$ *and* $\vec{p} \perp \overrightarrow{Ox}$

So $\boldsymbol{\Delta E_C} = W_{\vec{F}} = \frac{1}{2}ka^2 = \frac{1}{2}mv_o^2 - \frac{1}{2}mv_A^2$ with $v_A=0$

Hence, $\boldsymbol{v_o} = \boldsymbol{a}\sqrt{\frac{k}{m}}$

2- Case of friction

We also have: $\sum_i \boldsymbol{W_i} = \boldsymbol{\Delta E_C} = W_{\vec{p}} + W_{\vec{R}} + W_{\overrightarrow{F_f}} + W_{\vec{F}}$



with: $W_{\vec{p}} = W_{\vec{R}} = \vec{0}$

so $\boldsymbol{\Delta E_C} = W_{\vec{F}} + W_{\overrightarrow{F_f}} = \frac{1}{2}ka^2 - a.F_f = \frac{1}{2}ka^2 - a.\mu_c.mg = \frac{1}{2}mv_o^2$ because $v_A=0$

Hence, $\boldsymbol{v_o} = \sqrt{\frac{ka^2}{m} - 2\mu_c.a.g} = \boldsymbol{a}\sqrt{\frac{k}{m} - \frac{2\mu_c.g}{a}}$

Proposed exercises about chapter VI

Exercise 1

A body is subjected to a force $\vec{F}$ such that: $\vec{F} = (y^2 - x^2)\vec{\imath} + (3xy)\vec{\jmath}$

Find the work of force $\vec{F}$ if the body moves from point A(0,0) to point B(1,3) Following the following trajectories:

1. On the (Ox) axis from A to C(1,0) then parallel to (Oy) from C to B.
2. On the (Oy) axis from A to D(0.3), then parallel to (Ox) from D to B.
3. On the straight line [AB].
4. On trajectory $y=x^2$.

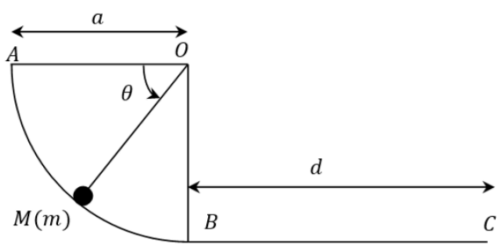
Exercise 2

A particle of mass m, initially at rest in A, slides without friction on the circular surface AOB of radius a.

1) Determine the work of weight from A to M.

2) Determine the work of the surface-particle contact force m.

3) Determine the potential energy E_p of m at the point M ($E_p(B) = 0$).



4) Use the kinetic energy theorem to determine the speed of m at point M, deduce its kinetic energy E_c .

5) Calculate the mechanical energy E_m.

6) Show E_c ,E_p and E_m ($0 <\theta<\pi 2$). Discuss.

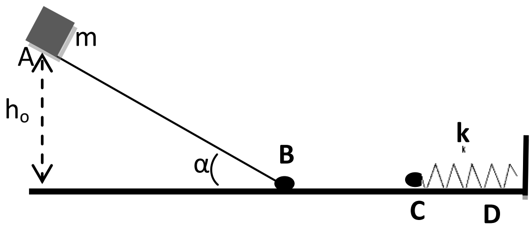
7) The circular surface AOB is connected to a horizontal part BC, there is friction between B and C, the particle stops at a distance d from B. Determine the coefficient of kinetic friction.

Given d = 3a = 3m.

Exercise 3

Consider a small block of mass m =5kg dropped without initial velocity at point A of an inclined plane at an angle α=30° to the horizontal. Point A is at a height h_0=5m from the horizontal.

1- Knowing that the coefficient of dynamic friction on plane AB is μ_d=0.2, applying the fundamental principle of dynamics:



- What is the nature of the motion on plane AB?
- Calculate the speed of the block when it reaches point B.

2- After passing through point B at speed V_B, the mass arrives at point C. Knowing that the coefficient of friction is negligible on plane BC :

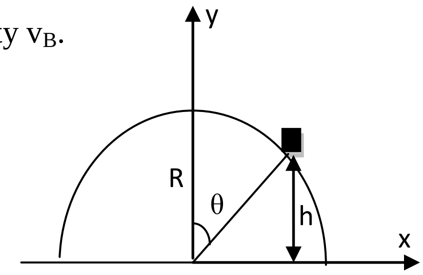
- Deduce the speed at point C?
- Calculate the maximum compression of the spring, given a stiffness constant equal to k=100N/m? (g =10 m/s^2).

Exercise 4

A piece of ice M of mass m slides without friction over the outer surface of an igloo, which is a half-sphere of radius r with a horizontal base.

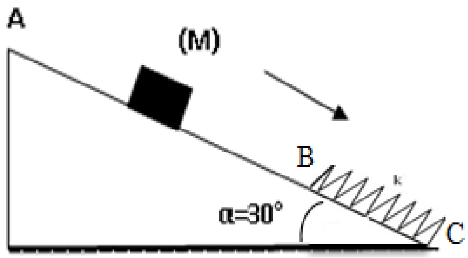
At t=0, it is released from point A without any initial velocity.

1. Find the expression for the velocity at point B, as a function of g, r and θ.
2. Using the fundamental relation of dynamics, determine the expression of $|\vec{N}|$ the reaction of the igloo on M at point B as a function of velocity v_B.
3. At what height does M leave the sphere?
4. At what speed does M arrive at the axis (Ox)?



Exercise 5

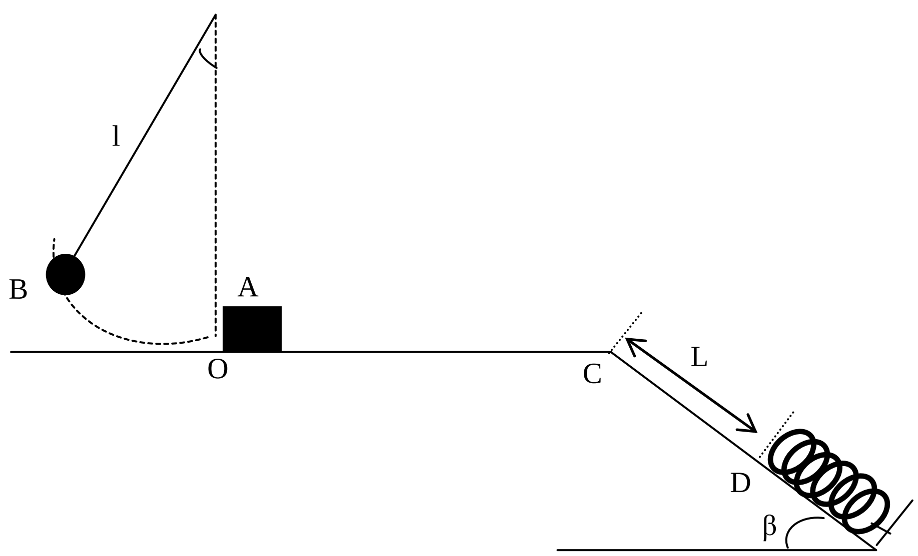
Consider a small block of mass m =2kg dropped without initial velocity at point A of an inclined plane at an angle α=30° to the horizontal. Point A is at a height h_A=5m from the horizontal.



1. Knowing that the coefficient of dynamic friction on plane AB is **μ_d=0.2**, applying the fundamental principle of dynamics, what is the acceleration of the block on plane AB=8m?
2. Calculate the speed of the block when it reaches point B.
3. Using the kinetic energy theorem, find the speed of the block at point B.
4. At point B, the block hits a spring with stiffness constant k=100N/m at speed V_B. Calculate the maximum compression (x) of the spring (given g =10 m/s^2).

Exercise 6

- A ball B of mass m, attached to an inextensible wire of length l, is moved away from its equilibrium position by an angle α. It is dropped without initial velocity. Passing through the vertical position, the ball strikes (touches) a body A of the same mass and stops, body A passes from point O to point C (OC=d) on a rough horizontal plane of friction coefficient μ.



1. Show the forces exerted on body A.
2. What is the nature of the motion on the horizontal plane?
3. Express the velocity of ball B just before touching body A.
4. Using the principle of conservation of momentum, determine the velocity of body A after the interaction.
5. If v_A=v_B at point O, give the velocity of body A at point C as a function of g, l, d, α and μ.
6. By what angle must ball B be moved away for body A to arrive at point C with zero velocity.

- From point C, body A approaches the perfectly smooth (no friction) path CD=L, inclined at an angle β to the horizontal. It arrives, without initial velocity, on a perfect spring of length l0 and stiffness constant k.

  1. Show the forces exerted on A as the spring compresses.
  2. What is the value of the spring's maximum compression?

We give m=200g, d=OC=1m, l=10 cm, L=1m, μ=0.1, g=10m/s^2, k=140N/m, β=30°.

Exercise 7

A solid body S of mass m is linked on one side to a spring of stiffness K, while the other side of the spring is fixed. The body is moved horizontally from its equilibrium position by a distance x and then released (μ =tg φ : coefficient of friction).

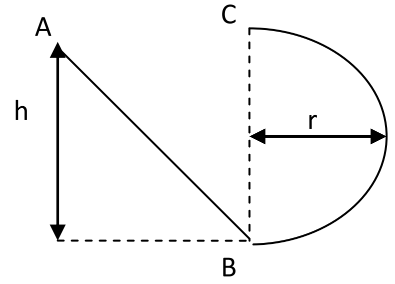
1. Show the forces applied to body S.
2. Calculate the speed V_B corresponding to the movement of S from its equilibrium position.

Exercise 8

A ball slides without friction inside a gutter.

Find the smallest height h_{min} from which the ball is

launched to reach point C, without leaving the gutter.

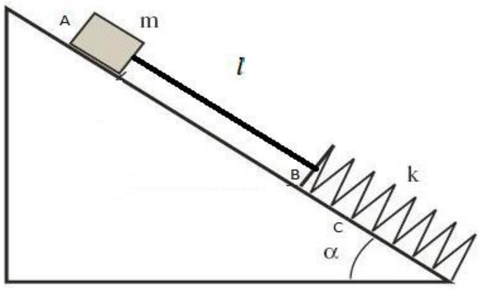


Exercise 9

A block of mass m is dropped without initial velocity onto an inclined plane making an angle α with the horizontal, at a distance l above a light, uncompressed spring of stiffness k. The motion of the block is frictionless.

1. Using the principle of conservation of mechanical energy, find the expression for the velocity of the block when it first touches the spring.

2. Using the fundamental principle of dynamics, find the expression for the maximum compression of the spring as a function of m, g, α and k.



Correction of exercises about chapter VI

Exercise 1

$$\vec{F} = (y^2 - x^2)\vec{\imath} + (3xy)\vec{\jmath}$$

The work of force $\vec{F}$ when the body moves from point A(0,0) to point B(2,4) along the trajectories:

$$\mathrm{dW} = \vec{\mathrm{F}}\,\overrightarrow{\mathrm{dr}} \text{ with } \overrightarrow{\mathrm{dr}} = \mathrm{dx}\vec{\imath} + dy\vec{\jmath}$$

Then

$$\mathrm{dW} = \vec{\mathrm{F}}.\overrightarrow{\mathrm{dr}} = \begin{pmatrix} y^2 - x^2 \\ 3xy \end{pmatrix}.\begin{pmatrix} \mathrm{dx} \\ \mathrm{dy} \end{pmatrix} \Rightarrow \mathrm{dW} = (y^2 - x^2)\mathrm{dx} + 3xydy$$

So $W = \int \vec{\mathrm{F}}.\overrightarrow{\mathrm{dr}} = \int \mathrm{F_x dx} + \int F_y dy \Rightarrow W = \int (y^2 - x^2)\mathrm{dx} + \int 3xydy$

1- Following axis (Ox) from A(0,0) to C(2,0):

The variation is on the Ax axis, so y=0; therefore dx=0 and x varies from 0 to 2.

$$W = \int (y^2 - x^2)\mathrm{dx} + \int 3xydy = \int_0^2 -x^2\, dx \Rightarrow W = -\frac{x^3}{3} = -\frac{8}{3}j$$

2- Following axis (Oy) from C(2,0) to B(2,4):

The variation is parallel to Oy so x is constant (x=2) then dx=0 and y varies from 0 to 4:

$$W = \int_0^4 3xy\, dy = \int_0^4 6y\, dy \Rightarrow W = 6\frac{y^2}{2} = 48\, j$$

3- On line AB:

The equation of a straight line is generally of the form: y=a.x+b

If the line passes through the two points A(0,0) and B(2,4) then b=0 and $a = \frac{y_B - y_A}{x_B - x_A} = 2$

Then the equation of the line is of the form y=2.x, so dy= 2.dx.

In this case, the expression for work becomes:

(x varies from 0 to 2);

$$W = \int ((2x)^2 - x^2)\mathrm{dx} + \int 3x(2x)2dx = 15\int_0^2 x^2 dx \Rightarrow W = 15\frac{x^3}{3} = 40j$$

4- On the trajectory $y=x^2$:

$$y=x^2 \Rightarrow dy=2x.dx$$

Then the work formula becomes:

$$W=\int((x^2)^2-x^2)dx+\int 3x(x^2)2xdx=\int 7x^4-x^2\,dx$$

The body moves from A(0,0) and B(2,4) then x varies from 0 to 2; then,

$$W=\frac{7}{5}x^5-\frac{x^3}{3}=42{,}13j$$

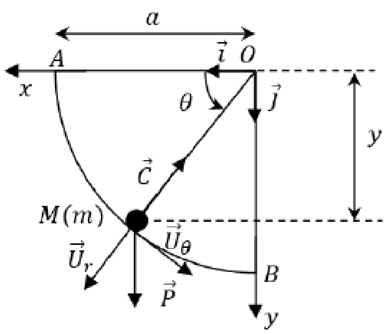
Exercise 2

1) The work of $\vec{p}$ from A to M is:

$$dW=\vec{p}.\vec{dl}\quad \text{with } p=mg\vec{j}$$

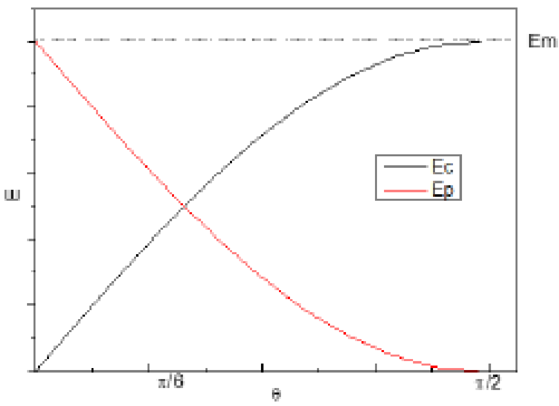
$$\vec{dl}=dx\vec{i}+dy\vec{j}\quad \text{so } dW=mgdy$$

$$W=mg\int_0^y dy=mgy=mg\,a\sin\theta$$



2) The work of R_N force is:

$$W_R=\int_0^y \overrightarrow{R_N}.\vec{dl}=\vec{0}\quad \text{Because } \overrightarrow{R_N}\perp\vec{dl}$$



3) Potential energy:

$dEp=-dW \Rightarrow Ep=-mg\,a.\sin\theta+c$

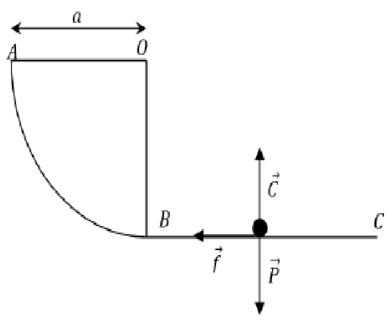
$E_p(B)=0$, $\theta=\pi/2$ so $c=mga$

$\Rightarrow E_p=mga(1-\sin\theta)$

4) $\Delta E_C=\sum W \Rightarrow \frac{1}{2}mv_M^2=mga\sin\theta$

$$V_M=\sqrt{2ga\sin\theta}$$

5) $E_m=E_c+E_P=mg\,a=cste$



6) When E_P decreases E_c increases while E_m remains constant.

7) $\mu=\frac{f}{R_N}=\frac{f}{p}\Rightarrow f=\mu mg$

So $\Delta E_C=W_f=\int_B^C\vec{f}.\vec{dl}=-\mu mgd \Rightarrow \frac{1}{2}mv_B^2=-\mu mgd$

$$\text{Then} \quad v_B = \sqrt{2ag}$$

Note : Replace θ = π/2 in the formula for v_M, we find :

$$v_B = \sqrt{2ag}$$

We cane used also: $E_{m_B} = E_{m_A} \Rightarrow E_{C_A} + E_{P_A} = E_{C_B} + E_{P_B}$

Calculation of μ:

$$\sum \vec{F} = m\vec{\gamma} = \vec{f} + \vec{P} + \overrightarrow{R_N}$$

We have $\mu = \frac{f}{R_N} = \frac{f}{mg}$ because R_N = mg (with projection on (oy))

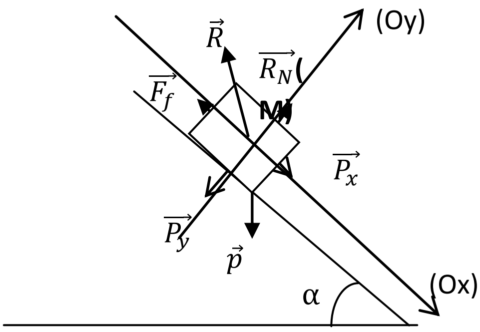
Projection on (ox): -f = m.γ

We have also: $v_c^2 - v_{\boldsymbol{B}}^2 = 2\gamma.d$ (v_C =0)

$-v_{\boldsymbol{B}}^2 = 2\gamma.d = -2ag$ so: $\gamma = \frac{-ag}{d}$ with -f = m.γ = $\frac{-mag}{d}$ so f = $\frac{m.a.g}{d}$

Then $\mu = \frac{f}{R} = \frac{f}{mg} = \frac{mag}{mg.d} = \frac{a}{d} = \frac{1}{3}$

Exercise 3



1. Knowing that the coefficient of dynamic friction on plane AB is μ_d=0.2, apply the fundamental principle of dynamics:

- What is the nature of the motion on AB? **a= ?**

$$\Sigma\vec{F} = m\vec{a} = \vec{p} + \overrightarrow{R_N} + \overrightarrow{F_f}$$

Following (Ox): $-F_f + p_x = -F_f + m\,g\sin\alpha = ma$

Following (Oy): $R_N - p_y = 0 \Rightarrow R_N = m\,g\cos\alpha$

$\mu_d = tg\varphi = F_f/R_N \Rightarrow F_f = R_N\, tg\,\varphi = \mu_d\, m\, g\cos\alpha$

$- \mu_d\, m\, g \cos\alpha + m\, g \sin\alpha = m.a$

$$\Rightarrow a = g.(\sin\alpha - \mu_d \cos\alpha)$$

So: **$a = 3.26\ m/s^2$.**

- Calculate the speed of the block when it reaches point B.

$$v_B^2 - v_A^2 = 2al \Rightarrow v_B^2 = 2al = 2a\left(\frac{h}{\sin\alpha}\right)$$

$$v_B = \sqrt{2a\left(\frac{h}{\sin\alpha}\right)} = 8.074\ m/s$$

2. $V_c = V_B$ because we have an MRU (principle of inertia or Newton's 1st law)

We calculate the compression distance of the spring:

$$\Delta Ec = \Sigma W_{f_{ext}} \Rightarrow Ec_D - Ec_C = W_p + W_{Fr} + W_{RN}$$

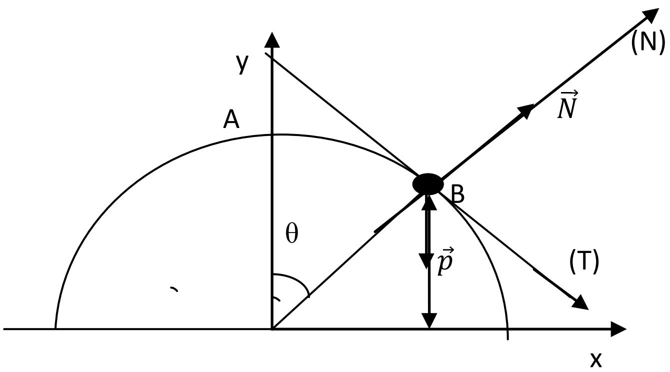
$-\frac{1}{2}kx^2 = -\frac{1}{2}mv_c^2$ so: $x = \sqrt{\frac{mv_c^2}{k}} = 1.8\ m$

2nd Method: Between points C and D

$E_{M_C} = E_{M_D} \Rightarrow E_{C_C} + E_{P_C} = E_{C_D} + E_{P_D} \Rightarrow \frac{1}{2}kx^2 = \frac{1}{2}mv_c^2$

So; $x = \sqrt{\frac{mv_c^2}{k}} = 1.8\ m$

Exercise 4



1- According to the principle of conservation of mechanical energy between two points A and B:

$$E_{M_A} = E_{M_B} \Rightarrow E_{C_A} + E_{P_A} = E_{C_B} + E_{P_B}$$

So: $E_{C_A} = E_{C_B} + E_{P_B}$ (*)

Because $E_{C_A} = 0$ $(v_A = O)$ *because the material point is launched without initial velocity*

With $h_B = Rcos\theta$

So; (*)$\Rightarrow mgR = \frac{1}{2}mv_B^2 +$ mg Rcos θ

Then: $gR = \frac{1}{2}v_B^2 +$ g Rcosθ $\Rightarrow$ $v_B^2 = 2(gR - gRcos\theta)$

$$\Rightarrow v_B = \sqrt{2(gR - gRcos\theta)}$$

2-According to the fundamental principle of dynamics:

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{N} + \vec{p} = m\vec{a}$$

We choose a reference frame consisting of the axis (OT) tangent to the half-sphere and the axis (ON) following the radius and in the direction of $\vec{N}$:

Projecting on (ON):

N-p cos θ =m a_N $\Rightarrow N - mg\cos\theta = -m\frac{v^2}{R}$

3- When point P leaves the sphere N=0 so:

$mg.\cos\theta = m\frac{v_p^2}{R} \Rightarrow v_p^2 = Rg.\cos\theta$

(*)$\Rightarrow$ $R = \frac{1}{2}R\ g\cos\theta +$ g Rcos $\theta \Rightarrow cos\theta = \frac{2}{3}$ so $\theta_0 = 48°$

The material point P leaves the sphere at height: $h_p = \frac{2}{3}R$

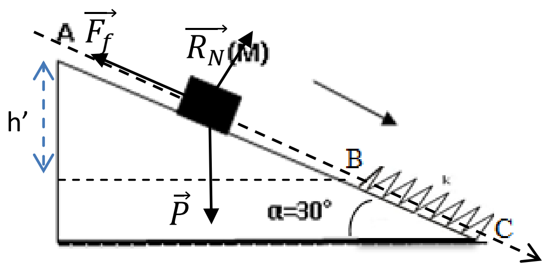
The angle relative to the horizontal at which the point leaves the half-sphere is: *90-48=52*

4- The velocity of the material point at this point:

$$v_p^2 = Rg.\cos\theta$$

$$\Rightarrow v_p = \sqrt{\frac{2}{3}Rg}$$

Exercise 5



1- The acceleration of mass m on AB:

By applying the FPD: $\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \overrightarrow{\mathrm{R}_N} + \vec{f} = m\vec{a}$

Following (Ox): $-$ f +p_x= -f + m g sinα=ma….(1)

Following (Oy): R_N-p_y=0 ⇒ R_N=m g cosα……(2)

μ_d=tgφ=F_f/R_N ⇒F_f=N tg φ so F_f= μ_d m g cosα

(1): - μ_d m g cosα + m g sinα = m.a ⇒ a= g (sinα- μ_d cosα) = 3.27m/s^2

2- The velocity at point B:

we have v_A=0

$and\ v_B^2 - v_A^2 = 2a(AB) \Rightarrow v_B^2 = 2a(AB)$
with (AB)=8m

$$\Rightarrow v_B = \sqrt{2(3.27)(8)} = 7.23 m/s^{-1}$$

3- Applying the kinetic energy theorem, find the speed of the block when it reaches point B.

$$\Delta Ec = \Sigma W_{f\,ext} \Rightarrow Ec_B - Ec_A = W_p + W_{Ff} + W_{RN}$$

$\frac{1}{2}mv_B^2 = mgsin\alpha\ AB - F_f AB$

And

$\frac{1}{2}mv_B^2 = mgsin\alpha\ AB - \mu_d\ \mathrm{m\ g\ cos\alpha}\ AB$

so $v_B = \sqrt{g.2.AB\ sin\alpha - 2\mu_d\ \mathrm{g\ cos\alpha}\ AB}$

$$v_B = \sqrt{g.2.AB(sin\alpha - \mu_d\ \mathrm{cos\alpha})}$$

4- At point B, the block touches a spring with stiffness constant k=100N/m at speed VB.

Calculate the maximum compression (x) of the spring?

(we give g =10 m/s^2).

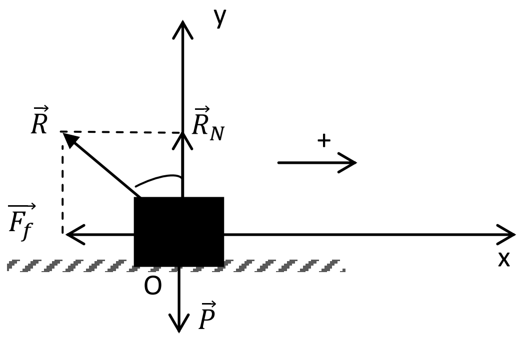
$\Delta E_M = E_{M_C} - E_{M_B} = \Sigma W_{f_{NC}} \Rightarrow (E_{C_C} + E_{P_C}) - \left(E_{C_B} + E_{P_B}\right) = W_{Ff}$

$\Rightarrow \frac{1}{2}kx^2 - \frac{1}{2}mv_B^2 - mg\,h' = \frac{1}{2}kx^2 - \frac{1}{2}mv_B^2 - mg\,(AB\,sin\alpha) = -\mu_d\,\mathrm{m\,g\,cos\alpha}\,AB$

So;

$$x = \sqrt{\frac{m(v_B^2 + g\,2(AB\,sin\alpha) - \mu_d\,2\,\mathrm{g\,cos\alpha}\,AB)}{k}} = 1.07\mathrm{m}$$

Exercise 6



1- Representation of forces on the figure:

2- Acceleration:

According to the fundamental principle of dynamics:

$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{R}_N + \vec{F_f} + \vec{P} = m\vec{a}$

Following (Ox): - F_f =ma (1)

Following (Oy): R_N -P=0

$\mu = \frac{F_f}{R_N} \Rightarrow F_f = \mu\,R_N$; with P=R_N so $F_f = \mu\,\mathrm{m.g}$

(1) $\Rightarrow -\mu\,mg = m.a$

so a= $-\mu\,g = -1\ \mathrm{m}/s^2$

a = -1 so we have uniformly decelerated rectilinear motion.

3- Il n’y a pas de frottement donc d’après le principe de conservation de l’énergie mécanique.

$E_{Mi} = E_{Mf} \Rightarrow E_{Ci} + E_{Pi} = E_{Cf} + E_{Pf}$ with v_i=0 so $E_{Ci} = 0$

$E_{Pi} = mgl(1 - cos\alpha)$, $E_{Cf} = \frac{1}{2}mv_f^2$ and $E_{Pf} = 0$

$mgl(1-cos\alpha) \ = \frac{1}{2}mv_f^2 \Rightarrow \frac{1}{2}mv_B^2 = mgl(1-cos\alpha)$

So $v_B = \sqrt{2gl(1-cos\alpha)}$

4- According to the principle of conservation of momentum:

P_i (before impact) = P_f (after impact)

$$m\overrightarrow{v_B} + \vec{0} = \vec{0} + m\overrightarrow{v_A} \Rightarrow v_A = v_B = \sqrt{2gl(1-cos\alpha)}$$

5- According to the principle of kinetic energy between O and C:

$$\Delta E_C = \Sigma W_{(Fext)} \ \Rightarrow E_{C_C} - E_{C_O} = W_P + W_{R_N} + W_{F_f}$$

$$\frac{1}{2}mv_C^2 - \frac{1}{2}mv_O^2 = -F_f(OC)$$

With F_f= μ mg, $v_O^2 = 2gl(1-cos\alpha)$ and OC=d

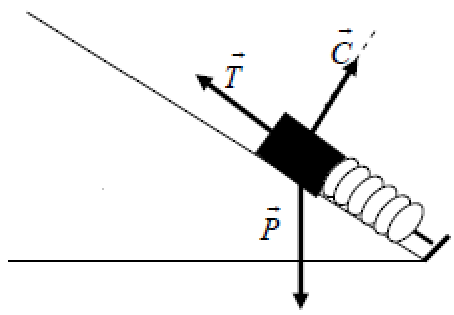
So $v_C = \sqrt{2gl(1-cos\alpha) - 2\mu gd}$

6- Find the angle α so that v_C=0

$$v_C = 0 \Rightarrow \sqrt{2gl(1-cos\alpha) - 2\mu gd} = 0$$

So $\cos\alpha = 1 - \frac{\mu d}{l} \Rightarrow \alpha_{min} = \frac{\pi}{2}$

- 1- Representation of forces on the figure:



2- There is no friction, so according to the principle of conservation of mechanical energy between point C and D:

E_{MC}=E_{MD} $\Rightarrow$ $\Delta E_c = -\Delta E_c \Rightarrow E_{Cc} + E_{Pc} = E_{CD} + E_{PD}$

With v_D=0 because body A changes direction at point D, so $E_{CD} = 0$

The body approaches the CD section without any initial speed, so v_C=0 and E_{Cc}=0

$$E_{Pc} = mgh \text{ with } h = (L+x)sin\beta \text{ so } E_{Pc} = mg\,(L+x)\,sin\beta$$

$$E_{PD} = \frac{1}{2}kx^2$$

So $\frac{1}{2}kx^2 = mg\,(L+x)sin\beta \Rightarrow \frac{1}{2}kx^2 - mg\,Lsin\beta - mgxsin\beta = 0$

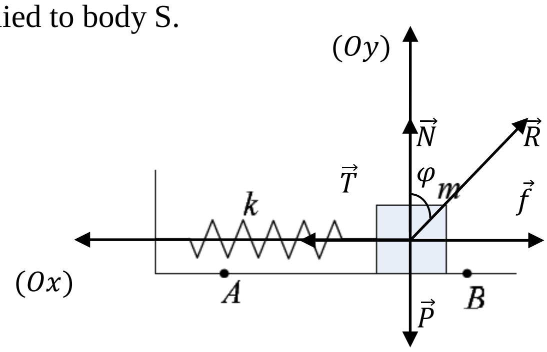
$\Rightarrow 70x^2 - x - 1 = 0$

So x=12.7cm

Exercise 7

A solid body S of mass m is linked on one side to a spring of stiffness K, while the other side of the spring is fixed. The body is moved horizontally from its equilibrium position by a distance x and then released (μ =tg φ : coefficient of friction).

1. Show the forces applied to body S.



2- Calculate the speed V_B corresponding to the movement of S from its equilibrium position.

$$\Delta Ec = \Sigma W_{f_{ext}} \Rightarrow Ec_B - Ec_A = W_p + W_T + W_N + W_f$$

With, $W_{\vec{p}} = W_{\vec{R}} = \vec{0}$ because $\vec{R}$ *and* $\vec{p} \perp \overrightarrow{Ox}$

So $Ec_B = W_T + W_f$ (*) $\quad$ ($Ec_A = 0$ *because the initial velocity is null*)

We have, $\vec{F} = -kx\vec{\imath}$ and $\overrightarrow{dl} = dx\vec{\imath}$

$$\Rightarrow W_{\vec{F}} = \int dW_{\vec{F}} = \int \vec{F}\,.\overrightarrow{dl} = -k\int_x^0 xdx = \frac{1}{2}kx^2$$

so $W_T = \frac{1}{2}kx^2$ and $W_f = -f\ x$

According to the fundamental principle of dynamics:

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{N} + \vec{p} + \vec{T} + \vec{f} = m\vec{a}$$

$$\mu = tg\varphi = \frac{f}{N}$$

We choose a frame of reference composed of the (Ox) axis following the axis of motion, i.e. in the same direction as $\vec{T}$, and the (Oy) axis following $\vec{N}$.

By projecting onto the Oy axis

N-p=0 $\Rightarrow$ $N = mg\ and\ \ \mu = tg\varphi = \frac{f}{mg}$

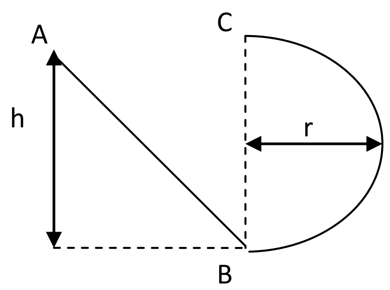
So f=mg tg φ

Then (*)$\Rightarrow \frac{1}{2}mv_B^2 = \frac{1}{2}kx^2 - \text{mg x } \mu$

Hence,

$$v_B^2 = \frac{2}{m}\left(\frac{1}{2}kx^2 - \text{mg x } \mu\right)$$

Exercise 8



A ball slides without friction inside a gutter.

Find the smallest height h_{min} from which the ball is launched to reach point C, without leaving the gutter.

According to the principle of conservation of mechanical energy:

- Between two points A and B:

$$E_{M_A} = E_{M_B} \Rightarrow E_{C_A} + E_{P_A} = E_{C_B} + E_{P_B}$$

Then $E_{P_A} = E_{C_B}$(*)

Because $E_{C_A} = 0$

$since\ v_A = O\ \ because\ the\ ball\ is\ launched\ without\ initial\ velocity$

$and\ \ E_{P_B} = 0\ hence\ E_{P_B} = mgh\ and\ h = 0$

So (*)$\Rightarrow mgh = \frac{1}{2}mv_B^2$

- Between points B and C:

$$E_{M_B} = E_{M_C} \Rightarrow E_{C_B} + E_{P_B} = E_{C_C} + E_{P_C}$$

Then $E_{C_B} = E_{C_C} + E_{P_C}$

So $mgh = \frac{1}{2}mv_C^2 + 2mgr$ (*')

The ball leaves the gutter at point C when N=0,

According to the fundamental principle of dynamics:

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{N} + \vec{p} = m\vec{a}$$

We choose a reference frame consisting of the axis (OT) tangent to the half-sphere and the axis (ON) following the radius and in the direction of $\vec{N}$ and $\vec{p}$.

Projecting onto the ON axis:

N+p = m.a_N $\Rightarrow N + mg = m\frac{v^2}{r}$

At point C for N=0 the speed will be:

$mg = m\frac{v_C^2}{r} \Rightarrow v_C^2 = r\,g$

(*')$\Rightarrow mgh_C = \frac{1}{2}mgr + 2mgr = \frac{5}{2}mgr$

So $h_C = \frac{5}{2}r$

h_C is the minimum value at which the ball reaches point C without leaving the gutter.

for h<h_C the ball does not reach point C.

for h>h_C the ball reaches point C and leaves the trough.

Exercise 9

1. There are no non-conservative forces. We can therefore use the principle of conservation of mechanical energy between points A and B:

$\Delta E_m = 0$ so $E_{TA} + E_{ppA} = E_{TB} + E_{ppB}$

Taking the origin of the potential energies in B, and knowing that the velocity in A is equal to 0, we obtain: gh =(½)$m.v_B^2$

So: $v_B = \sqrt{2gh}$ with $h = l\ sin\alpha$

Then $v_B = \sqrt{2g.l\sin\alpha}$

2. Study of the system between points B and C:

If we choose the origin of abscissas at B (x_B = 0) and therefore x_C = d (maximum compression distance), we find:

$$v_c^2 - v_B^2 = 2ax \quad (*)$$

To find the expression for acceleration, we use the fundamental principle of dynamics:

$$\Sigma \overrightarrow{F_{ext}} = m\vec{a}$$

The forces are weight, reaction R and spring return force Fe:

$$\vec{P} + \vec{R} + \vec{F_e} = m\vec{a}$$

Projection on axis (Ox):

$$mg\ sin\alpha - Fe = mg\ sin\alpha - kx = ma$$

Then: $a = g\ sin\alpha - (k/\ m)\ x$

Replace in (*), to find:

$$-v_B^2 = 2(\text{g sin}\alpha - (\text{k}/\ \text{m})\ \text{x})x \Rightarrow (k/\ m)\ d^2 - 2g\ sin.d - v_B{}^2 = 0$$

$$(\text{k}/2)\ d^2 - (g\ \text{m}\ sin\alpha)\ d - (\text{m g. l sin}\alpha) = 0$$

Solving this second-degree equation in d yields two solutions. One is negative; we retain only the positive solution (the physical solution):

$$d = \frac{mgsin\alpha + \sqrt{m^2g^2sin^2\alpha + 2kmlsin\alpha}}{k}$$

EXAMS

Continuous Assessment N°01

Continuous assessment in mechanics

Exam duration: 01 h 30mn

Exercise 1: (05 Pts)

A. The momentum P (P=mϑ where m is a mass and ϑ is a velocity) associated with a photon depends on its frequency f according to the following expression:

$$P = \sigma^{\alpha} f^{\beta} c^{\gamma}$$

Where c is the speed of light and σ has the following dimension $[\sigma] = M.L^2.T^{-1}$.

Using dimensional analysis, find the exponents α, β and γ.

B. The average velocity of the molecules of a gas is written in the following formula:

$$\vartheta = \sqrt{\frac{PV}{m}}$$

m being the mass of the molecule, V the volume, and p the pressure of the gas.

Calculate the relative uncertainty in ϑ as a function of **Δp, Δm and ΔV**.

Exercise 2: (05 Pts)

A. $\vec{i},\ \vec{j}\ and\ \vec{k}$ being the unit vectors of an orthonormal reference frame (Oxyz), consider the vectors. $\vec{r_1} = 2\,\vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{r_2} = \vec{i} + \vec{j} + \vec{k}$

1- Calculate the vector product $\vec{r_1} \wedge \vec{r_2}$.

2- Deduce the angle θ formed by the two vectors $\vec{r_1}$ and $\vec{r_2}$.

B. Let be a polar coordinate system with origin O and unit vectors $\vec{u_\rho}, \vec{u_\theta}$.

M is a point with coordinates $\begin{cases} \rho = 2t^3 + 1 \\ \theta = \omega t \end{cases}$ (ω constant).

1- Using a detailed diagram, give the expression of the position vector $\overrightarrow{OM}$ and calculate the velocity vector of point M in polar coordinates.

2- Write this velocity vector $\vec{v}$ (M) in cartesian coordinates ($\vec{i}, \vec{j}, \vec{k}$).

Exercise 3: (05 Pts)

A particle moves along a trajectory whose equation is $x^2 + y^2 = 4$ such that $x(t) = 2\sin(\omega t)$.

Knowing that ω is constant and at t=0, the mobile is at point M (0, R), Determine:

1) The component y(t).
2) Velocity and acceleration vector components and their moduli.
3) Tangential and normal accelerations.
4) The nature of the motion.

Good luck

Continuous Assessment N°02

Continuous assessment in mechanics

Exam duration: 01 h 30mn

Exercise 1: (06 pts)

A. Check the homogeneity of this formula:

$$p = \rho g h_1 + h_2 F$$

Such as: P a pressure, ρ the density, g an acceleration of gravity, h_1 and h_2 are heights and F a force.

B. The period T of oscillation of a pendulum of length l in a gravitational field (gravitational acceleration) g is proposed in the following form:

$$\boldsymbol{T = k.\, l^{\alpha}.\, g^{\beta}}$$

1. Find α and β such that k is a dimensionless constant. Write the law e the period T.
2. Calculate the relative uncertainty on T as a function of Δl and Δg.

Exercise 2: (07 pts)

A. Let the points M_1 (+1,+1,+1), M_2 (+2,+2,+1) et M_3 (+2,+1,0) ;

1. Find the angle $\mathbf{M_1\widehat{M_2}M_3}$.
2. Calculate the area of the triangle $M_1M_2M_3$

B. A material point M is identified by its Cartesian coordinates (x, y, z).

1. Write down the relationship between Cartesian coordinates and cylindrical coordinates (using a diagram).
2. Write the position vector in cylindrical coordinates and deduce the velocity vector in the same coordinate system.
3. If the position of the point is marked in cylindrical coordinates by: $\begin{cases} \rho = 4t^2 \\ \theta = \omega t \\ z = \sqrt{t} \end{cases}$

Find the expression of the velocity vector $\vec{v}$ in cylindrical coordinates.

Exercise 3: (06 pts)

The coordinates x and y of a moving point M in the plane (oxy) vary with time t according to the following relationships: $x = 2t$, $y = 2t^2$

Find:

1. The equation of the trajectory.
2. Velocity components and modulus v.
3. Components of acceleration and its modulus a.
4. Nature of motion.
5. Tangential acceleration a_T and normal acceleration a_N; deduce the radius of curvature.

Good luck

Continuous Assessment N°03

Exam to replace the Continuous assessment in mechanics

Exam duration: 01 h 30mn

Exercise 01: (05 Pts)

The limiting **velocity** ϑ of a ball of radius r of **mass m** and **density ρ**, falling into a viscous medium (a fluid) of **density ρ_f** and viscosity coefficient η معامل الميوعة , is given by :

$$\vartheta = \frac{2r^2(\rho - \rho_f)}{9\eta} g$$

g is the acceleration of gravity.

1. Using the dimensional equations, determine the dimension of η and derive its unit in the MKSA system.
2. Determine the relative uncertainty on η, as a function of Δr, Δρ, $\Delta\rho_f$ and Δυ.

Exercise 0 2 : (05 Pts)

A. Let the points A (+1,+1,+1), B (+2,+2,+1) et C (+2,+1,0)

Calculate the scalar product $\overrightarrow{AB}.\overrightarrow{AC}$ and the vector product $\overrightarrow{AB} \wedge \overrightarrow{AC}$. What do these two products represent? Deduce the angle between the vectors $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$.

B. Define cylindrical coordinates and give the transition relationships between Cartesian coordinates and cylindrical coordinates.

Write the position vector in cylindrical coordinates and calculate the velocity vector in this coordinate system.

Exercise 03: (05 Pts)

From the ground, a balloon rises with a constant initial velocity v_0 (following y). The wind gives the balloon a horizontal velocity $\mathbf{V_x=\gamma \cdot y}$ (γ constant).

a- Determine the equations of motion x(t) and y(t). Deduce the equation of the trajectory y=f(x).

b- Calculate the accelerations a, a_N and a_T. Deduce the radius of curvature.

Good luck

Continuous Assessment N°04

Continuous assessment in mechanics

Exam duration: 01 h 30mn

Exercise 1: (6 pts)

The velocity limit reached by a weighted parachute is a function of its weight P and its surface area S, and is given by: $v = \sqrt{\frac{P}{K.S}}$

1) Give the dimension of the constant k.
2) Calculate the limiting speed of a parachute with the following characteristics : M=90 kg, S=80 m^2, g=9,81 m/s^2, and k=1,15 MKS.
3) With the weight known to within 2% and the surface area to within 3%, calculate the relative uncertainty $\Delta v/v$ on the velocity v, and the absolute uncertainty Δv and deduce the condensed form of this velocity.

Exercise 2: (5 pts)

A. In the vector space related to the orthonormal basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$, consider the vectors $\vec{U}(0, 3, 1)$, $\vec{V}(0, 1, 2)$.

1) Calculate the scalar product $\vec{U}.\vec{V}$ and the angle φ acute between $\vec{U}$ *and* $\vec{V}$.
2) Determine the components of the vector $\vec{W} = \vec{U} \wedge \vec{V}$ then calculate $\|\vec{W}\|$ by tow methods. What does the latter represent.
3) Calculate the mixed product $(\vec{U}, \vec{V}, \vec{W})$, what does this product represent.

B. Does each of the following expressions have a meaning? If yes, specify whether it is a vector or a real. If not, say why (without calculation).

1) $\vec{A}.(\vec{B} \wedge \vec{C})$	**2)** $\vec{A} \wedge (\vec{B} . \vec{C})$	**3)** $\vec{A} \wedge (\vec{B} \wedge \vec{C})$
4) $\vec{A} . (\vec{B}.\vec{C})$	**4)** $(\vec{A} \wedge \vec{B}) \wedge (\vec{C} \wedge \vec{B})$	**5)** $(\vec{A} \wedge \vec{B}).(\vec{C} \wedge \vec{C})$

Exercise 3: (8 pts)

Let be a cylindrical reference frame with origin O, unit vectors $\overrightarrow{\mathbf{u_\rho}}, \overrightarrow{\mathbf{u_\theta}}, \overrightarrow{\mathbf{u_z}}$. M is any point with coordinates (ρ, θ, z).

Using a detailed diagram, give the expression for the position vector $\overrightarrow{\mathbf{OM}}$ as a function of the unit vectors $\overrightarrow{\mathbf{u_\rho}}, \overrightarrow{\mathbf{u_\theta}}, \overrightarrow{\mathbf{u_z}}$.

1. Find the velocity vector in cylindrical coordinates.
2. Express the elementary displacement vector in cylindrical coordinates.
3. Write the expression for the elementary volume in this frame of reference and deduce the volume of a cylinder.

Good luck

Continuous Assessment N°05

Continuous assessment in mechanics

Exam duration: 01 h 30mn

Exercise 1: (6pts)

A) A particle of mass m enclosed in a cubic box of side L, has a kinetic energy E such that:

$$E = \frac{\pi^2 \sigma^2}{2mV^{\frac{2}{3}}} n^2$$

Where **V** the volume of the box and n a dimensionless number.
Using the dimensional equations, find the dimension of σ.

B) The average velocity of the molecules in a gas can be written as:

$$\vartheta = \sqrt{\frac{PV}{m}}$$

m being the mass of the molecule, **V** the volume, and **p** the pressure of the gas.
Calculate the relative uncertainty in ϑ as a function **of Δp, Δm and ΔV**.

Exercise 2 : (8pts)

A material point M is identified by its Cartesian coordinates (x, y).

1. Write the relationship between Cartesian coordinates and polar coordinates.
2. Give the expression of the unit vectors $\overrightarrow{U_r}$ *and* $\overrightarrow{U_\theta}$ as a function of the unit vectors $\vec{\imath}$ *and* $\vec{\jmath}$.
3. Find the expression of the velocity vector $\vec{v}$ of point M in polar coordinates.
4. Give the expression of the vector $\vec{A} = 2x\vec{\imath} - y\vec{\jmath}$ in polar coordinates.

Exercise 3: (6pts)

The coordinates x and y of a moving point M in the plane (oxy) vary with time t according to the following relationships: $x = t + 1$, $y = \frac{t^2}{2} + 2t$

Find :

1. The equation of the trajectory.
2. Components of velocity and its modulus v.
3. Components of acceleration and its modulus.
4. Nature of motion.
5. Tangential and normal accelerations.
6. Radius of curvature R.

Good luck

Continuous Assessment N°06

Exam to replace the Continuous assessment in mechanics

Exam duration: 01 h 30mn

Exercise 1: (6pts)

The energy of a photon is given by the expression E=hυ, where h is the Planck constant and υ the photon frequency.

1- Give the dimension of h.

2- Find the expression for the wavelength λ, assuming it to be of the form **λ=k.**$\boldsymbol{h^x m^y V^z}$ .Where k is a dimensionless constant, m and V represent the photon's mass and velocity respectively.

3- Determine the relative uncertainty on λ as a function of Δm, Δh and ΔV.

Exercise 2: (8pts)

1. Define cylindrical coordinates.
2. Write the unit vectors $\vec{u}_\rho$, $\vec{u}_\theta$ *and* $\vec{u}_z$ in the cylindrical coordinate system in terms of the unit vectors ($\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$).
3. Write the elementary displacement vector in cylindrical coordinates.
4. Deduce the volume of a cylinder.
5. Write the vector $\vec{A} = x\vec{\imath} + 2y\vec{\jmath} - z\vec{k}$ in cylindrical coordinates.

Exercise 3: (6pts)

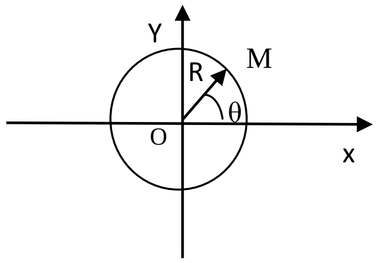
A. Let a moving point M describe a circle of radius R and center O with angular velocity ω=dθ/dt. At time t=0 the point M is at A.

1. Write the coordinates of M as a function of R and θ.
2. Calculate the modulus of the velocity of point M.
3. Determine the components of the acceleration on the axes Ox and Oy (Cartesian coordinates) on the one hand, and on the axes parallel and perpendicular to OM on the other (polar coordinates).

B. Assume α=dω/dt (α is a non-zero constant). Give the expressions for ω and θ as a function of time.

Recall that at t=0, $\theta_0=0$ and $\omega=\omega_0$.

What relationship exists between ω and θ.



Good luck

Continuous Assessment N°07

Continuous assessment in mechanics

Exam duration: 01 h 30mn

Exercise 1: (06pts)

A) Experience has shown that the velocity v of sound in a gas is a function only of the gas density ρ and its coefficient of compressibility χ . It is given by :

$$\boldsymbol{v = k\,\rho^{x}\chi^{y}}$$

Recall that χ is homogeneous to the inverse of a pressure, with k a dimensionless constant. Determine the velocity of sound relationship.

B) The focal length f of a lens is determined from the formula:

$$f = \frac{D^2 - a^2}{4D}$$

Calculate the absolute uncertainty Δf as a function of ΔD and Δa.

Exercise 2: (08pts)

Let two vectors $\vec{A}$ and $\vec{B}$ in the orthonormal reference frame (Oxyz), be defined by :

$$\vec{A} = \mathbf{2} \cdot \vec{\boldsymbol{i}} + \mathbf{4} \cdot \vec{\boldsymbol{j}} - \mathbf{5} \cdot \vec{\boldsymbol{k}} \;\; , \; \vec{B} = -\vec{\boldsymbol{i}} + \vec{\boldsymbol{j}} - \mathbf{2} \cdot \vec{\boldsymbol{k}}$$

1. Calculate and represent the two vectors ($\vec{A} + \vec{B}$) and ($\vec{A} - \vec{B}$).
2. Calculate their moduli.
3. Calculate the scalar product ($\vec{A} . \vec{B}$) . Determine the angle $\theta = (\vec{A}, \vec{B})$.
4. Calculate the vector product ($\vec{A} \wedge \vec{B}$), what present $|\vec{A} \wedge \vec{B}|$?
5. Write the vector $\vec{\boldsymbol{C}} = \boldsymbol{y}\,.\,\vec{\boldsymbol{i}} - \mathbf{2}.\boldsymbol{x}.\vec{\boldsymbol{j}} + \boldsymbol{z}.\vec{\boldsymbol{k}}$ in cylindrical coordinates i.e. as a function of $\boldsymbol{\rho}$, $\boldsymbol{\theta}$, $\boldsymbol{z}$ and $\overrightarrow{\mathbf{u}_\rho}$, $\overrightarrow{\mathbf{u}_\theta}$, $\overrightarrow{\mathbf{u}_z}$. (**indication:**: use the relations for passing coordinates (x,y,z) and unit vectors $\vec{i}, \vec{j}, \vec{k}$ as a function of unit vectors $\overrightarrow{u_\rho}$, $\overrightarrow{u_\theta}$, $\overrightarrow{u_z}$).

Exercise 2: (06pts)

The coordinates x and y of a moving point M in the plane (oxy) vary with time t according to the following relationships: $x = t$, $y = t^2 - t$

Find:

1. The equation of the trajectory.
2. Velocity components and modulus v.
3. Components of acceleration and its modulus.
4. Tangential and normal accelerations as a function of the modulus of v.
5. Radius of curvature R.

Good luck

Final Exam N°01

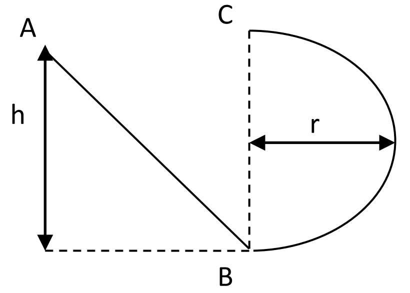
Final Exam of Mechanics

Exam duration: 01 h 30mn

Course questions: (6pts)

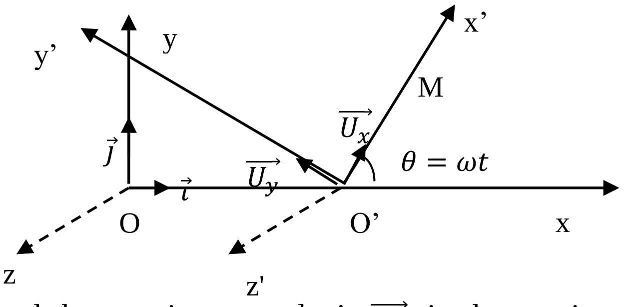
1- State and demonstrate the kinetic energy theorem.

2- A ball slides without friction inside a gutter.
Find the smallest height h_{min} from which the
ball is launched to reach point C, without leaving the gutter.



Exercise 1 : (7pts)

In the (Oxy) plane, a point O' (the origin of the moving reference frame) moves along the (Ox) axis such that |OO'|=t. The reference frame (O'X'Y') rotates around Oz with a constant angular velocity ω. A moving point M (O'M=r) moves along the axis (O'X') according to the law $r = r_0\,(\cos\omega t + \sin\omega t)$ with r_0= constant. Determine at time t as a function of r_0 and ω.



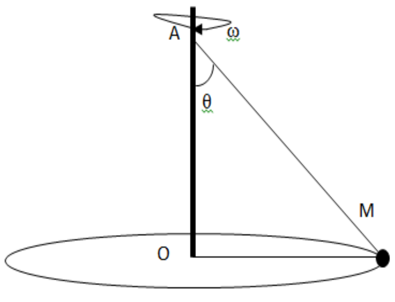
1- The velocity $\overrightarrow{v_r}$ and the entrainment velocity $\overrightarrow{v_e}$, in the moving reference frame (OX'Y'), deduce the absolute velocity $\overrightarrow{v_a}$ in the same reference frame.

2- Relative acceleration $\overrightarrow{a_r}$, entrainment acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$ in the moving frame of reference, deduce the absolute acceleration $\overrightarrow{a_a}$ in this frame of reference.

Exercise 2 : (7pts)

A ball of mass m is attached by two wires (Am and Om) to a vertical pole. The whole system rotates with a constant angular velocity ω around the axis of the pole (we know g the acceleration of gravity, θ and $L = |\overrightarrow{OM}|$)

1. Assuming ω is large enough to keep both wires taut, find the force (wire tension) each wire exerts on the ball.
2. What is the minimum angular velocity ω_{min} for which the bottom wire remains taut.



Good luck

Final Exam N°02

Final Exam of Mechanics

Exam duration: 01 h 30mn

Course questions: (5pts)

1- Why use dimensional analysis?

2- What can it say about the total mechanical energy of a system in the presence of frictional forces?

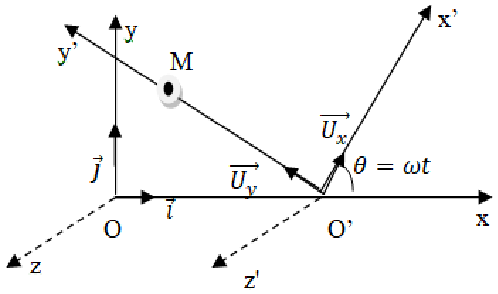
3- What is the difference between a conservative force (قوة منحفضة) and a non-conservative force? Give an example for each one.

4- Calculate the work of a force F=1.5 10^4N supplied to move a body a height (AB) of 3 meters (vertically).

5- Calculate the work of the spring return force with stiffness constant k ($\overrightarrow{dl} = dx.\vec{\imath}$).

Exercise 1: (7pts)

Consider the fixed reference frame R(Oxyz) where point O' moves **along axis (Ox)** with **constant velocity v_0**. Linked to O' is the moving reference frame (O'x'y'z') which rotates **around (Oz)** with **constant** angular velocity ω. A moving point M moves **along the (O'y')** axis with constant **acceleration γ**.



At time t=0, the axes (Ox) and (O'x') are coincident and M is at O.

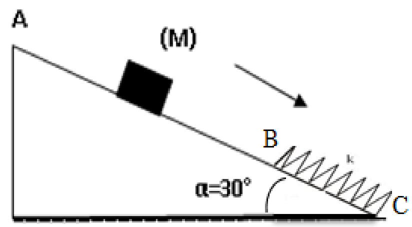
Calculate in the moving frame:

1- The relative velocity $\overrightarrow{v_r}$ and the entrainment velocity $\overrightarrow{v_e}$, deduce the absolute velocity $\overrightarrow{v_a}$.

2- The relative acceleration $\overrightarrow{a_r}$, the entrainment acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$, deduce the absolute acceleration $\overrightarrow{a_a}$.

Exercise 2: (8pts)

Consider a small block of mass m =2kg dropped without initial velocity at point A of an inclined plane at an angle α=30° to the horizontal. Point A is at a height h_A=5m from the horizontal.



1- Knowing that the coefficient of dynamic friction on **plane AB is μ_d=0.2**, applying the fundamental principle of dynamics, what is the acceleration of the block on plane AB=8m?

2- Calculate the speed of the block when it reaches point B.

3- Using the kinetic energy theorem, find the speed of the block at point B.

4- At point B, the block hits a spring with stiffness constant k=100N/m at speed V_B. Calculate the maximum compression (x) of the spring (given g =10 m/s^2).

Good luck

Final Exam N°03

<u>Final Exam of Mechanics</u>

<u>Exam duration: 01 h 30mn</u>

<u>Exercise 1</u>: (6 pts)

A) The position of a moving body, subjected to **a force F**, is marked by its **abscissa x**, at any **instant t** according to the relation: **x= $a.(bt)$ +$F.c$**

Give the dimensions of the different quantities a, b and c. What might the quantity c represent?

B) The average velocity of the molecules of a gas is written in the following formula:

$$\boldsymbol{\vartheta} = \sqrt{\frac{\boldsymbol{PV}}{\boldsymbol{m}}}$$

m being the mass of the molecule, **V** the volume, and **p** the pressure of the gas.

Calculate the relative uncertainty on **ϑ** as a function of **Δp, Δm and ΔV.**

<u>Exercise 2</u>: (8 pts)

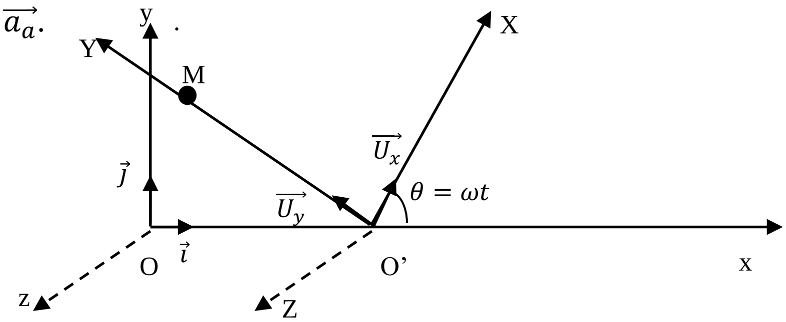
Let the reference frame be R(Oxyz) and the point O' moves on the axis (Ox) with a constant velocity v0. We link to O' the reference frame (O'XYZ) which rotates around (Oz) with a constant angular velocity ω. Mobile M moves along axis (O'Y) such that $\left|\overrightarrow{O'M}\right| = (t^2 + 2)$ (without initial velocity).

At time t=0, axis (O'X) is coincident with (Ox) and point M is at O'.

Calculate in the **moving reference frame** :

1- The relative velocity $\overrightarrow{v_r}$ and the entrainment velocity $\overrightarrow{v_e}$, deduce the absolute velocity $\overrightarrow{v_a}$.

2- The relative acceleration $\overrightarrow{a_r}$, the entrainment acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$, deduce the absolute acceleration $\overrightarrow{a_a}$.



<u>Exercise 3:</u> (6 pts)

A particle moves along a trajectory whose y-coordinate equation is given by: $y(t) = t^2+1$ such that at each instant $v_x=v_0=cste$.

If at t=0, $x_0=0$; determine :

1- The equation of the particle's trajectory.

2- The particle's velocity and acceleration.

3- Normal and tangential accelerations and radius of curvature.

Good luck

Final Exam N°04

Final Exam of Mechanics

Exam duration: 01 h 30mn

Cours Questions: (5pts)

I. What is the work done by a force $\vec{F} = 2t\,\vec{i}$ acting on a particle of mass m=2kg for a displacement along the horizontal (dl = dx).

1. Is this force conservative?
2. Name three conservative forces.
3. Deduce the power of this force.

II. Define the dynamic and static coefficients of friction. What is the relationship between these two coefficients? what is the dimension of the friction coefficient?

Exercise 1 : (8 pts)

In the Oxy plane, consider a system of moving axes (O'XY), such that (Ox) makes a variable **angle θ with (O'X).** Point O' moves **along axis (Ox)** with constant **acceleration a**. A point M moving along the O'X axis is marked by **O'M = r.** We call relative motion of M its motion with respect to (O'XYZ) and absolute motion with respect to (Oxyz). (see figure 1)
Calculate in the moving reference frame :

1- The relative velocity $\vec{v_r}$ and the entrainment velocity $\vec{v_e}$, deduce the absolute velocity $\vec{v_a}$.
2- The relative acceleration $\vec{a_r}$, the entrainment acceleration $\vec{a_e}$ and Coriolis acceleration $\vec{a_c}$, deduce the absolute acceleration $\vec{a_a}$.

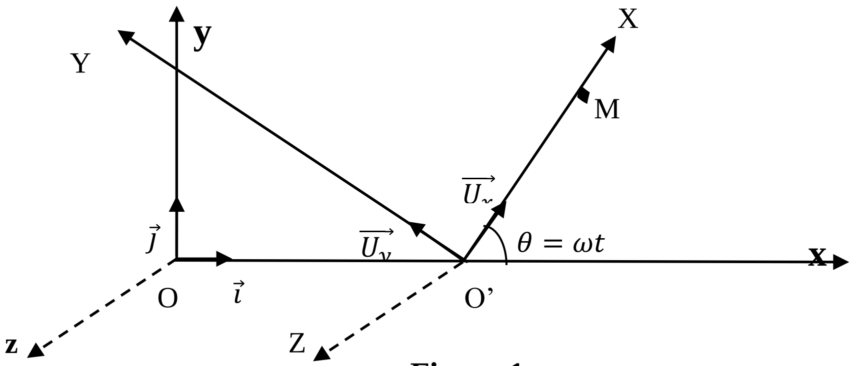


Figure 1

Exercise 2 : (7 pts)

- A ball B of mass m, attached to an inextensible wire of length l, is moved away from its equilibrium position by an angle α. It is dropped without initial velocity.

Passing through the vertical position, the ball strikes (touches) a body A of the same mass and stops, body A passes from point O to point C **(OC=d)** on a rough horizontal plane **of friction coefficient μ**.

1- Show the forces exerted on body A.
2- What is the nature of the motion on the horizontal plane?
3. Express the velocity of ball B just before it touches body A.

4. Using the principle of conservation of momentum of the system, determine the velocity of body A after the interaction.

5. If $v_A=v_B$ at point O, give the velocity of body A at point C as a function of g, l, d, α and μ.

6. By what angle must ball B be moved away for body A to arrive at point C with zero velocity.

- From point C, body A approaches the perfectly smooth (no friction) path CD=L, inclined at an angle β to the horizontal (Fig. 2). It arrives, without initial velocity, on a perfect spring of length l0 and stiffness constant k.

1. Show the forces exerted on A as the spring compresses.
2. What is the value of the spring's maximum compression?

We give m=200g, d=OC=1m, l=10 cm, L=1m, μ=0.1, g=10m/s^2, k=140N/m, β=30°.

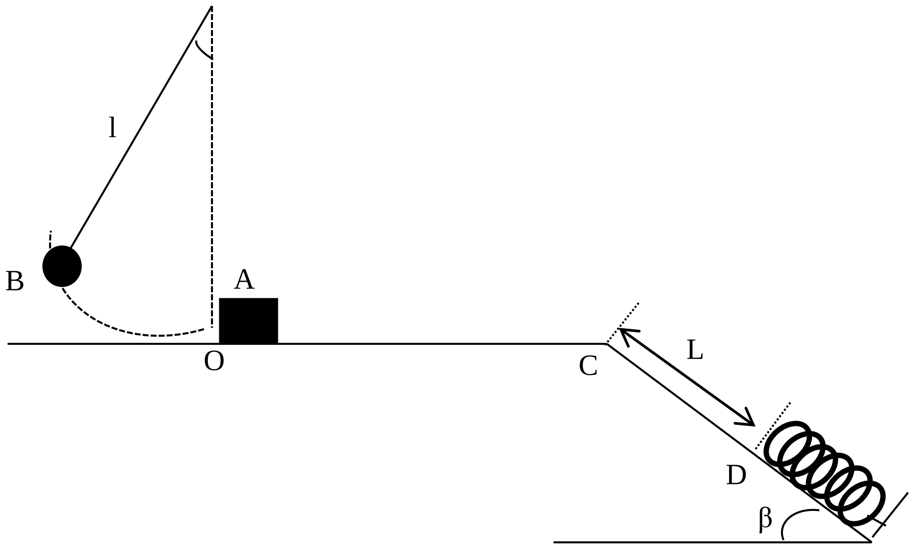


Figure 2

Final Exam N°05

Final Exam of Mechanics

Exam duration: 01 h 30mn

Exercise 1 : (6pts)

KEPLER's third law relates the period T to the semi-major axis a of a planet's orbit around the sun as follows: $\frac{T^2}{a^3} = \frac{4\pi^2}{GM_S}$

With G the universal gravitational constant and Ms the mass of the sun.

We give: G = $(6.668\pm0.005).10^{-11}$SI

For the earth: T = $(365.25636567\pm0.00000001)$ days and a=$(1.4960\pm0.0003).10^{-11}$m

1- Determine the dimension and unit of G.
2- Determine the mass of the sun Ms and the absolute uncertainty Δ Ms on this mass.

Exercise 2: (8pts)

Let the reference frame be R(Oxyz) and the point O' moves along the **axis (Ox**) with a constant **acceleration γ;** and a **positive initial velocity V_0**. We link to O' the reference frame (O'XYZ) which rotates around (Oz) with a constant angular velocity ω. The coordinates of a moving body M in the moving frame of reference are: **X'=t^2+2 and Y'=2t.**

At time t=0, the axis (O'X) coincides with (Ox). (Fig 1)

Calculate in the **moving frame** of reference (O'XYZ) :

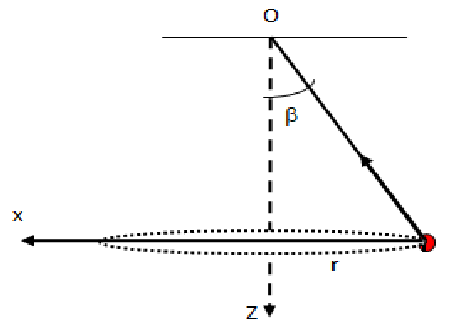
1- The relative velocity $\overrightarrow{v_r}$ and the entrainment velocity $\overrightarrow{v_e}$, deduce the absolute velocity $\overrightarrow{v_a}$.

2- The relative acceleration $\overrightarrow{a_r}$, the entrainment acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$, deduce the absolute acceleration $\overrightarrow{a_a}$.

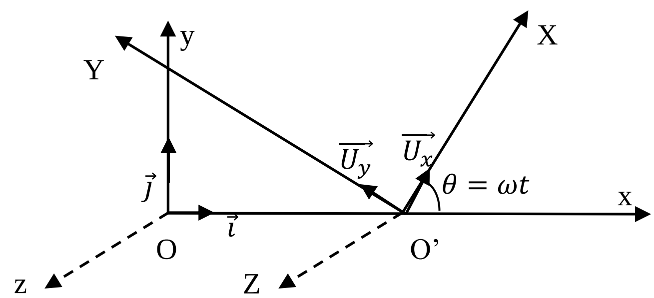
Exercise 3: (6pts)

A material point, of mass m, is suspended at a fixed point O by a wire of length l inextensible and negligible mass. By rotating it around axis Oz, it acquires a constant angular velocity ω. It describes a horizontal circle of radius r. (Fig 2)

1. Find the expression for the wire tension.
2. Find the expression for the inclination β of the wire with respect to the vertical.



(Fig 2)



(Fig 1)

Good luck

Final Exam N°06

Final Exam of Mechanics

Exam duration: 01 h 30mn

Exercise 1 (6pts)

1. A moving body whose motion is uniformly circular is subject to an acceleration:

a- constant c- whose modulus is constant

b- zero d- directed towards the center of the trajectory

Give the correct answers.

2. A particle M moves along a parabolic trajectory with equation :

$y = x^2$ with $x(t) = 2t$

a- Determine the components of velocity and acceleration, and calculate their moduli.

b- Determine the tangential and normal accelerations, and deduce the radius of curvature R.

Exercise 2 (8pts)

A point M moves with constant velocity V_0 on the axis (OX) of a reference frame (OXYZ) which rotates with constant angular velocity ω around (Oz) in the plane (Oxy) ($\overrightarrow{OO'} = \vec{0}$).

1- What is the expression of $\overrightarrow{OM}$ in the fixed reference frame. Calculate absolute velocity and absolute acceleration.

2- Calculate relative velocity and entrainment velocity, check that $\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e}$.

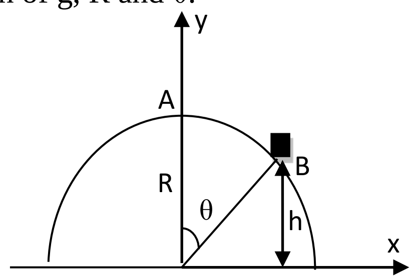
3- Calculate the relative acceleration $\overrightarrow{a_r}$, entrainment acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$, check that $\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_e} + \overrightarrow{a_c}$.

Exercise 3 (6pts)

A piece of ice M of mass m slides frictionlessly over the outer surface of an igloo, which is a half-sphere of radius R with a horizontal base.

At t=0, it is released from point A without any initial velocity.

1. Find the expression for the velocity at point B, as a function of g, R and θ.
2. Using the fundamental relation of dynamics, determine the expression of $|\vec{N}|$ the reaction of the igloo on M at point B as a function of velocity v_B.
3. At what height does M leave the sphere?
4. At what speed does M arrive at the axis (Ox)?



Good luck

Final Exam N°07

Final Exam of Mechanics

Exam duration: 01 h 30mn

Cours Questions: (6pts)

1- Give Newton's three laws.

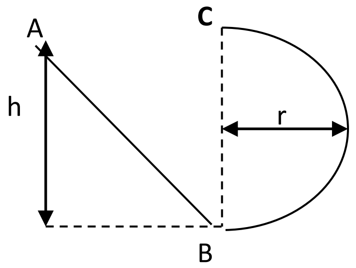
2- What is the difference between a conservative force and a non-conservative force, with examples for each case?

3- In which case do we have conservation of mechanical energy, and what do we have in the opposite case?

4- State the kinetic energy theorem and demonstrate it.

5- A ball is thrown without initial velocity and without friction inside a gutter.

Find the height at which the ball reaches point C and changes direction.



Exercise 1 (7 pts) :

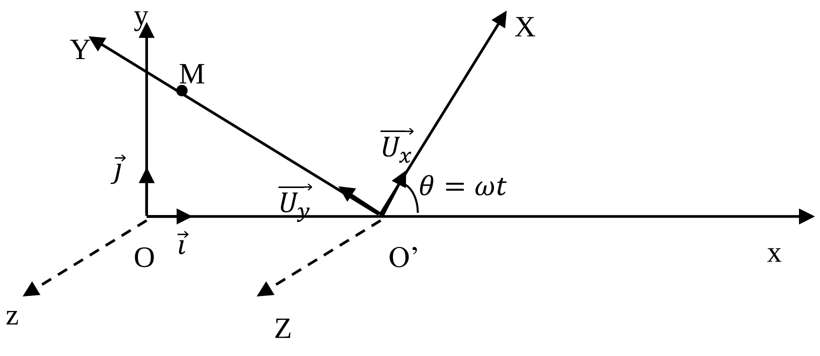
Let the reference frame be R(Oxyz) and the point O' moves on **the axis (Ox)** with a constant **velocity v_0**. We link to O' the reference frame (O'XYZ) which rotates around (Oz) with a constant angular velocity ω. Mobile M moves along **axis (O'Y)** with constant **acceleration γ** (no initial velocity).

At time t=0, axis (O'X) is coincident with (Ox) and point M is at O'.

Calculate in the **moving reference** frame:

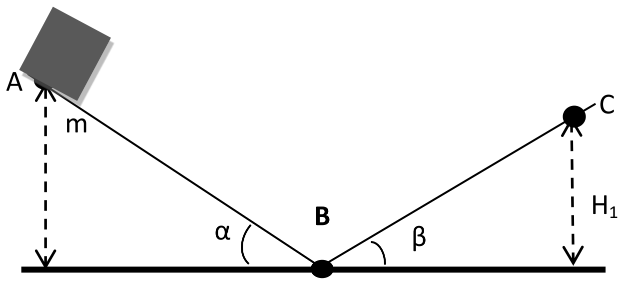
1- The relative velocity $\overrightarrow{v_r}$ and the entrainment velocity $\overrightarrow{v_e}$, deduce the absolute velocity $\overrightarrow{v_a}$.

2- The relative acceleration $\overrightarrow{a_r}$, the entrainment acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$, deduce the absolute acceleration $\overrightarrow{a_a}$.



Exercise 2 (7 pts) :

Consider a small block of mass m =5kg dropped without initial velocity at point A of an inclined plane at an angle α=30° to the horizontal (see figure 2). Point A is at a height h_0=5m.



1- What is the value of the coefficient of static friction μs that keeps the mass in equilibrium at point A?

2- Knowing that the coefficient of dynamic friction on plane AB is μ_d=0.2, apply the fundamental principle of dynamics:

- What is the nature of the motion on AB?
- Calculate the speed of the block when it reaches point B.
- What can be said about the total mechanical energy of the mass m?

3- After passing through point B at speed V_B, the mass moves up the inclined plane BC (angle=20°), and stops at point C. Knowing that the coefficient of friction remains the same, determine the height h1 of point C?

Good luck

Final Exam N°08

Final Exam of Mechanics

Exam duration: 01 h 30mn

Course question (5pts)

1- Which physical quantity has a unit and no dimension?

2- What are the different coordinate systems? In which system are the components dependent?

3- In the case where we have only conservative forces, give the theorems we can use?

4- Define the dynamic and static coefficients of friction. Which is the most important?

Exercise 1 (8pts)

In the (Oxy) plane, consider a system of moving axes (OXY) with the same origin O, rotating with a constant angular velocity **ω around (OZ).** A moving point M moves along **axis (OX)** with constant **acceleration γ** and no initial velocity. We call relative motion of M its motion with respect to (OXY), and absolute motion with respect to (Oxy).

At time t=0, axes (Ox) and (OX) are coincident and M is at O.

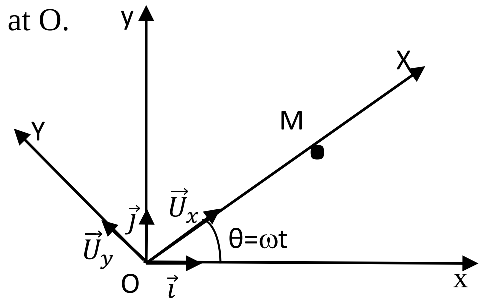
Calculate in the moving reference frame :

1- The velocity and relative acceleration of M.

2- Entrainment velocity of M and acceleration.

3- Coriolis acceleration.

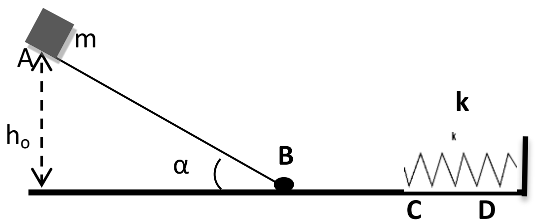
4- Deduce its absolute velocity and acceleration.



Exercise 2 (7 pts)

Consider a small block of mass m =5kg dropped without initial velocity at point A of an inclined plane at an angle α=30° to the horizontal. Point A is at a height h_0=5m from the horizontal.

. What is the value of the coefficient static friction coefficient μ_s that keeps the mass the mass in equilibrium at point A.



2. Knowing that the coefficient of dynamic friction on plane AB is μ_d=0.2, apply the fundamental principle of dynamics:

- What is the nature of the motion on plane AB?
- Calculate the speed of the block when it reaches point B.

3. After passing through point B at speed V_B, the mass arrives at point C. Knowing that the coefficient of friction is negligible on the plane BC :

- Deduce the velocity at point C?
- Calculate the maximum compression of the spring, given a stiffness constant equal to k=100N/m? (we given g =10 m/s^2)

Good luck

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