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Sur le thème

Geometry of Nearly Sasakian Manifolds and their Submanifolds

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ملخص

ينقسم هذا البحث إلى قسمين مختلفين.

القسم الأول: هو حول دراسة الأسطح في الكرة ذات خمسة أبعاد، مع هيكل تقريبا ساساكين أو هيكل قريبا كوسامبليكتيك. وقد برهنا أن هذه الأسطح هي داما في حد أدنى. وكذا وجود توافق هذه الأسطح مع أسطح لاقرونجيان في الفضاء الإسقاطي المركب.

القسم الثاني: تم تصنيف كلي لهيبار-أسطح تآلفية ذات الأبعاد الأربعة، متجانسة و محدبة محليا بقوة، في الحالة أين مشغل الشكل لديه قيمتان ذاتيتان متميزة، من خلال النظر في حالة أين تعدد كل من القيم الذاتية هو 2.

Résumé

Cette thèse est divisé en deux parties différentes. La première est consacrée pour l'étude des surfaces de la sphère de dimension cinq, qui a la structure nearly Sasakienne, et aussi la structure nearly cosymplectique. La seconde partie donne la classification des hypersurfaces affines de dimension quatre, localement fortement convexes, dans le cas où l'opérateur de forme a deux valeurs propres distinctes, en considérant qu'ils ont la même multiplicité 2.

Partie I :

Nous étudions les surfaces dans la sphère S^5 nearly Sasakienne dans le champ vectoriel de la structure ξ est normal à la surface et anti-invariant qui respecte la structure nearly sasakienne. Ainsi nous allons démontrés le théorème suivant :

Théorème : *La surface totalement réelle de la sphère nearly Sasakienne de dimension* 5 *est toujours minimale.*

Nous allons montrés également que ce résultat est valable pour les surfaces dans la sphère nearly co-symplectique de dimension 5.

Comme conséquence de cette minimalité, on peut avoir aussi une correspondance local entre les surfaces de la sphère S^5 avec la structure nearly Sasakienne, oubien la structure nearly cosymplectique, et les surfaces Lagrangiennes minimales de l'espace projective complexe $\mathbb{C}P^2$.

Part II :

Nous étudions les hypersurfaces affine de dimension quatre localement fortement convexe, ou l'opérateur de forme a deux valeurs propres distinctes. Dans le cas ou une des valeurs propre a la dimension 1 ces hypersurfaces ont étés étudies auparavant par Dillem, Vrancken, Hu, Li and Zhang, où ils ont classifiés les hypersurfaces de dimension 4 et 5 avec l'hypothèse supplémentaire que la multiplicité de l'une des valeurs propres est 1. Nous complétons la classification de la dimension 4 en considérant le cas où la multiplication des deux valeurs propres est égal à 2, voici son théorème :

Théorème : *Soit l'hypersurface affine, localement fortement convexe, M^4 de $\mathbb{R}^5$. Nous supposons que M a deux distinctes valeurs propre, de même multiplicité 2. Alors M est équivalente à la partie convexe d'une des hypersurfaces suivantes :*

$$\begin{aligned} &\left(x_1 - x_4^2\right)^3 \left(x_2 - x_5^2\right)^3 x_3^2 = 1, \\ &x_2^3 \left(x_1 - (x_3^2 + x_4^2) - \frac{x_5^2}{x_2}\right)^5 = 1, \\ &2x_2x_3x_4 - x_4^2 - x_1\left(x_3^2 - 2x_5\right) - 2x_2^2x_5 = 1, \end{aligned}$$

où $(x_1, x_2, x_3, x_4, x_5)$ sont les coordonnées de $\mathbb{R}^5$.

abstract

This thesis is divided into two different parts. The first is about the study of the surface in the sphere of dimension five, with the nearly Sasakian structure or the cosymplectic structure. The second part is the classification of 4-dimensional locally strongly convex homogeneous affine hypersurfaces, in the case how the shape operator have two distinct eigenvalues, by considering that the multiplicity of both eigenvalues is 2.

Part I :

We investigate surfaces in the nearly Sasakian 5-sphere for which the structure vector field ξ is normal to the surface and which are anti-invariant with respect to the nearly Sasakian structure. We show the flowing theorem :

Theorem : *A totally real surface of the nearly Sasakian S^5 is always minimal.*

We show also that this result is also valid for the surfaces in nearly cosymplectic 5-sphere.

As a consequence of the minimality, we can also obtain a local correspondence between totally real surfaces of the S^5 with nearly Sasakian structure, or nearly cosymplectic structure, and minimal Lagrangian surfaces of the complex projective space $\mathbb{C}P^2$.

Part II :

We study 4 dimensional locally strongly convex, locally homogeneous, hypersurfaces whose affine shape operator has two distinct principal curvatures. In case that one of the eigenvalues has dimension 1 these hypersurfaces have been previously studied by Dillem, Vrancken, Hu, Li and Zhang, in which a classification of such submanifolds was obtained in dimension 4 and 5 under the additional assumption that the multiplicity of one of the eigenvalues is 1. We complete the classification in dimension 4 by considering the case that the multiplicity of both eigenvalues is 2, this is the theorem :

Theorem : *Let M^4 be a locally strongly convex, locally homogeneous, affine hypersurface in $\mathbb{R}^5$. Assume that M has two distinct eigenvalues, both of multiplicity 2. Then M is equivalent to the convex part of one of the following hypersurfaces:*

$$\begin{aligned}
&\left(x_1 - x_4^2\right)^3 \left(x_2 - x_5^2\right)^3 x_3^2 = 1,\\
&x_2^3 \left(x_1 - (x_3^2 + x_4^2) - \frac{x_5^2}{x_2}\right)^5 = 1,\\
&2x_2x_3x_4 - x_4^2 - x_1\left(x_3^2 - 2x_5\right) - 2x_2^2x_5 = 1,
\end{aligned}$$

where $(x_1, x_2, x_3, x_4, x_5)$ are the coordinates of $\mathbb{R}^5$.

Contents

Table of notations

$\mathbb{R}^n$	n-dimensional vector space over the field of the real numbers.
M^n or M	n-dimensional differentiable manifold.
$C^k(M)$	Set of all functions of class C^k on M.
$C^\infty(M)$ or C^∞	Set of all functions of class C^∞ on M.
$\mathcal{D}^n$ or $\mathcal{D}$	Set of differentiable functions on M.
T_pM	Tangent space of M at p.
$\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_1}\}$	Orthonormal basis of T_pM.
$X, Y, Z, X_i, \ldots$	Vector fields in M.
$[X, Y]$	Lie bracket of X and Y.
T_p^*M	Co-tangent space of M at p.
TM	Tangent bundle on M.
$\{dx_1, \ldots, dx_n\}$	Orthonormal basis of TM.
ω	Differential form.
df , $d\omega$	Differential of the function f and differential of the k-form ω.
$\mathfrak{X}(M)$ or $\Gamma(M)$	Set of all vector fields on M.
∇	Affine connection, in the first part it is Levi-Civita connection.
$\hat{\nabla}$	Levi-Civita connection.
D_XY	Canonic connection on $\mathbb{R}^n$.
$T(X, Y)$	Torsion of the connection ∇.
Γ_{ij}^k	Christoffel symbols of the connection.
$\mathbb{S}^n$	n-dimensional sphere.
g or $< \ . , \ . \ >$	Riemannian metric.
NpM	Normal space on M at p.
$h(X, Y)$	The second fundamental form.
$Tr\ h$	Trace of h.
γ	Differentiable curve on M.
$R(X, Y)Z$ or R	Curvature tensor of the manifold M.
$Ric(Y, Z)$	Ricci tensor on M.
T	k-tensor fields.
η	1-form of the contact structure.
φ	tensor field of the contact structure.

ξ	In the first part, it is vector field of the contact structure.
ξ	In the second part, it is the transversal vector field.
$\hookrightarrow$	Immersion.
C	The cubic form.
h	In the second part is affine fundamental form.
S	Shape operator.
τ	Transversal connection form.
θ	Induced volume element on M.
ω_h	Volume element of the nondegenerate metric h.
λ , λ_i	Eigenvalues of the shape operator S.
A	Equiaffine transformation.
$K(X,Y)$ or $K_X Y$	Difference tensor.
$G\,ijkl$	the component of Gauss equation.
$\{E_1,\ldots,E_n\}$	A local orthonormal vector fields basis.
$\mathcal{E}_1$ and $\mathcal{E}_2$	Sub-spaces of dimension 2 of M.

Introduction

This thesis is divided into two different parts. The first is about the study of the surface in the sphere of dimension five, with the nearly Sasakian structure or the cosymplectic structure, completed with the paper of Belkhelfa and Chikh Salah [BS16].
The second part is the classification of 4-dimensional locally strongly convex homogeneous affine hypersurfaces, in the case how the shape operator have two distinct eigenvalues, by considering that the multiplicity of both eigenvalues is 2.

Part I

The notion of a nearly Sasakian structure on an almost contact metric manifold has been introduced by Blair, Showers and Yano in [BSY76]. The basic properties of such a manifold will be recalled in Section 2. They also give necessary and sufficient condition for when a hypersurface of a nearly Kaehler manifold inherits a nearly Sasakian structure. An example of such hypersurface is the 5-dimensional sphere S^5, with radius $\frac{1}{\sqrt{2}}$ umbilically embedded at an angle of $\frac{\pi}{4}$. As on this sphere all sectional curvatures are equal to 2, it immediately follows that its inherited structure is not a Sasakian structure.

The notion of nearly cosymplectic structure on an almost contact metric manifold was introduced and studied by Blair and Showers some years earlier in [Bla71] and [BS74]. They show also that the totally geodesic 5-sphere in the nearly Kaehler 6-sphere, has a nearly cosymplectic structure.

Submanifolds of the nearly Kaehler sphere S^6 have been investigated by many authors leading to many classification results. The existence of such a structure for the 6-sphere was proved by Fukami and Ishihara [FI55] by using the properties of the Cayley algebra. The study of the surfaces in nearly Keahlerian 6-dimensional sphere were already realized by Bolton, Dillen, Opozda, Verstraelen, Vrancken and Woodward (see [BVW97], [DOV88]).

In contrast to previous submanifolds of the nearly Kaehler sphere S^6, we have few results about nearly Sasakian manifolds. For example, Cappelletti-Montano and Dileo focused on the 5-dimensional case and proved that there exists a one-to-one correspondence between nearly Sasakian structures and some special class of SU(2)-structures, (see [CMD15]). Moreover, almost nothing is known about submanifolds of the nearly Sasakian S^5. This despite the fact that it is by far the easiest example of a non trivial nearly Sasakian manifold.

In this thesis we will focus on surfaces of the nearly Sasakian S^5 and the nearly cosymplectic S^5 for which the structure vector field ξ is normal to the surface. In the Sasakian case, submanifolds of dimension n of a $(2n+1)$-dimensional Sasakian sphere for which ξ is normal are called, depending on the literature, C-totally real or horizontal submanifolds. In that case, it is known that such submanifolds are always anti invariant, i.e. the structure φ maps tangent vectors to normal vectors. In this thesis we first show that this is no longer the case in the nearly Sasakian 5-sphere or the nearly cosymplectic 5-sphere.

In view of this, we suggest to call a surface of the 5-sphere with nearly Sasakian structure, or nearly cosymplectic structure, totally real if and only if the structure vector field ξ is normal to the surface and φ maps tangent vectors to normal vectors. Note that in the Sasakian case the second condition is redundant. The main result we will prove about such surfaces is the following:

Theorem 2.2.1 : *A totally real surface of the nearly Sasakian S^5 is always minimal.*

Note that this result is also valid for the surfaces in nearly cosymplectic 5-sphere, (see theorem 3.2.1). This result is neither true for C-totally real surfaces in Sasakian manifolds or for totally real surfaces of the nearly Kaehler 6-sphere.

As a consequence of the minimality, we can also obtain a local correspondence between totally real surfaces of the S^5 with nearly Sasakian structure, or nearly cosymplectic structure, and minimal Lagrangian surfaces of the complex projective space $\mathbb{C}P^2$ (see theorem 2.2.2 and theorem 3.2.2).

Part II

In this thesis we study locally strongly convex, locally homogeneous, hypersurfaces in the affine space $\mathbb{R}^{n+1}$. In case that the dimension is 2 these hypersurface were first studied in the book of Guggenheimer [Gug77]. Their result was completed in the paper of Nomizu and Sasaki [NS91] thus giving a complete classification in dimension 2.

Starting from dimension 2 only partial results exist so far. Apart from the results of [Oog13] and [MV95] most of these results deal with the case that the affine hypersurface is locally strongly convex, i.e. the induced affine metric is a positive definite metric. In that case such hypersurfaces have been studied in the paper of Sasaki [Sas80] in the case that the hypersurface is an affine sphere and in a series of papers by Dillen and Vrancken [DV93a] giving a complete classification in dimension 3.

The only other results are those of [DV93b], [HLZ14] which give a classification in dimension 5, provided that the affine hypersurface is locally strongly convex and quasi umbilical. This means that the affine shape operator has only two distinct eigenvalues and that the multiplicity of one of those two eigenvalues is 1. In dimension 4 this result state

Theorem 0.0.1 *([DV94b]) Let M^4 be a locally strongly convex, locally homogeneous, proper quasiumbilical affine hypersurface in $\mathbb{R}^5$. Then M is equivalent to the convex part of one of the following hypersurfaces:*

$$\left(y - \frac{1}{2}(x^2 + z^2 + u^2)\right)^5 w^2 = 1,$$
$$\left(y - \frac{1}{2}(x^2 + u^2)\right)^4 \left(z - \frac{1}{2}w^2\right)^3 = 1,$$
$$\left(y - \frac{1}{2}x^2\right)^3 z^2u^2w^2 = 1,$$
$$\left(y - \frac{1}{2}x^2 - \frac{1}{2}(w^2/z + u^2/z)\right)^5 z^4 = 1,$$
$$\left(y - \frac{1}{2}x^2 - \frac{1}{2}w^2/z\right)^4 z^3u^2 = 1,$$
$$\left(y - \frac{1}{2}x^2\right)\left(z^2 - (w^2 + u^2)\right) = 1,$$

where (x, y, z, w, u) are the coordinates of $\mathbb{R}^5$

Here, we investigate locally strongly convex affine hypersurface of dimension 4 whose affine shape operator has two distinct eigenvalues. In view of the previous result, in order to obtain a complete classification it is sufficient to deal with the case that the multiplicity of both eigenvalues is two. By combining the theorems 7.3.1, 7.3.2 and 8.2.1, we show that there exist just three such hypersurfaces and we prove the following theorem:

Theorem A :

Let M^4 be a locally strongly convex, locally homogeneous, affine hypersurface in $\mathbb{R}^5$. Assume that M has two distinct eigenvalues, both of multiplicity 2. *Then*

M is equivalent to the convex part of one of the following hypersurfaces:

$$\left(x_1 - x_4^2\right)^3 \left(x_2 - x_5^2\right)^3 x_3^2 = 1, \tag{0.1}$$

$$x_2^3 \left(x_1 - (x_3^2 + x_4^2) - \frac{x_5^2}{x_2}\right)^5 = 1, \tag{0.2}$$

$$2x_2x_3x_4 - x_4^2 - x_1\left(x_3^2 - 2x_5\right) - 2x_2^2x_5 = 1, \tag{0.3}$$

where $(x_1, x_2, x_3, x_4, x_5)$ *are the coordinates of* $\mathbb{R}^5$.

Note in [DV94a] several constructions were introduced which permitted to introduce several classes of affine homegeneous locally homogeneous hypersurfaces. For example all the examples of [DV93b] and [HLZ14] can be recovered using these constructions. The same is true for the first two examples in our main theorem. However the final example introduces a new class of affine homogeneous hypersurfaces which is not obtained as one of the Calabi constructions.

Finally note that there is a conjecture of Dillen-Vrancken, together with an additional version of Niebergall-Ryan (for isoparametric hypersurfaces, [NR94]), see [LMSS96], stating that for an positive definite affine homogeneous affine hypersurface or a positive definite isoparametric affine hypersurface the affine shape operator S has at most one non zero eigenvalue.

Part I

Surfaces in the nearly Sasakian 5-sphere

Chapter 1

Preliminaries

We begin by recalling some fundamental definitions and proprieties of manifolds.

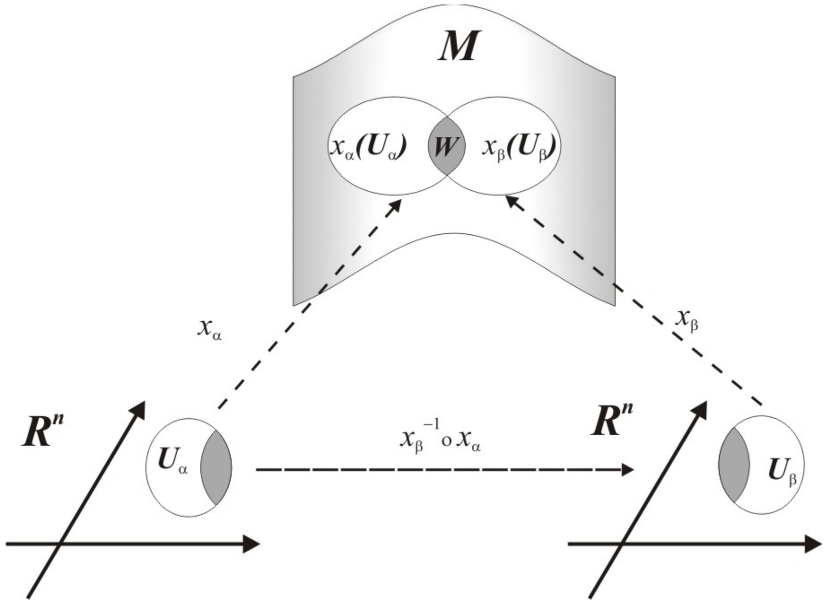
1.1 Basic differential geometry

we can find more details for this section in many books we quote as example: Yano, Kon [YK84], Do Carmo [DC92], Kobayashi, Nomizu [KN69], Gallot, Hulin, Lafontaine [GHL80].

1.1.1 Differentiable manifolds

Definition 1.1.1 *A* ***differential manifold*** *of dimension n is a set M and family of injective mappings $x_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open sets U_α of $\mathbb{R}^n$ into M such that :*
1. $\cup_\alpha x_\alpha(U_\alpha) = M$.
2. for any pair α, β, with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \varnothing$,the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$, and mappings $x_\beta^{-1} \circ x_\alpha$ are differentiable.
3. the family $\{(U_\alpha, x_\alpha)\}$ is maximal relative to the conditions 1. and 2.

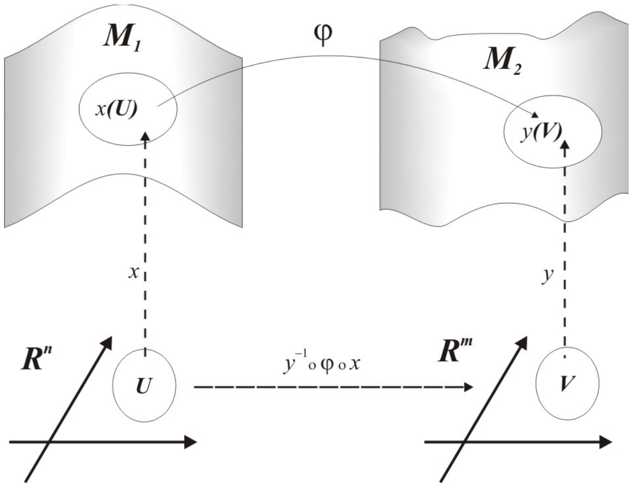
The pair (U_α, x_α) (or the mapping x_α), with $p \in x_\alpha(U_\alpha)$, is called de ***parametrization*** *(or* ***system of coordinates****) of M at p. And we note M^n to say that M is of dimension n.*



Remark 1.1.1 *A differential structure on a set M induces a natural topology on M. it suffices to define $A \subset M$ to be an open set in M if and only if $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ is an open set in $\mathbb{R}^n$ for all α.*

Definition 1.1.2 *Let M_1^n and M_2^m be differential manifolds, A mapping $\varphi : M_1 \to M_2$ is differentiable at $p \in M_1$, if given parametrization $y : V \subset \mathbb{R}^m \to M_2$ at $\varphi(p)$ there exists a parametrization $x : U \subset \mathbb{R}^n \to M_1$ at p such that $\varphi(x(U)) \subset y(V)$ and the mapping $y^{-1} \circ \varphi \circ x : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x^{-1}(p)$.*

φ is differentiable on an open set of M_1 if it is differentiable at all points of this open set.

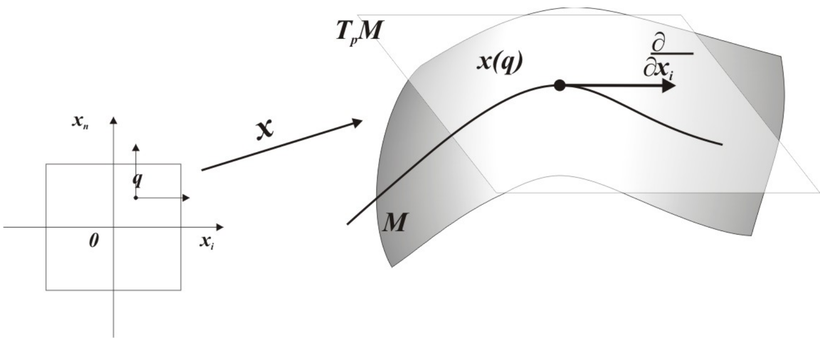


Definition 1.1.3 *Let M be the differentiable manifold. A differential function $\alpha : (-\epsilon, \epsilon) \subset \mathbb{R} \to M$ is called a* ***differentiable curve*** *in M, if $\alpha(t) \in M$ for all $t \in (-\epsilon, \epsilon)$.*
Suppose that $\alpha(0) = p \in M$, and let $\mathcal{D}$ be the set of functions on M that are differentiable at p.
The ***tangent vector*** *to the curve α at $t = 0$ is the function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ given by*

$$\alpha'(0)f = \frac{d(f \circ \alpha)}{dt}|_{t=0}, \qquad f \in \mathcal{D}.$$

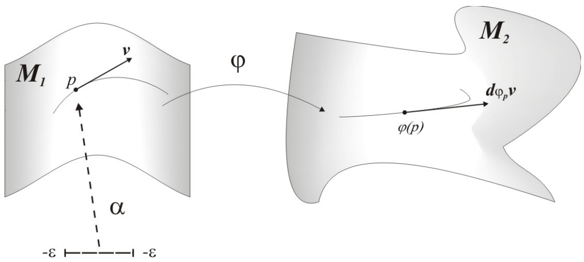
A tangent vector at $p \in M$ is the ***tangent vector*** *at $t = 0$ of some curve $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$.*
The set of all tangent vectors to M at p is the ***tangent space*** *of M at p, will be indicated by T_pM.*

Remark : The set T_pM with the usual operations of functions, form a vector space of dimension n, and that the choice of a parametrization $x : U \to M$ determines an associated basis $\{\frac{\partial}{\partial x_1}|_0, ..., \frac{\partial}{\partial x_n}|_0\}$ in T_pM.
It is immediate that the linear structure in T_pM defined above does not depend on the parametrization x.



Proposition 1.1.1 *Let M_1^n and M_2^m be differentiable manifolds and let $\varphi : M_1 \to M_2$ be a differential mapping. For every $p \in M_1$ and for each $v \in T_pM_1$, choose a differentiable curve $\alpha : (-\epsilon, \epsilon) \to M_1$ with $\alpha(0) = p$, $\alpha'(0) = v$. Take $\beta = \varphi \circ \alpha$. The mapping $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ given by $d\varphi_p(v) = \beta'(0)$ is linear mapping that does not depend on the choice of α.*

The linear mapping $d\varphi_p$ is called ***the differential*** *of φ at p.*



Definition 1.1.4 *Let M_1 and M_2 be differentiable manifold. A mapping $\varphi : M_1 \to M_2$ is a diffeomorphism if it is differentiable, bijective, and its inverse φ^{-1} is differentiable.*
φ is said to be a local diffeomorphism at $p \in M_1$ if there exist neighborhoods U of p and V of $\varphi(p)$ such that $\varphi : U \to V$ is a diffeomorphism.

The notion of diffeomorphism is the natural idea of equivalence between differential manifolds.
It is immediate consequence of the chain rule that if $\varphi : M_1 \to M_2$ is a diffeomorphism, then $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is an isomorphism for all $p \in M_1$. In particular, the dimensions of M_1 and M_2 are equal.
A local converse to this is the following theorem.

Theorem 1.1.1 *Let $\varphi : M_1^n \to M_2^n$ be a differential mapping and let $p \in M_1$ be such that $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is an isomorphism. then φ is a local diffeomorphism at p.*

1.1.2 Immersions, submersion

Definition 1.1.5 *Let M_1^n and M_2^m be differentiable manifolds. Let mapping $\varphi : M_1 \to M_2$.*

1. *φ is said to be an* ***immersion*** *if*
   (a) *φ differentiable,*
   (b) *$d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is injective for all $p \in M_1$.*

   We note the immersion $\varphi : M_1 \hookrightarrow M_2$.

2. *If, in addition, φ is homeomorphism onto $\varphi(M_1) \subset M_2$, where $\varphi(M_1)$ has the subspace topology induced from M_2, we say that φ is an* ***embedding****.*

3. *If $M_1 \subset M_2$ and the inclusion $i : M_1 \to M_2$ is an embedding, we say that M_1 is a* ***submanifold*** *of M_2.*

4. *It can be seen that if $\varphi : M_1^n \hookrightarrow M_2^m$ is an immersion, then $n \leq m$, the difference $m - n$ is called the* ***codimension*** *of the immersion φ.*

5. *φ is said to be an* ***submersion*** *if*
   (a) *φ differentiable,*
   (b) *$d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is subjective for all $p \in M_1$.*

Proposition 1.1.2 *Let $\varphi : M_1^n \hookrightarrow M_2^m$, $n \leq m$, be an immersion of the differential manifold M_1 in the differential manifold M_2. For every point $p \in M_1$, there exists a a neighborhood $V \subset M_1$ of p such that the restriction $\varphi|_V : V \to M_2$ is an embedding.*

1.1.3 The tangent bundle and orientation

This can be an example of differentiable manifolds construction.
Let M^n be a differentiable manifold and let $TM = \{(p, v);\ p \in M, v \in T_pM\}$.

Proposition 1.1.3 *Let M^n be a differentiable manifold. The set TM have differentiable structure, it is a manifold of dimension $2n$.*
*TM will be called the **tangent bundle** of M.*

Let $\{(U_\alpha, X_\alpha)\}$ be the maximal differential structure in M. Let some $U \in \{U_\alpha\}$, denoted by $(x_1, ...x_n)$ the coordinates of U and by $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ the associated bases to the tangent space of $x(U)$, we define

$$y : U \times \mathbb{R}^n \to TM, \quad \text{by}$$

$$y(x_1, ..., x_n, u_1, ...u_n) = (x(x_1, ..., x_n), \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}), \quad (u_1, ...u_n) \in \mathbb{R}^n.$$

Geometrically this means that we are taking as coordinates of a point $(p, v) \in TM$ the coordinates $x_1, ..., x_n$ of p together with the coordinates of v in the bases $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$, and $\{(U_\alpha \times \mathbb{R}^n, y_\alpha)\}$ is the differentiable structure of TM.

Definition 1.1.6 *Let M be a differentiable manifold. We say that M is **orientable** if M admits a differential structure $\{(U_\alpha, x_\alpha)\}$ such that:*

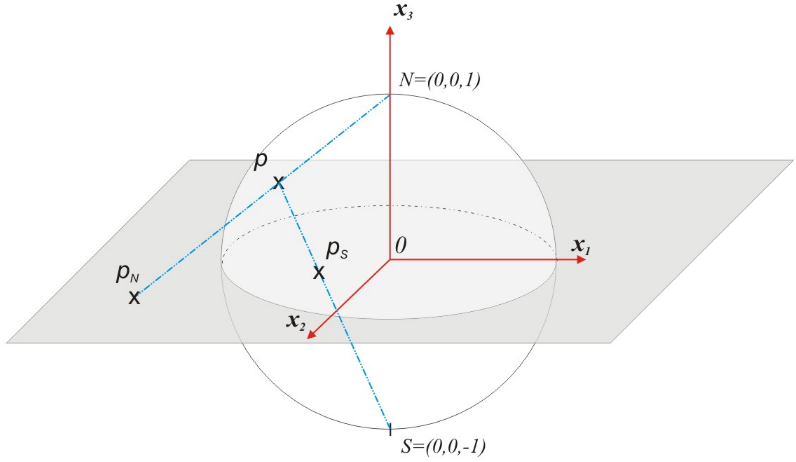
1. *For every pair α, β, with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \varnothing$, the differential of the change coordinates $x_\beta^{-1} \circ x_\alpha$ has positive determinant.*

Now let M_1 and M_2 be differentiable manifolds and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then M_1 is orientable if and only of M_2 is orientable. If additionally, M_1 and M_2 are connected and are oriented, φ induces an orientation on M_2 which may or may not coincide with the initial of M_2. In the first case, we say that φ **preserves the orientation** and in the second case, that φ **reverses the orientation**.

Example
Let the unit sphere of dimension n defined by:

$$\mathbb{S}^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}; \sum_{i=1}^{n+1} x_i^2 = 1 \right\} \subset \mathbb{R}^{n+1}$$

is orientable. Indeed, Let $N = (0, \dots, 0, 1)$ be the north pole and $S = (0, \dots, 0, -1)$ the south pole of $\mathbb{S}^n$. Define a mapping $\pi_1 : \mathbb{S}^n - \{N\} \to \mathbb{R}^n$ (called **Stereographic projection from the north pole**) that takes $p = (x_1, \dots, x_{n+1}) \in \mathbb{S}^n - \{N\}$ into the intersection of the hyperplane $x_{n+1} = 0$ with de lane that passes through p and N.



Stereographic projection from de north and south poles in the hyperplane $x_3 = 0$

It verify that

$$\pi_1(x_1, \dots, x_{n+1}) = (\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}).$$

The mapping π_1 is differentiable, and injective maps from $\mathbb{S}^n - \{N\}$ into the hyperplane $x_{n+1} = 0$. The stereographic projection $\pi_2 : \mathbb{S}^n - \{S\} \to \mathbb{R}^n$ from the south pole onto the hyperplane $x_{n+1} = 0$ has the same properties.

Therefore, the family $\{(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})\}$ is the differentiable structure on $\mathbb{S}^n$. Observe that intersection $\pi_1^{-1}(\mathbb{R}^n) \cap \pi_2^{-1}(\mathbb{R}^n) = \mathbb{S}^n - \{N, S\}$ is connected, thus $\mathbb{S}^n$ is orientable and the family given determines an orientation of $\mathbb{S}^n$.

1.1.4 Vector fields, brackets

Definition 1.1.7 *A vector field X on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. In terms of mappings, X is a mapping of M into tangent bundle TM. The field is differentiable if $X : M \to TM$ is differentiable.*
We denotes by $\mathfrak{X}(M)$ the set of all vector fields of class $\mathcal{C}^\infty$.

Considering a parametrization $x : U \subset \mathbb{R}^n \to M$ we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i},$$

where each $a_i : U \to \mathbb{R}$ is a functions on U and $\left\{ \frac{\partial}{\partial x_i} \right\}$ is the basis associated to x.

Lemma 1.1.1 *Let X and Y be differentiable vector fields on a differentiable manifold M. Then there exist a unique vector field Z such that, for all $f \in \mathcal{D}(M)$,*

$$Zf = (XY - YX)f,$$

where $\mathcal{D}(M)$ is the set of all differentiable functions on M.

The vector field Z given by this lemma is called the **bracket** of X and Y noted

$$[X, Y] = XY - YX.$$

The bracket operation has the following properties :

Proposition 1.1.4 *If X, Y and Z are differentiable vector fields on M, a, b are real numbers, and f, g differentiable functions, then :*

a) $[X, Y] = -[Y, X]$ *(anti-commutativity),*

b) $[aX + bY, Z] = a[X, Z] + b[Y, Y]$ *(linearity),*

c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *(Jacobi identity),*

d) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$

Definition 1.1.8 *Let M differentiable manifold,*

1. ***Hausdorff axiom** : Given two points of M there exist neighborhoods of these two points that do not intersect.*

2. ***Countable basis axiom** : M can be covered by a countable number of coordinate neighborhoods (we say then that M has a **countable basis**).*

1.2 Riemannian manifolds

For the remainder of this section, the differentiable manifolds considered will be assumed to be Hausdorff spaces with countable bases. "Differentiable" will signify "of class $\mathcal{C}^\infty$", and when $M^n = M$ denotes a differentiable manifold, n denoted the dimension of M (for more details of this section see [DC92, GHL80])

1.2.1 Riemannian metric and isometry

A Riemannian metric on an open set U of $\mathbb{R}^n$ is a family positive definite quadric forms on $\mathbb{R}^n$, depending smoothly on $m \in U$. The general definition is not fundamentally different.

Definition 1.2.1 *A* ***Riemannian metric*** *(or* ***Riemannian structure****) on a differential manifold M is a smooth and positive-definite section g of the bundle S^2T^*M of the symmetric bilinear 2-forms on M.*

Let $x : U \subset \mathbb{R}^n \to M$ is a system of coordinates in a local chart around a point $p \in M$, with $x(x_1, x_2, \cdots, x_n) = q \in x(U)$.
And $\left(\frac{\partial}{\partial x_i}\right)_{(1\leq i\leq n)}$ be the coordinate vector fields. Let $u, v \in T_qM$ with

$$u = \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}_{|q} \quad \text{and} \quad v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}_{|q}.$$

Then $g_q(u, v) = \sum_{i,j} g_{ij}(q)u_iv_j$, where

$$g_{ij}(q) = g\left(\frac{\partial}{\partial x_i}_{|q}, \frac{\partial}{\partial x_j}_{|q}\right).$$

We will denoted $g = \sum_{i,j} g_{ij}\, dx_i \otimes dx_j$, or shortly

$$g = \sum_{i,j} g_{ij}\, dx_i dx_j.$$

Theorem 1.2.1 *There exists at least one Riemannian metric on any manifold M.*

Remark 1.2.1

1. *This definition does not depend on the choice of coordinate system.*
2. *Another way to express the differentiability or the Riemannian metric is to say that for any pair of vector fields X and Y, which are differentiable in a neighborhood V of M, the function $g(X, Y)$ is differentiable on V.*
3. *We note sometime $< ., . >_{|p}$ a Riemannian metric at p.*

Definition 1.2.2 *A differentiable manifold with a given Riemannian metric will be called a* ***Riemannian manifold****, noted (M, g).*

After introducing any type of mathematical structure, we must introduce a notion of when two objects are the same.

Definition 1.2.3 *Let M and N be Riemannian manifolds. A diffeomorphism $f : M \to N$ (that is, f is a differentiable bijection with a differentiable inverse) is called an* ***isometry*** *if :*

$$\langle u, v \rangle = \langle df_p(u), df_p(v) \rangle_{f(p)}, \quad \text{for all } p \in M,\ u, v \in T_pM. \tag{1.1}$$

Definition 1.2.4 *Let M and N be Riemannian manifolds. A differentiable mapping $f : M \to N$ is a* ***local isometry*** *at $p \in M$ if there is a neighborhood $U \subset M$ of p such that $f : U \to f(U)$ is a diffeomorphism satisfying* (1.1).

It is common to say that a Riemannian manifold M is ***locally isometric*** *to a Riemannian manifold N if for every $p \in M$ there exists a neighborhood U of p in M and a local isometry $f : U \to f(U) \subset N$.*

Definition 1.2.5 *Let (M, g) be a Riemannian manifold. (N, h) is a* ***Riemannian submanifold*** *of (M, g) if :*

1. *N is a submanifold of M.*
2. *for any $p \in M$, h_p is the restriction of g_p to T_pN.*

Proposition 1.2.1 *Let M^n and N^{n+m} differentiable manifolds and $f : M^n \to N^{n+m}$ is an immersion. If N has a Riemannian structure $< . , . >$, then f induces a Riemannian structure on M by defining*

$$\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}, \quad \text{for all } p \in M,\ u, v \in T_pM.$$

This metric on M is then called the ***metric induced*** *by f, and f is an* ***isometric immersion****.*

1.2.2 Affine connection and Riemannian connection

A vector field Y on $\mathbb{R}^n$ can be seen as a smooth map from $\mathbb{R}^n$ to itself. the derivative of Y at $p \in \mathbb{R}^n$ in the direction $v \in T_p\mathbb{R}^n$ is the vector $D_pY.v = d_pY.v$. If X is another vector field, the derivative of Y at p in the direction X_p will be $D_pY.X_m$. This yields, when p goes through $\mathbb{R}^n$, the vector field $dY(X)$.
It is clear that for any $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, $dY(fX) = fdY(X)$, and $d(fY)X = df(X)Y + fdY(X)$. Furthermore, an explicit computation (using coordinates) shows that $dY(X) - dX(Y) = [X, Y]$.
On the manifold, there is no natural way to define the directional derivative of a vector field. the job is done by connections.

Definition 1.2.6 *An* ***affine connection*** ∇ *on a differentiable manifold M is a mapping :*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

which is denoted by $(X, Y) \overset{\nabla}{\to} \nabla_X Y$ *and which satisfies the following properties :*

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z,$
2. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z,$
3. $\nabla_X(fY) = X(f)Y + f\nabla_X Y,$

in which $X, Y, Z \in \mathfrak{X}(M)$ *and* $f, g \in \mathcal{D}(M)$.

This definition is not as transparent as the Riemannian structure. the following proposition, nevertheless,

Proposition 1.2.2 *Let M de a differentiable manifold with affine connection ∇. There exists a unique correspondence which associates to a vector field V along the differential curve $c : I \to M$ another vector field $\frac{DV}{dt}$ along c, such that :*

a) $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}.$

b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt},$

where W is a vector field along c and f a differentiable function on I.

c) If V is induced by a vector field $Y \in \mathfrak{X}(M)$, i.e., $V(t) = Y(c(t))$, then

$$\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y.$$

Remark 1.2.2 *The last line c) of the proposition, make sense, since $\nabla_X Y(p)$ depends on the value of $X(p)$ and the value Y along a curve, tangent to X at p. In effect, part* 3) *of the definition 1.2.6 allows us to show that the notion of affine connection is actually a local notion. Choosing a system of coordinates $(x_1, \cdots, x_n)$ about $p \in M$ and writing*

$$X = \sum_i x_i X_i, \quad Y = \sum_j y_j X_j,$$

where $X_i = \frac{\partial}{\partial x_i}$, setting

$$\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma_{ij}^k X_k, \tag{1.2}$$

we conclude that the Γ_{ij}^k are differentiable functions and that

$$\nabla_X Y = \sum_k \left(\sum_{ij} x_i y_j \Gamma_{ij}^k + X(y_k) \right) X_k,$$

Definition 1.2.7 *Let M de the differentiable manifold with an affine connection ∇. A vector field V along a curve $c : I \to M$ is called* ***parallel*** *when $\frac{DV}{dt} = 0$, for all $t \in I$.*

Proposition 1.2.3 *Let M be the differentiable manifold with an affine connection ∇. Let $c : I \to M$ be the differentiable curve in M and let V_0 be a vector tangent to M at $c(t_0)$, $t_0 \in I$ (i.e. $V_0 \in T_{c(t_0)}M$). Then there exist a unique parallel vector field V along c, such that $V(t_0) = V_0$, ($V(t)$ is called the* ***transport parallel*** *of $V(t_0)$ along c).*

Definition 1.2.8 *A connection ∇ on a differentiable manifold M is* ***torsion-free*** *(sometimes called* ***symmetry****), if for all $X, Y \in \mathfrak{X}(M)$*

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

Remark 1.2.3 *For any connection ∇ on a differentiable manifold M, it can be easily checked that the map*

$$T : (X, Y) \to \nabla_X Y - \nabla_Y X - [X, Y]$$

defines a tensor. This tensor is called the ***torsion*** *of the connection ∇.*

Theorem 1.2.2 *(Levi-Civita). Given a Riemannian manifold (M, g), there exist a unique affine connection ∇ on M satisfying the conditions :*

1. *∇ is with torsion-free :*

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

2. *∇ is compatible with the Riemannian metric g i.e. :*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(y, \nabla_X Z),$$

where X, Y, Z are vector fields on M.

Definition 1.2.9 *The connection given by the theorem will be referred to, from now on, as the* ***Levi-Civita*** *(or* ***Riemannian****)* ***connection*** *of the metric g on M.*

Locally in a coordinate system (U, x), it is customary to call the functions Γ_{ij}^k defined on U in (1.2), the **coefficients of the connection** ∇ on U or the the **Christoffel symbols** of the connection.
If we put $g_{ij} = g(X_i, X_j)$. Since the matrix $(g_{ij})_{1 \le i,j \le n}$ admits an inverse $(g^{ij})_{1 \le i,j \le n}$, we obtain that

$$\Gamma_{ij}^m = \frac{1}{2} \sum_{k=1}^{n} \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_i} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km}$$

1.2.3 Induced connection ans second fundamental form

Let M^n an n-dimensional isometrically immersed in an m-dimensional Riemannian manifold $\bar{M}^m$. We put $m = n + p$, $p > 0$. Since the discussion is local, we may assume, if we want, that M is embedded in $\bar{M}$.

We denoted by g the Riemannian metric tensor field of $\bar{M}$. Then the submanifold M is also a Riemannian manifold with the Riemannian metric h given by

$$h(X, Y) = g(X, Y), \qquad \text{for any vector feilds } X, Y \text{ in } M.$$

The Riemannian metric h on M is called the **induced metric** on M.

Throughout this, the induced metric h will be denoted by the same g as that of the ambient manifold $\bar{M}$ to simplify the notation because it may cause no confusion.

Definition 1.2.10 *If a vector V of $\bar{M}$ at a point x of M satisfies $g(X, V) = 0$ for any vector X of M in x, then V is called a* ***normal vector*** *of M in $\bar{M}$ at x. A unit normal vector field of M in $\bar{M}$ is sometimes called a* ***normal section*** *on M.*
NM denote the vector bundle of all normal vectors of M in $\bar{M}$.

Remark 1.2.4 *The tangent bundle of $\bar{M}$, restricted to M, is the direct sum of the tangent bundle TM of M and the normal bundle NM of M in $\bar{M}$*

$$T\bar{M} = TM \oplus NM.$$

We denote by $\bar{\nabla}$ the operator of covariant differentiation on $\bar{M}$.

Lemma 1.2.1 *Let X and Y be vector fields on M and let $\bar{X}$ and $\bar{Y}$ be extension of X and Y, respectively. Then $[\bar{X}, \bar{Y}]|_M$ is independent of the extensions and*

$$[\bar{X}, \bar{Y}]|_M = [X, Y],$$

and $(\bar{\nabla}_{\bar{X}}\bar{Y})|_M$ is also independent of the extension.

We denote $(\bar{\nabla}_{\bar{X}}\bar{Y})|_M$ by $\bar{\nabla}_X Y$. We put

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.3}$$

where $\nabla_X Y$ and $h(X, Y)$ are, respectively, the tangential and normal components of $\bar{\nabla}_X Y$.

Proposition 1.2.4 *From the previous equation, we have*

1. *∇ is the operator of covariant differentiation with respect to the metric on M and define an affine connection on M.*

2. *h is bilinear symmetric in X and Y.*

3. *∇ has no torsion and $\nabla g = 0$. Since $\bar{\nabla}$ has no torsion*

$$\bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] = 0, \qquad \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

Definition 1.2.11 *We call the Riemannian connection ∇ the **induced connection** and h the **second fundamental form** of M (or the immersion ι).*
For each point $x \in M$, $h(X, Y)$ at x depends only X_x and Y_x.

Proposition 1.2.5 *Let V be a normal vector field on M and X, Y vector fields on M. we put*

$$\bar{\nabla}_X V = -A_V X + D_X V, \tag{1.4}$$

where $-A_V X$, $D_X V$ are, respectively, the tangential component and the normal component of $\bar{\nabla}_X V$, then

1. *$-A_V X$ and $D_X V$ are differentiable vector fields on M.*

2. *$A_V X$ is bilinear in V and X and $A_V X$ at $x \in M$ depends only on V_x and X_x.*

3. *$g(h(X, Y), V) = g(A_V X, Y)$.*

*A is called the **associated second fundamental form** to h or the **Weingarten operator**.*

Definition 1.2.12 *We have the first set of basic formulas for submanifolds, namely,*

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.5}$$
$$\bar{\nabla}_X V = -A_V X + D_X V. \tag{1.6}$$

*The first formula is called the **Gauss formula**,*
*the second formula is called the **Weingarten formula**.*

We give now some fundamental definitions relative to the Gauss formula and the Weingarten formula.

Definition 1.2.13 *Let V a normal vector field on M and X, Y vector fields on M,*

1. *V is said to be **parallel** in NM, or simply parallel, if*

$$D_X V = 0 \quad \text{for all } X \in \mathfrak{X}(M).$$

2. *A submanifold M is said to be* ***totally geodesic*** *if the second fundamental form vanishes identically,that is $h \equiv 0$ or equivalently $A \equiv 0$.*

3. *If A_V is everywhere proportional to the identity transformation I, that is, $A_V = aI$ for some function a, then V is called an* ***umbilical section*** *on M, or M is said to be umbilical with respect to V.*

4. *If the submanifold M is umbilical with respect to every local normal section of M, then M is said to be* ***totally umbilical****.*

Let $\{e_1, \cdots, e_n\}$ be an orthogonal basis in T_xM in each point $x \in M$.

5. *The* ***mean curvature vector*** *μ of M is defined by*

$$\mu = \frac{1}{n} Tr\, h, \qquad \text{where } Tr\, h = \sum_{i=1}^{n} h(e_i, e_i),$$

with is independent of the choice of a basis.

6. *If $\mu = 0$, then M is said to be* ***minimal****,*

$$\mu = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) = 0.$$

Proposition 1.2.6 *Any submanifold M which is minimal and totally umbilical is totally geodesic.*

1.2.4 Geodesic, curvature

In what follows, M will be a Riemannian manifold, together with its Riemannian connection.

Definition 1.2.14 *A parametrized curve $\gamma : I \to M$ is a* ***geodesic*** *at $t_0 \in I$ if $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ at the point t_0;*
If γ is a geodesic at t for all $t \in I$, we say that γ is a ***geodesic****.*
If $[a, b] \subset I$ and $\gamma : I \to M$ is a geodesic, the restriction of gamma to $[a, b]$ is called a ***geodesic segment joinin****g $\gamma(a)$ to $\gamma(b)$.*

Theorem 1.2.3 *If X is a vector field on the open set V in the Riemannian manifold M and $p \in M$ then there exist an open set $V_0 \subset V$, $p \in V_0$, a number $\delta > 0$, and a C^∞ mapping $\varphi : (-\delta, \delta) \times V_0 \to V$ such that the curve$t \to \varphi(t, q)$, $t \in (-\delta, \delta)$, is the unique trajectory of X which at the instant $t = 0$ passes through the point q, for every $q \in V_0$.*

The mapping $\varphi_t : V_0 \to V$ given by $\varphi_t(q) = \varphi(t, q)$ is called the **flow** of X on V.

Lemma 1.2.2 *There exist a unique vector field Y on TM whose trajectories are of the form $t \to (\gamma(t), \gamma'(t))$, where γ is a geodesic on M.*

Definition 1.2.15 *The* ***curvature*** *R of a Riemannian manifold M is a correspondence that associates to every pair $X, Y \in \mathfrak{X}(M)$ a mapping $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad Z \in \mathfrak{X}(M),$$

where ∇ is the Riemannian connection of M.

Observe that if $M = \mathbb{R}^n$, then $R(X, Y)Z = 0$ for all $X, Y, Z \in \chi(\mathbb{R}^n)$.

Proposition 1.2.7 *The curvature R of a Riemannian manifold M has the following properties :*

1. *R is bilinear in $\mathfrak{X}(M) \times \mathfrak{X}(M)$, that is,*

$$R(fX_1 + gX_2, Y) = fR(X - 1, Y) + gR(X_2, Y)$$
$$R(X, fY_1 + gY_2) = fR(X, Y_1) + gR(X, Y_2),$$

   where $f, g \in \mathcal{D}(M), \quad X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(M)$.

2. *For any $X, Y \in \mathfrak{X}(M)$, the curvature $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is linear, that is,*

$$R(X, Y)(fZ + gW) = fR(X, Y)Z + gR(X, Y)W,$$

   where $f, g \in \mathcal{D}(M), \quad Z, W \in \mathfrak{X}(M)$.

3. *(Bianchi identity)*

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0, \quad \textit{for all } X, Y, Z \in \mathfrak{X}(M).$$

For now on, we shall write $g(R(X, Y)Z, T) = R(X, Y, Z, T)$, for all $X, Y, Z, T \in \mathfrak{X}(M)$.

Proposition 1.2.8 *The curvature R of a Riemannian manifold M has the following properties :*

a. *R(X,Y,Z,T)+R(Y,Z,X,T)+R(Z,X,Y,T)=0,*

b. *R(X,Y,Z,T)=-R(Y,X,Z,T),*

c. *R(X,Y,Z,T)=- R(X,Y,T,Z),*

d. $R(X,Y,Z,T) = R(Z,T,X,Y)$,

for all $X,Y,Z,T \in \mathfrak{X}(M)$.

It is convenient to express what was seen above in a coordinate system (U,x) based at the point $p \in M$. Let us indicate, as usual, $\frac{\partial}{\partial x_i} = X_i$. We put

$$R(X_i, X_j)X_k = \sum_{l=1}^{n} R^l_{ijk} X_l.$$

Thus R^l_{ijk} are the components of the curvature R in (U,x). Which by a direct calculation yields

$$R^s_{ijk} = \sum_l \Gamma^l_{ik}\Gamma^s_{jl} - \sum_l \Gamma^l_{jk}\Gamma^s_{il} + \frac{\partial}{\partial x_j}\Gamma^s_{ik} - \frac{\partial}{\partial x_i}\Gamma^s_{jk}.$$

If we write the vector fields in the local coordinate

$$X = \sum_i u^i X_i, \quad Y = \sum_j v^j X_j, \quad Z = \sum_k w^k X_k,$$

we obtain, from the linearity of R,

$$R(X,Y)Z = \sum_{i,j,k,l} R^l_{ijk} u^i v^j w^k X_l.$$

1.2.5 Tensors on Riemannian manifolds

The notion of curvature is a particular case of of the idea of a tensor, which is a useful object in differential geometry. we present here a rapid introduction to the study of tensors on a Riemannian manifolds. The idea of a tensor is a natural generalization of the idea of a vector field, an important point being that analogously to vector fields, tensors can differentiated covariantly.

For what follows it is useful to observe that $\mathfrak{X}(M)$ is a module over $\mathcal{D}(M)$, that is, $\mathfrak{X}(M)$ has a linear structure when we take as "scalars" the elements of $\mathcal{D}(M)$.

Definition 1.2.16 *A* ***tensor*** T *of* ***order*** r *on a Riemannian manifold* M *is a* $\mathcal{D}(M)$*-multilinear mapping*

$$T : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r \text{ factors}} \to \mathcal{D}(M)$$

A tensor T is a pointwise object in a sense that we now explain. Fix a point $p \in M$ and let U be a neighborhood of p in M on which it is possible to define vector fields $E_1, \cdots, E_n \in \mathfrak{X}(M)$, in such a fashion that at each point $q \in M$, the vectors $\{E_i\}$, $i = 1, \cdots, n$, form a basis of T_qM. We say, in this case, that $\{E_i\}$ is a **moving frame** on U. Let

$$Y_1 = \sum_{i_1=1}^{n} y_{i_1} E_{i_1}, \cdots, Y_r = \sum_{i_r=1}^{n} y_{i_r} E_{i_r}, \qquad \text{for } i_1, \cdots, i_r = 1, \cdots, n,$$

be the restrictions to U of the vector fields $Y_1, \cdots, Y_r$, expressed in the moving frame $\{E_i\}$.By linearity

$$T(Y_1, \cdots, Y_r) = \sum_{i_1, \cdots, i_r} y_{i_1} \cdots y_{i_r} T(E_{i_1}, \cdots, E_{i_r})$$

Examples :

1. The **curvature tensor**

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{D}(M)$$
$$(X, Y, Z, T) \longmapsto R(X, Y, Z, T) = g(R(X, Y)Z, T).$$

It is easy to verify that R is a tensor of order 4.

2. The metric tensor

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{D}(M)$$
$$(X, Y) \longmapsto g(X, Y) = g(X, Y).$$

g is a tensor of order 2 and its components in the frame $\{X_i\}$ are the coefficients g_{ij} of the Riemannian metric in the given system of coordinates.

3. The Riemannian connection

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{D}(M)$$
$$(X, Y, Z) \longmapsto \nabla(X, Y, Z) = g(\nabla_X Y, Z),$$

is not a tensor, because ∇ is not linear with respect to the argument Y.

Definition 1.2.17 *Let T be a tensor of order r. The* ***covariant differential*** *∇T of T is a tensor of order $(r+1)$ given by*

$$\nabla T(Y_1, \cdots, Y_r, Z) = Z(T(Y_1, \cdots, Y_r)) - T(\nabla_Z Y_1, \cdots, Y_r) - \cdots - T(Y_1, \cdots, \nabla_Z Y_r).$$

For each $Z \in \mathfrak{X}(M)$, the covariant derivative $\nabla_Z T$ of T relative to Z is a tensor of order r given by

$$\nabla_Z T(Y_1, \cdots, Y_r) = \nabla T(Y_1, \cdots, Y_r, Z)$$

We give now a abstract definition of a Killing vector field in a Riemannian manifold and a theorem to use it. (*for detail of this definition see [YK84] pages 40-42*)

Definition 1.2.18 *Let M a Riemannian manifold, a vector field X on M is called a Killing vector field if a local 1-parameter group of local transformations generated by X in a neighborhood of each point of M consists of local isometries.*

Theorem 1.2.4 *Let (M, g) a Riemannian manifold and X a vector field on M, then X is Killing vector field if and only if*

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0, \qquad \textit{for all } Y, Z \in \mathfrak{X}(M)$$

where $\mathcal{L}_X$ is a Lie differentiation with respect to X.

1.3 Contact manifolds and almost contact manifolds

In this chapter we give the basic definitions and properties concerning contact and almost contact manifolds given by a contact structure and the almost contact structure, (for more details see Books : D.E.Blair [Bla76], S.Kobayashi-K.Nomizu [KN69] and K.Yano-M.Kon [YK84]).

1.3.1 Contact manifolds

Definition 1.3.1 *Let M^{2n+1} be a smooth manifold of dimension $2n+1$. A smooth 1-form η in M is called a **contact form** if*

$$\eta \wedge (d\eta)^n \neq 0, \quad \textit{everywhere on } M,$$

where $(d\eta)^n = d\eta \wedge \cdots \wedge d\eta$, n-factors.

*A smooth manifold M^{2n+1} together with the 1-form η is called a **contact manifold** and we noted it (M^{2n+1}, η).*

Proposition 1.3.1 *Let M^{2n+1} a contact manifold with contact form η*

1. *$\eta \wedge (d\eta)^n$ is a volume element of M, so that a contact manifold is orientable.*
2. *$d\eta$ has rank $2n$ on the Grassmann algebra $\bigwedge T_p^* M$ at each point $p \in M$.*
3. *there exist a vector field ξ satisfying*

$$d\eta(\xi, X) = 0, \quad \eta(\xi) = 1, \qquad \textit{for all } X \in \mathfrak{X}(M).$$

*ξ is called the **characteristic vector field** (or **Reeb vector field**) of the contact structure η.*

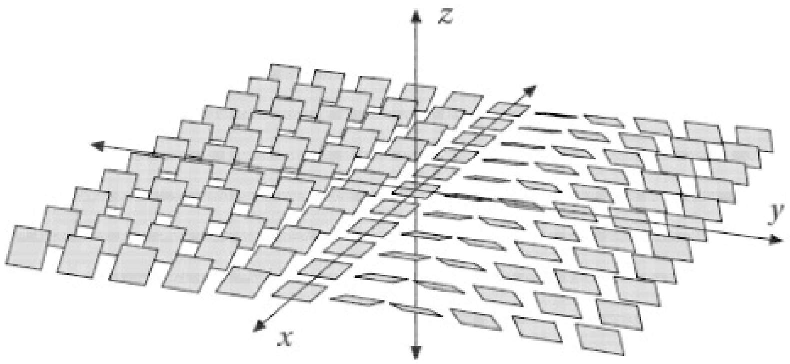


Figure 1.1: The standard contact structure in $\mathbb{R}^3$.

1.3.2 Almost contact manifolds

Definition 1.3.2 *Let M be a $2n+1$-dimensional manifold and φ, ξ, η be a tensor field of type (1.1), a vector field, a 1-form on M respectively. If φ, ξ and η satisfy the conditions*

$$\eta(\xi) = 1,$$
$$\varphi^2 X = -X + \eta(X)\xi$$

*for any vector field X in M, then M said to have an **almost contact structure** (φ, ξ, η) and is called an **almost contact manifold**.*

Theorem 1.3.1 *Suppose M^{2n+1} has a (φ, ξ, η)-structure. Then*

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad rank\,\varphi = 2n,$$

Proposition 1.3.2 *Every almost contact manifold M^{2n+1} admits a Riemannian metric tensor field g such that*

$$\eta(X) = g(X, \xi),$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in \mathfrak{X}(M)$.

We say that M^{2n+1} has an **almost contact metric structure** (φ, ξ, η, g) and M is called **almost contact metric manifold**. g is called a **compatible metric**.

Proposition 1.3.3 *Every almost contact manifold is orientable.*

Remark 1.3.1 *For a manifold M^{2n+1} with an almost contact metric structure (φ, ξ, η, g) we can also construct a useful local orthonormal basis. Let U be a coordinate neighborhood on M and X_1 any unit vector field on U orthogonal to ξ. Then φX_1 is a unit vector field orthogonal to both X_1 and ξ. Now choose a unit vector field X_2 orthogonal to ξ, X_1 and φX_1. Then φX_2 is also a unit vector field orthogonal to $\xi, X_1, \varphi X_1$ and X_2. Proceeding in this way we obtain a local orthonormal basis $\{X_i, \varphi X_i, \xi\}$, called a φ-**basis**.*

Definition 1.3.3 *For an almost contact metric structure (φ, ξ, η, g) on a manifold M^{2n+1} we put*

$$\Phi(X, Y) = g(X, \varphi Y), \qquad \text{for all } X, Y \in \mathfrak{X}(M).$$

*We call Φ the **fundamental** 2-**form** of the almost contact metric structure.*

Theorem 1.3.2 *Let M be a $2n+1$-dimensional manifold with contact structure η. Then there exists an almost contact metric structure (φ, ξ, η, g) on M such that*

$$g(X, \varphi Y) = d\eta(X, Y), \qquad \text{for all } X, Y \in \mathfrak{X}(M).$$

This structure is called a **contact metric structure** associated to η and a manifold with a structure is called a **contact metric manifold**.

Proposition 1.3.4 *An almost metric structure with $\Phi = d\eta$ is a contact metric structure.*

1.3.3 Torsion tensor of almost contact manifolds

Let M be a $(2n+1)$-dimensional almost contact manifold with almost contact structure (φ, ξ, η). We consider a product manifold $M \times \mathbb{R}$, where $\mathbb{R}$ denotes a real line. Then a vector field on $M \times \mathbb{R}$ is given by $(X, f(\frac{d}{dt}))$, where X is a vector field in M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. We define a linear map J on the tangent space of $M \times \mathbb{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right).$$

Then we have $J^2 = -I$ and hence J is an almost complex structure on $M \times \mathbb{R}$. (*for definition and properties of almost complex structure see [Bla76, YK84]*).
The almost complex structure J is said to be **integrable** if its Nijenhuis tensor $[J, J]$ vanishes, where a **Nijenhuis torsion** is defined by :

$$[J, J](X, Y) = J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY].$$

We can see the other notation for Nijenhuis tensor in the literature, *exp: N_J in the book of Yano-Kon [YK84].*

Definition 1.3.4 *If the almost complex structure J on $M \times \mathbb{R}$ is integrable, we say that the almost contact structure (φ, ξ, η) is **normal**.*
And wee define the Nienhuis torsion $[\varphi, \varphi]$ of φ by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

Proposition 1.3.5 *The almost contact structure (φ, ξ, η) of M is normal if and only if*

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0.$$

1.3.4 Sasakian manifolds and cosymplectic manifolds

Definition 1.3.5 *Let M be a $(2n+1)$-dimensional contact metric manifold with contact metric structure (φ, ξ, η, g). If the normal metric structure of M is normal, then M is said to have a **Sasakian structure** (or **normal contact metric structure**) and M is called a **Sasakian manifold** (or **normal contact metric manifold**).*

We denote by ∇ the operator of covariant differentiation with respect to g. then we have

Theorem 1.3.3 *An almost contact metric structure (φ, ξ, η, g) on M is a Sasakian structure if and only if*

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

where X, Y are vector fields on M.

Proposition 1.3.6 *Let M a Sasakian manifold, then*

1. $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$,
2. $R(X, \xi)Y = -g(X, Y)\xi + \eta(Y)X$,

where R denotes the Riemannian curvature of M and $X, Y \in \mathfrak{X}(M)$.

Theorem 1.3.4 *Let M^{2n+1} be a Riemannian manifold admit a unit Killing vector field ξ. Then M is a Sasakian manifold if and only if*

$$R(X, \xi)Y = -g(X, Y)\xi + \eta(Y)X.$$

For the notion of cosymplectic structure there are at least three definitions in the literature, we give here just one, (for the other definitions see [Bla76]).

Definition 1.3.6 *Let M be a $(2n+1)$-dimensional manifold with a normal almost contact metric structure (φ, ξ, η, g), then M have a* ***cosymplectic structure*** *if η and Φ are closed.*
M with cosymplectic structure is called a ***cosymplectic manifold****.*

Theorem 1.3.5 *An almost contact metric structure (φ, ξ, η, g) is cosymplectic if and only if φ is parallel. i.e. :*

$$\nabla_X \varphi = 0, \quad \textit{for all } X \in \mathfrak{X}(M).$$

1.3.5 Hypersurface of almost Hermitian manifolds

For the notions of Hermitian, almost Hermitian, complex, almost complex, Kaehler and nearly Kaehler structures there are a lot of literature, we direct a readers to see [Bla76, YK84, Hub05] and others.

Let M^{2n+2} be an almost Hermitian manifold with structure (J, G) and Riemannian connection $\overline{\nabla}$. Let $\iota : M^{2n+1} \to M^{2n+2}$ be a $\mathcal{C}^\infty$ orientable hypersurface and C a unit normal. The induced metric g is given by

$$g(X, Y) = G(\iota_* X, \iota_* Y), \qquad \text{for all } X, Y \in \mathfrak{X}(M),$$

And its Riemannian connection ∇ is governed by the Gauss-Weingarten equations

$$\begin{aligned} \overline{\nabla}_{\iota_* X} \iota_* Y &= \iota_* \nabla_X Y + h(X, Y) C, \\ \overline{\nabla}_{\iota_* X} C &= -\iota_* H X, \end{aligned}$$

for all X, Y vector fields in M^{2n+1}, where h denotes the second fundamental form and H the corresponding Weingarten map.

Y.Tashiro [Tas63] showed that the almost Hermitian structure (J, G) on M^{2n+2} induces an almost contact metric structure (φ, ξ, η, g) on a hypersurface M^{2n+1} given by

$$\begin{aligned} J\iota_* X &= \iota_* \varphi X + \eta(X) C, \\ JC &= -\iota_* \xi, \end{aligned}$$

where X is a vector field on M^{2n+1} and g the induced metric.

1.3.6 Nearly Sasakian manifolds

The notion of nearly Sasakian was be used for the first time by Blair, Showers and Yano, in 1975 into their paper [BSY76]. There is very few literature on this subject, we can see some propriety in the book of Blair [Bla10].
All the proofs of this subsection are in the previous paper.

Definition 1.3.7 *Let M^{2n+1} be an almost contact metric manifold with the structure (φ, ξ, η, g), this structure is said to be* ***nearly Sasakian structure*** *if*

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = -2g(X, Y)\xi + \eta(X)Y + \eta(Y)X, \tag{1.7}$$

or, is equivalently,

$$(\nabla_X \varphi)X = -g(X, X)\xi + \eta(X)X, \tag{1.8}$$

for all vector fields X, Y.

An almost Contact metric manifold with a nearly Sasakian structure is called ***nearly Sasakian manifold****.*

Proposition 1.3.7 *On a nearly Sasakian manifold the vector field ξ is Killing.*

It follows from [BSY76] and [YK84] that

$$(\nabla_\xi \varphi)\xi = 0, \quad \varphi \nabla_\xi \xi = 0, \quad \nabla_\xi \xi = 0, \quad \nabla_\xi \eta = 0. \tag{1.9}$$

Theorem 1.3.6 *For a nearly Sasakian structure normality is equivalent to contact metric. In particular a normal nearly Sasakian structure is Sasakian.*

In [BSY76] the authors prove the following theorem, in which, they give the condition so that the hypersuface of the nearly Kahler manifold has an nearly Sasakian structure :

Theorem 1.3.7 *Let M^{2n+1} be a hypersurface of a nearly Kaehler manifold M^{2n+2}. Then the induced structure on M^{2n+1} is nearly Sasakian if and only if*

$$h(X, Y) = g(X, Y) + (h(\xi, \xi) - 1)\eta(X)\eta(Y), \tag{1.10}$$

for all X, Y vectors fields in M^{2n+1}, where h denotes the second fundamental form.

1.3.7 Nearly cosymplectic manifolds

The notion of nearly cosymplectic was be used for the first time by Davis E. Blair in 1971 into his paper [Bla71].

Definition 1.3.8 *Let M^{2n+1} be an almost contact metric manifold with the structure (φ, ξ, η, g), this structure is said to be* ***nearly cosymplectic structure*** *if whose tensors are Killing field and :*

$$(\nabla_X \varphi)X = 0, \quad \textit{or} \quad (\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0, \tag{1.11}$$

for all vector fields X, Y, and ∇ denotes covariant differentiation with respect to the Levi Civita connection of g.

Theorem 1.3.8 *Let M^{2n} a nearly Kaehler manifold and M^{2n-1} a C^∞ orientable hypersurface. Let (φ, ξ, η, g) denoted the induced almost contact metric structure on M^{2n-1}. Then φ is Killing if and only if the second fundamental form h is proportional to $\eta \otimes \eta$.*

On a nearly cosymplectic manifold we know that the vector field ξ is Killing. In [Bla71] the author proved the theorem:

Theorem 1.3.9 *Let M^{2n} a nearly Kaehler manifold and M^{2n-1} a C^∞ orientable hypersurface. Let η denote the induced almost contact form and suppose the second fundamental form h is proportional to $\eta \otimes \eta$. Then η is Killing and in particular, on M^{2n-1}, it is nearly cosymplectic.*

1.4 Cayley algebra on $\mathbb{R}^7$

We give a brief exposition of the multiplication on the Cayley number, we refer the reader to [DV96] for more details.
The multiplication on the Cayley numbers may be used to define a vector cross product on the purely imaginary Cayley numbers $\mathbb{R}^7$ using the formula

$$u \times v = 1/2(uv - vu), \tag{1.12}$$

while the standard inner product on $\mathbb{R}^7$ is given by

$$\langle u, v\rangle = -1/2(uv + vu). \tag{1.13}$$

It is easy to check that

$$\begin{aligned} u \times (v \times w) + (u \times v) \times w &= 2\langle u, w\rangle v - \langle u, v\rangle w - \langle w, v\rangle u, \\ \langle u, v \times w\rangle &= \langle v, w \times u\rangle = \langle w, u \times v\rangle, \\ u \times (u \times v) &= -\langle u, u\rangle v + \langle u, v\rangle u, \\ \langle u \times v, u \times w\rangle &= \langle u, u\rangle\langle v, w\rangle - \langle u, v\rangle\langle u, w\rangle. \end{aligned}$$

An ordered orthonormal basis, $e_1, e_2, ..., e_7$ is called a G_2-frame if

$$e_3 = e_1 \times e_2, \quad e_5 = e_1 \times e_4, \quad e_6 = e_2 \times e_4, \quad e_7 = e_3 \times e_4. \tag{1.14}$$

Using the previously mentioned properties, it follows that the multiplication table of a G_2 frame is given by

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

(1.15)

1.5 Complex projective space

In this section we give very short overview of the Complex projective space and the Lagrangian submanifold. (see [KN69], Vol.II, P133-134 and [BG88], P102).

Definition 1.5.1

- *The* ***complex Grassmann manifold*** $G_{p,q}(\mathbb{C})$ *of p-planes in* $\mathbb{C}^{p+p}$ *is the set of p-dimensional complex subspace in* $\mathbb{C}^{p+p}$ *with the structure of a complex manifold.*
- ***The group*** $GL(p+q;\mathbb{C})$ ***acting*** *in* $\mathbb{C}^{p+p}$ *sends every p-dimensional subspace into a p-dimensional subspace and hence can be considered as a transformation group acting on* $G_{p,q}(\mathbb{C})$*. The action is holomorphic and transitive.*

Remark 1.5.1 *The action of* $Gl(p+q;\mathbb{C})$ *in* $\mathbb{C}^{p+p}$ *is given by:*
If S_0 *denoted the p-dimensional subspace spanned by the first p elements of the natural basis of* $\mathbb{C}^{p+p}$*, then the isotropy subgroup at* S_0 *is given by*

$$H = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL(p+q;\mathbb{C}) \right\},$$

where 0 *denoted the zero matrix with p columns and q rows. Thus* $G_{p,q}(\mathbb{C})$ *is the quotient space* $GL(p+q;\mathbb{C})/H$ *of the complex Lie group* $GL(p+q;\mathbb{C})$ *by the closed Lie subgroup* H *and the natural projection* $GL(p+q;\mathbb{C}) \to G_{p,q}(\mathbb{C})$

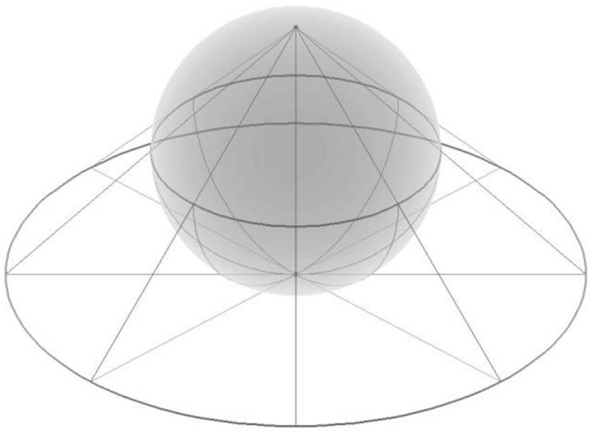
Definition 1.5.2 *The Grassmann manifold* $G_{n,1}(\mathbb{C})$ *is called an n-dimensional* ***complex projective space****, denoted* $\mathbb{P}_n(\mathbb{C})$ *or* $\mathbb{CP}^n$*.*

We have a equivalent definition :

Definition 1.5.3 *We consider $A = \mathbb{C}^{n+1}/\{0_{(origin)}\}$. Two elements z and w in A are equivalent if*

$$z \sim w \Leftrightarrow z = \lambda w,$$

where $\lambda \in \mathbb{C}$, It is straightforward to check that this is indeed an equivalence relation.
We call the quotient space the ***complex projective space****, denoted $\mathbb{CP}^n$.*



The one-dimensional complex projective space,
i.e. the complex projective line.

1.5.1 Lagrangian submanifolds

The fact that $\mathbb{CP}^n$ is an symplectic space, we use the definition of the Lagrangian as a submanifold of the symplectic manifold, [Aud05]:

Definition 1.5.4 *A Lagrangian submanifold of a symplectic manifold is a submanifold the tangent space of which is, at any point, a maximal totally isotropic subspace. If the symplectic form on the manifold W is denoted ω and the inclusion of the submanifold is*

$$j : L \hookrightarrow W,$$

to say that L is a Lagrangian is to say that $j^*\omega = 0$ and $L = \frac{1}{2}\dim W$.

Chapter 2

Surfaces in the nearly Sasakian 5-sphere

2.1 Nearly Sasakian structure on S^5

In [BSY76] the authors show how to induce a nearly Sasakian structure on S^5. In order to do so, they look at S^5 as a hypersurface in S^6 equipped with its nearly Kaehler structure. We have

$$S^5 \hookrightarrow S^6 \hookrightarrow \mathbb{R}^7, \tag{2.1}$$

where S^6 is the unit sphere in $\mathbb{R}^7$ with its cross product $\times$ induced by the Cayley algebra. We denote by P the unit outer normal. It is well known that S^6 has a nearly Kaehler structure with respect to the induced metric, we now consider S^5 umbilically embedded in S^6 at a latitude of 45°, and with normal unit N such that the second fundamental form $\tilde{h}(X,Y) = g(X,Y)$. Then we see that the induced structure on S^5 from the nearly Kaehler structure is nearly Sasakian. In this case we have: for $P \in S^5$, V tangent vector to S^5 and N the normal vector of S^5 in S^6

$$\begin{aligned}
P &= \frac{1}{\sqrt{2}}(x_1, x_2, x_3, 1, x_5, x_6, x_7),\\
N &= -\frac{1}{\sqrt{2}}(x_1, x_2, x_3, -1, x_5, x_6, x_7),\\
V &= \frac{1}{\sqrt{2}}(v_1, v_2, v_3, 0, v_5, v_6, v_7),
\end{aligned}$$

and ξ and φ from the nearly Sasakian structure are respectively given by

$$\begin{aligned}
\xi &= -P \times N,\\
\varphi(V) &= P \times V - \eta(V)N,\\
\eta(V) &= \langle \xi, V \rangle.
\end{aligned}$$

With the cross product, we obtain that ξ is given by

$$\xi = (x_5, x_6, x_7, 0, -x_1, -x_2, -x_3). \tag{2.2}$$

2.1.1 Totally real surfaces

Definition 2.1.1 *Let M a surface of S^5 with nearly Sasakian structure, we say that M is totally real submanifold of S^5 if for all $P \in M$ we have*

$$\xi \in N_p M \quad \text{and} \quad \varphi(T_p M) \subset N_p M, \tag{2.3}$$

where $N_P M$ and $T_P M$ denote respectively the normal space and the tangent space to M at the point P.

Let D be the standard Riemannian connection in $\mathbb{R}^7$. We denote the induced connections in S^6, S^5 and M by the previously mentioned immersions, respectively by $\tilde{\nabla}$, $\bar{\nabla}$ and ∇. Using the Gauss formula, we have

$$\begin{aligned} D_X Y &= \tilde{\nabla}_X Y - \langle X, Y \rangle P, \\ \tilde{\nabla}_X Y &= \bar{\nabla}_X Y + \tilde{h}(X,Y)N, \\ \bar{\nabla}_X Y &= \nabla_X Y + h(X,Y), \end{aligned}$$

where P denotes the position vector of the immersion of M into $\mathbb{R}^7$, and h, $\tilde{h}$ are the second fundamental forms of M and S^5 respectively, and X, Y are tangent vectors fields on M.

It then follows that

$$D_X Y = \underbrace{\underbrace{\underbrace{\underbrace{\nabla_X Y}_{T_pM} + \underbrace{h(X,Y)}_{N_pM}}_{T_pS^5} + \underbrace{\tilde{h}(X,Y)N}_{N_pS^5}}_{T_pS^6} - \underbrace{\langle X, Y\rangle P}_{N_pS^6}}_{T_p\mathbb{R}^7}. \tag{2.4}$$

Remarks : As we have previously mentioned, a n-dimensional manifold of a Sasakian manifold, for which ξ is normal, is always anti-invariant, i.e. $\varphi(T_P M) \subset N_P M$. In the following subsection we show that this result is no longer true in the 5-sphere with nearly Sasakian structure.

2.1.2 2-sphere in the nearly Sasakian 5-sphere

[BS16]
We look at S^2, which we parametrize in the usual way by

$$(cos\theta \cos\psi,\ \sin\theta\cos\psi,\ \sin\psi).$$

We define a 1-parameter family of immersions in the nearly Sasakian $S^5 \subset S^6(1) \subset \mathbb{R}^7$ by

$$\begin{aligned} P = \tfrac{1}{\sqrt{2}}(&\cos a\cos\theta\cos\psi,\ \cos a\sin\theta\cos\psi,\ \cos a\sin\psi, 1,\\ &\sin a\cos\theta\cos\psi,\ \sin a\sin\theta\cos\psi,\ \sin a\sin\psi),\end{aligned}$$

where a is an arbitrary constant. We also get that

$$\begin{aligned} N &= \tfrac{1}{\sqrt{2}}(\cos a\cos\theta\cos\psi,\ \cos a\sin\theta\cos\psi,\ \cos a\sin\psi, -1,\\ &\qquad \sin a\cos\theta\cos\psi,\ \sin a\sin\theta\cos\psi,\ \sin a\sin\psi),\\ \xi &= (-\sin a\cos\theta\cos\psi,\ -\sin a\sin\theta\cos\psi,\ -\sin a\sin\psi, 0,\\ &\qquad \cos a\cos\theta\cos\psi,\ \cos a\sin\theta\cos\psi,\ \cos a\sin\psi).\end{aligned}$$

In the case that ξ is normal, we get that for all $V \in T_PM$

$$\eta(V) = \langle \xi, V\rangle = 0 \quad \text{and} \quad \varphi(V) = P \times V.$$

The tangent vectors of S^2 are

$$\frac{\partial P}{\partial\theta} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos a\sin\theta\cos\psi \\ \cos a\cos\theta\cos\psi \\ 0 \\ 0 \\ -\sin a\sin\theta\cos\psi \\ \sin a\cos\theta\cos\psi \\ 0 \end{pmatrix}, \quad \frac{\partial P}{\partial\psi} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos a\cos\theta\sin\psi \\ -\cos a\sin\theta\sin\psi \\ \cos a\cos\psi \\ 0 \\ -\sin a\cos\theta\sin\psi \\ -\sin a\sin\theta\sin\psi \\ \sin a\cos\psi \end{pmatrix}.$$

With the cross product computations, we get that :

$$\varphi(\frac{\partial P}{\partial\theta}) = P \times \frac{\partial P}{\partial\theta} = \frac{1}{2}\begin{pmatrix} -\sqrt{2}\sin a\sin\theta\cos\psi - \frac{1}{2}\cos 2a\cos\theta\sin 2\psi \\ \cos\psi\left(\sqrt{2}\sin a\cos\theta - \cos 2a\sin\theta\sin\psi\right) \\ \cos 2a\cos^2\psi \\ 0 \\ \cos a\cos\psi\left(\sqrt{2}\sin\theta + 2\sin a\cos\theta\sin\psi\right) \\ \cos a\cos\psi\left(2\sin a\sin\theta\sin\psi - \sqrt{2}\cos\theta\right) \\ -2\cos a\cos^2\psi\sin a \end{pmatrix},$$

$$\varphi(\frac{\partial P}{\partial \psi}) = P \times \frac{\partial P}{\partial \psi} = \frac{1}{2}\begin{pmatrix} \cos 2a \sin\theta - \sqrt{2}\sin a\cos\theta\sin\psi \\ -\cos 2a\cos\theta - \sqrt{2}\sin a\sin\theta\sin\psi \\ \sqrt{2}\sin a\cos\psi \\ 0 \\ \cos a\left(\sqrt{2}\cos\theta\sin\psi - 2\sin a\sin\theta\right) \\ \cos a\left(2\sin a\cos\theta + \sqrt{2}\sin\theta\sin\psi\right) \\ -\sqrt{2}\cos a\cos\psi \end{pmatrix}.$$

from which it follows that

$$\begin{aligned}
&\langle\varphi(\frac{\partial P}{\partial \theta}), \xi\rangle = 0, \quad \langle\varphi(\frac{\partial P}{\partial \psi}), \xi\rangle = 0, \quad \langle\varphi(\frac{\partial P}{\partial \theta}), \varphi(\frac{\partial P}{\partial \psi})\rangle = 0,\\
&\langle\varphi(\frac{\partial P}{\partial \theta}), \frac{\partial P}{\partial \theta}\rangle = 0, \quad \langle\varphi(\frac{\partial P}{\partial \psi}), \frac{\partial P}{\partial \psi}\rangle = 0,\\
&\langle\varphi(\frac{\partial P}{\partial \theta}), \frac{\partial P}{\partial \psi}\rangle = -\frac{1}{2\sqrt{2}}\cos 3a\cos\psi,\\
&\langle\varphi(\frac{\partial P}{\partial \psi}), \frac{\partial P}{\partial \theta}\rangle = \frac{1}{2\sqrt{2}}\cos a(1 - 2\cos 2a)\cos\psi.
\end{aligned}$$

So we see that for all of these examples ξ is a normal vector to the immersion. However $\varphi(T_pM) \nsubseteq N_pM$, unless $a = \pm\frac{\pi}{6} + k\pi$, or $a = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$.

Therefore, in the nearly Sasakian case we define the notion of a totally real submanifold by demanding that ξ is normal and $\varphi(T_PM) \subset N_PM$. So for the immersions in our family which are totally real, i.e. when $a = \pm\frac{\pi}{6} + k\pi$, or $a = \frac{\pi}{2} + k\pi$, we will now compute the second fundamental form. Note that in this case, $\{\varphi(\frac{\partial P}{\partial \theta}), \varphi(\frac{\partial P}{\partial \psi}), \xi\}$ is a frame of normal space consisting of the mutually orthogonal vectors.

Proposition 2.1.1 *A totally real 2-sphere in an nearly Sasakian 5-sphere is always totally geodesic.*

Proof
Therefore the second fundamental form of the immersion is given by

$$h(\frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial \theta}) = \frac{\langle\frac{\partial^2 P}{\partial \theta^2}, \varphi(\frac{\partial P}{\partial \theta})\rangle}{\left|\frac{\partial P}{\partial \theta}\right|^2}\varphi(\frac{\partial P}{\partial \theta}) + \frac{\langle\frac{\partial^2 P}{\partial \theta^2}, \varphi(\frac{\partial P}{\partial \psi})\rangle}{\left|\frac{\partial P}{\partial \psi}\right|^2}\varphi(\frac{\partial P}{\partial \psi}) + \langle\frac{\partial^2 P}{\partial \theta^2}, \xi\rangle\xi,$$

$$h(\frac{\partial P}{\partial \psi}, \frac{\partial P}{\partial \psi}) = \frac{\langle\frac{\partial^2 P}{\partial \psi^2}, \varphi(\frac{\partial P}{\partial \theta})\rangle}{\left|\frac{\partial P}{\partial \theta}\right|^2}\varphi(\frac{\partial P}{\partial \theta}) + \frac{\langle\frac{\partial^2 P}{\partial \psi^2}, \varphi(\frac{\partial P}{\partial \psi})\rangle}{\left|\frac{\partial P}{\partial \psi}\right|^2}\varphi(\frac{\partial P}{\partial \psi}) + \langle\frac{\partial^2 P}{\partial \psi^2}, \xi\rangle\xi,$$

$$h(\frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial \psi}) = \frac{\langle \frac{\partial^2 P}{\partial \theta \partial \psi}, \varphi(\frac{\partial P}{\partial \theta})\rangle}{\left|\frac{\partial P}{\partial \theta}\right|^2}\varphi(\frac{\partial P}{\partial \theta}) + \frac{\langle \frac{\partial^2 P}{\partial \theta \partial \psi}, \varphi(\frac{\partial P}{\partial \psi})\rangle}{\left|\frac{\partial P}{\partial \psi}\right|^2}\varphi(\frac{\partial P}{\partial \psi}) + \langle \frac{\partial^2 P}{\partial \theta \partial \psi}, \xi\rangle \xi.$$

Before fixing a, straightforward computations, show that

$$\begin{aligned}
&\langle \frac{\partial^2 P}{\partial \theta^2}, \varphi(\frac{\partial P}{\partial \psi})\rangle = \langle \frac{\partial^2 P}{\partial \theta^2}, \xi\rangle = 0,\\
&\langle \frac{\partial^2 P}{\partial \psi^2}, \varphi(\frac{\partial P}{\partial \theta})\rangle = \langle \frac{\partial^2 P}{\partial \psi^2}, \varphi(\frac{\partial P}{\partial \psi})\rangle = \langle \frac{\partial^2 P}{\partial \psi^2}, \xi\rangle = 0,\\
&\langle \frac{\partial^2 P}{\partial \psi \partial \theta}, \varphi(\frac{\partial P}{\partial \theta})\rangle = \langle \frac{\partial^2 P}{\partial \psi \partial \theta}, \xi\rangle = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\langle \frac{\partial^2 P}{\partial \theta^2}, \varphi(\frac{\partial P}{\partial \theta})\rangle = -\frac{1}{2\sqrt{2}}\cos a(1 - 2\cos 2a)\cos^2\psi \sin\psi,\\
&\langle \frac{\partial^2 P}{\partial \psi \partial \theta}, \varphi(\frac{\partial P}{\partial \psi})\rangle = -\frac{1}{2\sqrt{2}}\cos a(1 - 2\cos 2a)\sin\psi.
\end{aligned}$$

Then, if M is totally real we find

$$h(\frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial \theta}) = h(\frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial \psi}) = h(\frac{\partial P}{\partial \psi}, \frac{\partial P}{\partial \psi}) = 0.$$

Therefore we obtain our totally real surfaces S^2 which are totally geodesic in the nearly Sasakian S^5. □

2.2 Surfaces in the nearly Sasakian 5-sphere

[BS16]
In this subsection, M will always denote a totally real surface of the 5-dimension nearly Sasakian sphere S^5 which we consider as a subset of $\mathbb{R}^7$. The structure of M is, as previously, built with the immersions :

$$M \hookrightarrow S^5 \hookrightarrow S^6 \hookrightarrow \mathbb{R}^7. \quad (2.5)$$

We now will give the important theorem of this section and his proof.

Theorem 2.2.1 *A totally real surface of the nearly Sasakian S^5 is always minimal.*

To prove this theorem, we divide it into three lemmas.

Lemma 2.2.1 *Let M be a totally real surface of the 5-dimensional nearly Sasakian sphere and let $\{u, v\}$ be a local orthonormal basis of tangent vector fields on M. Then $\{\xi, \varphi u, \varphi v\}$ is an orthonormal basis of the normal space N_pM. Moreover a basis of $\mathbb{R}^7$ is given by*

$$\{\underbrace{u, \quad v}_{T_pM}, \underbrace{\xi \quad , \quad \varphi u \quad , \quad \varphi v}_{N_pM}, \underbrace{-N}_{N_pS^5}, \underbrace{P}_{N_pS^6}\},$$

where we denote $\varphi V := \varphi(V)$.

Proof : We have $\xi = -P \times N$ and $\varphi u = P \times u - \eta(u)N$. As $\eta(u) = \langle u, \xi \rangle = 0$, it follows

$$\xi = -P \times N, \quad \varphi u = P \times u, \quad \varphi v = P \times v.$$

Using the properties of the cross product we get :

$$\begin{aligned}
\langle \xi, \varphi u \rangle &= \langle -P \times N, P \times u \rangle = -\langle P, P \rangle \langle N, u \rangle + \langle P, N \rangle \langle P, u \rangle, \\
\langle \xi, \varphi v \rangle &= \langle -P \times N, P \times v \rangle = -\langle P, P \rangle \langle N, v \rangle + \langle P, N \rangle \langle P, v \rangle, \\
\langle \varphi u, \varphi v \rangle &= \langle P \times u, P \times v \rangle = \langle P, P \rangle \langle u, v \rangle - \langle P, u \rangle \langle P, v \rangle.
\end{aligned}$$

or $u, v \in T_PM$ and $N \in N_PM$ in S^5, then $\langle N, u \rangle = 0$ and $\langle N, v \rangle = 0$, with the same reasoning we get $\langle P, u \rangle = \langle P, v \rangle = 0$. Then

$$\begin{aligned}
\langle \xi, \varphi u \rangle &= 0, \\
\langle \xi, \varphi v \rangle &= 0, \\
\langle \varphi u, \varphi v \rangle &= 0.
\end{aligned}$$

thus $\xi \perp \varphi u$, $\xi \perp \varphi v$ and $\varphi u \perp \varphi v$ and $\{\xi, \varphi u, \varphi v\}$ is an orthonormal basis of the normal space N_pM.
In order to be able to use the multiplication table for the cross product in $\mathbb{R}^7$ (1.15), we first remark that $\xi = -P \times N = u \times v$ and therefore

$$\{u = e_2,\ v = e_4, \quad P \times u = e_3, \quad P \times v = e_5, \quad \xi = e_6, \quad -N = e_7, \quad P = e_1\}$$

is a G_2 basis along the surface. □

Lemma 2.2.2 *On the surface M, for any tangent vector fields X, Y we have*

$$h(X, Y) \perp \xi.$$

Moreover, we get in each point P in M:

$$\begin{aligned}
\langle h(u, v), \varphi v \rangle &= \langle h(v, v), \varphi u \rangle, \\
\langle h(u, v), \varphi u \rangle &= \langle h(u, u), \varphi v \rangle.
\end{aligned}$$

Proof We know that ξ is in the normal space of M in S^5, from the lemma2.2.1, we have that $\{u, v, \xi, \varphi u, \varphi v, -N, P\}$ is an orthonormal basis of $\mathbb{R}^7$, then $\langle u, \xi\rangle = \langle v, \xi\rangle = 0$ and we have

$$\begin{aligned}
D_v\langle u,\xi\rangle &= 0\\
&= \langle D_v u, P\times N\rangle + \langle u, D_v P\times N\rangle + \langle u, P\times D_v N\rangle = 0\\
&= \langle \nabla_v u, P\times N\rangle + \langle h(u,v), P\times N\rangle + \langle u, v\times N\rangle + \langle u, P\times v\rangle = 0\\
&= \langle h(u,v), P\times N\rangle + \langle u, v\times N\rangle = 0\\
&= \underbrace{\langle h(u,v), P\times N\rangle}_{Symmetric} + \underbrace{\langle u\times v, N\rangle}_{antisymmetric} = 0\\
&\Rightarrow \langle h(u,v), P\times N\rangle = 0 \quad and \quad \langle u\times v, N\rangle = 0,\\
&\Rightarrow h(u,v)\perp\xi.\\
D_u\langle u,\xi\rangle &= 0\\
&= \langle D_u u, P\times N\rangle + \langle u, D_u P\times N\rangle + \langle u, P\times D_u N\rangle = 0\\
&= \langle \nabla_u u + h(u,u) - N - P, P\times N\rangle + \langle u, u\times N\rangle + \langle u, P\times u\rangle = 0\\
&= \langle h(u,u), P\times N\rangle,\\
&\Rightarrow h(u,u)\perp\xi.\\
D_v\langle v,\xi\rangle &= 0 \quad \text{in the same}\\
&\Rightarrow h(v,v)\perp\xi.
\end{aligned}$$

Therefore $h(X, Y)\perp\xi$, for any tangent vector fields X and Y on M. Next using that the immersion is anti-invariant, i.e. $\langle u, \varphi v\rangle = 0$ and $\langle v, \varphi u\rangle = 0$, we deduce that:

$$\begin{aligned}
D_v\langle u,\varphi v\rangle &= 0\\
&= \langle D_v u, P\times v\rangle + \langle u, D_v P\times v\rangle + \langle u, P\times D_v v\rangle = 0\\
&= \langle \nabla_v u, P\times v\rangle + \langle h(v,u), P\times v\rangle + \langle u, v\times v\rangle + \langle u\times P, D_v v\rangle = 0\\
&= \langle h(u,v), P\times v\rangle - \langle h(v,v) - N - P, P\times u\rangle = 0,\\
&\Rightarrow \langle h(u,v), P\times v\rangle = \langle h(v,v), P\times u\rangle.\\
D_u\langle v,\varphi u\rangle &= 0\\
&= \langle D_u v, P\times u\rangle + \langle v, D_u P\times u\rangle + \langle v, P\times D_u u\rangle = 0\\
&= \langle \nabla_u v, P\times u\rangle + \langle h(v,u), P\times u\rangle + \langle v, u\times u\rangle + \langle v\times P, D_u u\rangle = 0\\
&= \langle h(u,v), P\times u\rangle - \langle h(u,u) - N - P, P\times v\rangle = 0,\\
&\Rightarrow \langle h(u,v), P\times u\rangle = \langle h(u,u), P\times v\rangle.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.2.3 *We have*

$$\nabla_u u = \alpha\, v, \quad \nabla_v u = \beta\, v, \quad \nabla_v v = -\beta\, u, \quad \nabla_u v = -\alpha\, u,$$

where α and β are local functions on M.

Proof We have $\langle u, u\rangle = \langle v, v\rangle = 1$ and $\langle u, v\rangle = 0$. From

$$D_u\langle u, u\rangle = 2\langle D_u u, u\rangle = 0$$

we have

$$\langle \nabla_u u + h(u, u) - N - P, u\rangle = 0$$

i.e $\langle \nabla_u u, u\rangle = 0$. Then

$$D_u\langle u, u\rangle = 0 \Rightarrow \nabla_u u = \alpha\, v, \qquad \text{where } \alpha \in \mathcal{F}(M).$$

In the same way, we get that there exist local functions α', β and β' on M such that :

$$\begin{aligned} D_v\langle u, u\rangle = 0 &\Rightarrow \nabla_v u = \beta\, v, \\ D_v\langle v, v\rangle = 0, &\Rightarrow \nabla_v v = \alpha'\, u, \\ D_u\langle v, v\rangle = 0, &\Rightarrow \nabla_u v = \beta'\, u. \end{aligned}$$

Finally from $D_v\langle u, v\rangle = 0$

$$\begin{aligned} D_v\langle u, v\rangle &= 0 \\ &= \langle \nabla_v u + h(u, v), v\rangle + \langle u, \nabla_v v + h(v, v) - N - P\rangle = 0 \\ &= \langle \nabla_v u, v\rangle + \langle u, \nabla_v v\rangle = 0 \\ &= \langle \beta v, v\rangle + \langle u, \alpha' u\rangle = 0, \end{aligned}$$

we obtain that $\alpha' = -\beta$. In same from $D_u\langle u, v\rangle = 0$, we get $\beta' = -\alpha$.
This completes the proof of the lemma. □

Now we give the proof of the theorem 2.2.1.
Proof From lemmas 2.2.2 and 2.2.3, we have that $h \perp \xi$, therefore we get :

$$h(u, u) = a_1\varphi u + a_2\varphi v, \quad h(u, v) = b_1\varphi u + b_2\varphi v, \quad h(v, v) = c_1\varphi u + c_2\varphi v, \quad (2.6)$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are local functions on M.
As $\xi = -P \times N = u \times v$, we can compute the covariant derivative of ξ in two different ways. Indeed, we have that

$$D_u\xi = D_u(-P \times N) = D_u(u \times v).$$

So, we see that

$$\begin{aligned} -D_u(P \times N) &= -D_u P \times N - P \times D_u N = -u \times N - P \times u \\ &= -P \times v - P \times u, \end{aligned}$$

and

$$\begin{aligned} D_u(u \times v) &= D_u u \times v + u \times D_u v \\ &= (\nabla_u u + h(u,u) - N - P) \times v + u \times (\nabla_u v + h(u,v)) \\ &= (\alpha v + a_1 P \times u + a_2 P \times v - N - P) \times v \\ &\qquad + u \times (-\alpha u + b_1 P \times u + b_2 P \times v) \\ &= (-a_1 - b_2)N + (-a_2 + b_1)P - P \times u - P \times v. \end{aligned}$$

From these equalities, we get $(-a_1 - b_2)N + (-a_2 + b_1)P = 0$, so

$$\begin{cases} b_1 = a_2, \\ b_2 = -a_1. \end{cases}$$

In a similar way, we obtain :

$$D_v \xi = D_v(-P \times N) = D_v(u \times v) \Rightarrow \begin{cases} c_1 = b_2, \\ c_2 = -b_1. \end{cases}$$

Finally we get

$$\begin{cases} c_1 = b_2 = -a_1, \\ c_2 = -b_1 = -a_2. \end{cases}$$

Therefore we have $h(u,u) = -h(v,v)$ i.e M is a minimal surface. □

2.2.1 Minimal surfaces in the nearly Sasakian 5-sphere and minimal Lagrangian submanifolds

In this section we give the relation of our minimal surface proved before, and the minimal Lagrangian surface, for this we must go through several changes of variables [BS16].

As indicated in the following proposition, we can further improve our choice of basis.

Proposition 2.2.1 *Let M be a totally real surface of the nearly Sasakian sphere S^5. Then if necessary by restricting to an open dense subset there exists a local orthonormal frame $\{u, v\}$ of M at each point P of M such that*

$$\nabla_u u = \alpha\, v, \quad \nabla_v u = \beta\, v, \quad \nabla_v v = -\beta\, u, \quad \nabla_u v = -\alpha\, u,$$
$$h(u,u) = aP \times u, \quad h(u,v) = -aP \times v,$$

where α, β are the functions defined before and a is a function on this open dense subset of M satisfying :

$$\begin{cases} v(\alpha) - u(\beta) + 2a^2 - \alpha^2 - \beta^2 - 2 = 0, \\ v(\beta) + u(\alpha) = 0, \\ v(a) = 3\alpha a, \\ u(a) = -3\beta a, \end{cases} \tag{2.7}$$

First, note that if the immersion is totally geodesic on an open part, then we can take $a_1 = a_2 = 0$ (where a_1, a_2 are defined in (2.6)) and it follows from the Gauss equation that the first equations in (2.7) is satisfied.

Else, by restricting to an open dense subset, we may assume that $a_1^2 + a_2^2 \neq 0$.

Lemma 2.2.4 *Let M be a totally real surface. In a neighborhood of a non totally geodesic point of M, there exist a local function a such that the second fundamental form h can be written as*

$$h(u,u) = a\varphi u, \quad h(u,v) = -a\varphi v, \quad h(v,v) = -a\varphi u,$$

where $\{u, v\}$ is the frame of M at each point P of the neighborhood.

Proof Using a rotation of the orthonormal frame, we write

$$\begin{cases} U = \cos\theta\, u - \sin\theta\, v, \\ V = \sin\theta\, u + \cos\theta\, v, \end{cases}$$

where θ is a differentiable function. Then we have

$$h(U,U) = A_1 P \times U + A_2 P \times V.$$

We now want to find a function θ such that A_2 vanishes.
First we compute $h(U,U)$:

$$h(U,U) = h(\cos\theta\, u - \sin\theta\, v, \cos\theta\, u - \sin\theta\, v)$$

$$= \cos^2\theta\, h(u,u) + \sin^2\theta\, h(v,v) - 2\cos\theta\,\sin\theta\, h(u,v)$$

As
$h(u,u) = a_1\varphi u + a_2\varphi v, \quad h(u,v) = a_2\varphi u - a_1\varphi v, \quad h(v,v) = -a_1\varphi u - a_2\varphi v,$
we get :

$$h(U,U) = (a_1(\cos^2\theta - \sin^2\theta) - 2a_2\cos\theta\sin\theta)P\times u \\ + (a_2(\cos^2\theta - \sin^2\theta) + 2a_1\cos\theta\sin\theta)P\times v.$$

We have
$$P\times V = \sin\theta\, P\times u + \cos\theta\, P\times v$$
and $A_2 = \langle h(U,U), P\times V\rangle = 0$, or

$$\begin{aligned} A_2 =& \langle (a_1(\cos^2\theta - \sin^2\theta) - 2a_2\cos\theta\sin\theta)P\times u \\ &+ (a_2(\cos^2\theta - \sin^2\theta) + 2a_1\cos\theta\sin\theta)P\times v, \sin\theta\, P\times u + \cos\theta\, P\times v\rangle. \\ =&(a_1(\cos^2\theta - \sin^2\theta) - 2a_2\cos\theta\sin\theta)\sin\theta \\ &+ (a_2(\cos^2\theta - \sin^2\theta) + 2a_1\cos\theta\sin\theta)\cos\theta \\ =& a_1\sin\theta(3\cos^2\theta - \sin^2\theta) + a_2\cos\theta(\cos^2\theta - 3\sin^2\theta), \end{aligned}$$

then

$$A_2 = a_1\sin 3\theta + a_2\cos 3\theta = 0.$$

As by assumption a_1 and a_2 do not both vanish, we see that it is sufficient to take θ such that
$$\cos 3\theta = -\frac{a_1}{a_1^2 + a_2^2} \quad \text{and} \quad \sin 3\theta = \frac{a_2}{a_1^2 + a_2^2}$$
and take $a = A_1$. □

Using the previous lemma, we complete the proof of proposition 2.2.1.
Proof Using the frame vectors along our surface, we can compute the curvature tensor of $\mathbb{R}^7$. So, we take X and Y tangent vector fields to the surface and for Z we take any vector field belonging to our frame of $\mathbb{R}^7$. As the connection on $\mathbb{R}^7$ is flat, we have that

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z = 0.$$

Using lemma 2.2.3 and 2.2.4 and Gauss formula, straightforward computations in the nearly Sasakian case show that

$$\begin{aligned} R(u,v)P &= D_u D_v P - D_v D_u P - D_{[u,v]}P \\ &= D_u v - D_v u - D_{-\alpha u - \beta v}P \end{aligned}$$

$$\begin{aligned}&= \nabla_u v + h(u,v) - (\nabla_v u + h(u,v)) - (\alpha u - \beta v)\\&= -\alpha u - aP \times v - \beta v + aP \times +\alpha u + \beta v\\&= 0.\end{aligned}$$

At the same we get :

$$\begin{aligned}R(u,v)u &= (u(\beta) - v(\alpha) - 2a^2 + \alpha^2 + \beta^2 + 2)v\\&\quad + (3\alpha a - v(a))P \times u + (-3\beta a - u(a))P \times v,\\R(u,v)v &= (-u(\beta) + v(\alpha) + 2a^2 - \alpha^2 - \beta^2 - 2)u\\&\quad + (-3\beta a - u(a))P \times u + (-3\alpha a + v(a))P \times v,\\R(u,v)(P \times u) &= (-3\alpha a + v(a))u + (3\beta a + u(a))v\\&\quad + (u(\beta) - v(\alpha) - 2a^2 + \alpha^2 + \beta^2 + 2)P \times v,\\R(u,v)(P \times v) &= (3\beta a + u(a))u + (3\alpha a - v(a))v\\&\quad + (-u(\beta) + v(a) + 2a^2 - \alpha^2 - \beta^2 - 2)P \times u,\\R(u,v)(P \times N) &= 0.\end{aligned}$$

We deduce that :

$$\begin{cases} v(\alpha) - u(\beta) + 2a^2 - \alpha^2 - \beta^2 - 2 = 0,\\ v(a) = 3\alpha a,\\ u(a) = -3\beta a.\end{cases}$$

Note that, as we are working with the Levi Civita connection, we have that

$$[v,u](f) = v(u(f)) - u(v(f)) = (\nabla_v u - \nabla_u v)(f),$$

where f is an arbitrary function. With these two ways of computing the Lie bracket for the function a, we deduce that

$$-4\beta v(a) - 4\alpha u(a) - 3v(\beta)a - 3u(\alpha)a = 0.$$

From this equation, we obtain $v(\beta) + u(\alpha) = 0$.
This completes the proof of the proposition. □

In order to relate our surfaces to minimal Lagrangian surfaces in the complex projective space, we will use this proposition in order to introduce suitable coordinates on the surface.

Theorem 2.2.2 *Let M be totally real surface of the 5-dimension sphere S^5 with nearly Sasakian structure, then around each non totally geodesic point, M locally corresponds to a minimal Lagrangian in the complex projective space $\mathbb{CP}^2$ with the group $SU(3)$.*

Proof We have the differential system (2.7) :

$$\begin{cases} v(\alpha) - u(\beta) + 2a^2 - \alpha^2 - \beta^2 - 2 = 0, \\ v(\beta) + u(\alpha) = 0, \\ v(a) = 3\alpha a, \\ u(a) = -3\beta a. \end{cases}$$

As we are working on a neighborhood of a non totally geodesic point, we have that $a \neq 0$. We define a function ρ on M by $\rho = a^{-\frac{1}{3}}$. It then follows that

$$\begin{cases} u(\ln \rho) = \beta, \\ v(\ln \rho) = -\alpha, \end{cases} \qquad \text{or equivalently} \qquad \begin{cases} u(\rho) = \rho\beta, \\ v(\rho) = -\rho\alpha. \end{cases}$$

Computing now $\nabla_{\rho v}\rho u$ and $\nabla_{\rho u}\rho v$, we get :

$$\begin{aligned} \nabla_{\rho v}\rho u &= \rho \nabla_v \rho u \\ &= \rho(v(\rho)u + \rho\nabla_v u) \\ &= \rho\, v(\rho)u + \rho^2\beta v \\ &= -\rho^2\alpha u + \rho^2\beta v, \\ \nabla_{\rho u}\rho v &= \rho\, u(\rho)v - \rho^2\alpha u \\ &= \rho^2\beta v - \rho^2\alpha u. \end{aligned}$$

Hence

$$[\rho u, \rho v] = 0.$$

This implies that there exist local coordinates x and y on M such that

$$\frac{\partial}{\partial x} = \rho u \quad \text{and} \quad \frac{\partial}{\partial y} = \rho v.$$

It now follows that

$$u = \frac{1}{\rho}\frac{\partial}{\partial x} \quad \text{and} \quad v = \frac{1}{\rho}\frac{\partial}{\partial y}, \tag{2.8}$$

$$\alpha = -\frac{1}{\rho^2}\frac{\partial}{\partial y}(\rho) \quad \text{and} \quad \beta = \frac{1}{\rho^2}\frac{\partial}{\partial x}(\rho). \tag{2.9}$$

We compute $v(\alpha)$ and $u(\beta)$:

$$\begin{aligned} v(\alpha) &= \frac{1}{\rho}\frac{\partial}{\partial y}(\alpha) = \frac{2}{\rho^4}\left(\frac{\partial}{\partial y}(\rho)\right)^2 - \frac{1}{\rho^3}\frac{\partial^2}{\partial y^2}(\rho), \\ u(\beta) &= \frac{1}{\rho}\frac{\partial}{\partial x}(\beta) = \frac{-2}{\rho^4}\left(\frac{\partial}{\partial x}(\rho)\right)^2 + \frac{1}{\rho^3}\frac{\partial^2}{\partial x^2}(\rho). \end{aligned}$$

Replacing it all in the equation $v(\alpha) - u(\beta) - \alpha^2 - \beta^2 + 2a^2 - 2 = 0$,

$$
\begin{aligned}
v(\alpha) - u(\beta) - &\alpha^2 - \beta^2 + 2a^2 - 2 \\
&= \frac{2}{\rho^4}\left(\frac{\partial}{\partial y}(\rho)\right)^2 - \frac{1}{\rho^3}\frac{\partial^2}{\partial y^2}(\rho) + \frac{2}{\rho^4}\left(\frac{\partial}{\partial x}(\rho)\right)^2 - \frac{1}{\rho^3}\frac{\partial^2}{\partial x^2}(\rho) \\
&\quad - \left(-\frac{1}{\rho^2}\frac{\partial}{\partial y}(\rho)\right)^2 - \left(\frac{1}{\rho^2}\frac{\partial}{\partial x}(\rho)\right)^2 - 2\left(\rho^3\right)^2 - 2,
\end{aligned}
$$

it reduces to the following differential equation :

$$
-\rho\Delta\rho + \left(\frac{\partial}{\partial x}(\rho)\right)^2 + \left(\frac{\partial}{\partial y}(\rho)\right)^2 + 2\rho^{-2} - 2\rho^4 = 0. \tag{2.10}
$$

where Δ is the Laplace operator (i.e. $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$).

If we write now $\rho = e^\psi$, the differential equation (2.10) become :

$$
\begin{aligned}
-\rho\Delta\rho + \left(\frac{\partial}{\partial x}(\rho)\right)^2 &+ \left(\frac{\partial}{\partial y}(\rho)\right)^2 + 2\rho^{-2} - 2\rho^4 \\
&= -e^\psi\Delta(e^\psi) + \left(\frac{\partial}{\partial x}(e^\psi)\right)^2 + \left(\frac{\partial}{\partial y}(e^\psi)\right)^2 + 2e^{-2\psi} - 2e^{4\psi},
\end{aligned}
$$

using that $\rho = e^\psi$, we get :

$$
\begin{cases}
\frac{\partial}{\partial x}(\rho) = \frac{\partial}{\partial x}(\psi)e^\psi, \\
\frac{\partial}{\partial y}(\rho) = \frac{\partial}{\partial y}(\psi)e^\psi,
\end{cases}
\quad \text{and} \quad
\begin{cases}
\frac{\partial^2}{\partial x^2}(\rho) = \left(\frac{\partial}{\partial x}(\psi)\right)^2 e^\psi + \frac{\partial^2}{\partial x^2}(\psi)e^\psi, \\
\frac{\partial^2}{\partial y^2}(\rho) = \left(\frac{\partial}{\partial y}(\psi)\right)^2 e^\psi + \frac{\partial^2}{\partial y^2}(\psi)e^\psi,
\end{cases}
$$

and the differential equation, it reduces to :

$$
2\Delta\psi - 4e^{-4\psi} + 4e^{2\psi} = 0.
$$

In this last equation, applying the change of coordinates $\psi = \gamma + d$, where d is constant gives :

$$
2\Delta\gamma - 4e^{-4d}e^{-4\gamma} + 4e^{2d}e^{2\gamma} = 0.
$$

For $d = -\ln 2$, we get the final equation :

$$
2\Delta\gamma - 64\, e^{-4\gamma} + e^{2\gamma} = 0, \tag{2.11}
$$

with $\gamma = -\frac{1}{3}\ln a + \ln 2$.

The differential equations (2.11) is elliptic equations of Ţiţeica type of the form:

$$2\Delta\psi + \epsilon Q^2 e^{-4\psi} + \lambda e^{2\psi} = 0.$$

Using the classification in [LM13] for both cases, we obtain that our surface M is **minimal Lagrangian in** $\mathbb{CP}^2$ with $SU(3)$ group.
This completes the proof of the theorem. □

We give an example of totally real surfaces which are not totally geodesic in S^5, equipped with the nearly Sasakian structure .

Example 3 :
We consider the sphere S^5 equipped with the nearly Sasakian structure constructed before, and the surface M defined by the position vector :

$$P = \begin{pmatrix} \frac{\cos\left(\sqrt{2}(x-y)\right)+\cos\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)+\cos\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{3\sqrt{2}} \\ \frac{2\sin\left(\sqrt{2}(x-y)\right)+\left(1+\sqrt{3}\right)\sin\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)+\left(\sqrt{3}-1\right)\sin\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{6\sqrt{2}} \\ -\frac{2\cos\left(\sqrt{2}(x-y)\right)+\left(\sqrt{3}-1\right)\cos\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)-\left(1+\sqrt{3}\right)\cos\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{6\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{\sin\left(\sqrt{2}(x-y)\right)-\sin\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)+\sin\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{3\sqrt{2}} \\ -\frac{2\cos\left(\sqrt{2}(x-y)\right)-\left(1+\sqrt{3}\right)\cos\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)+\left(\sqrt{3}-1\right)\cos\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{6\sqrt{2}} \\ \frac{-2\sin\left(\sqrt{2}(x-y)\right)+\left(\sqrt{3}-1\right)\sin\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)+\left(1+\sqrt{3}\right)\sin\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{6\sqrt{2}} \end{pmatrix}.$$

We prove that this surface is totally real in S^5 but not totally geodesic. In fact, we find from the construction of this example that $a = 1$ and the second fundamental form h is given by

$$h(u,u) = \varphi u, \quad h(u,v) = -\varphi v, \quad h(v,v) = -\varphi u.$$

then M is minimal and no totally geodesique.
For all totally real surface M we have that

$$P = \frac{1}{\sqrt{2}}(x_1, x_2, x_3, 1, x_5, x_6, x_7), \quad N = -\frac{1}{\sqrt{2}}(x_1, x_2, x_3, -1, x_5, x_6, x_7),$$

then, we can write

$$N = \frac{1}{\sqrt{2}}(x_1, x_2, x_3, 1, x_5, x_6, x_7) = K - P,$$

where $K = \frac{2}{\sqrt{2}}(0, 0, 0, 1, 0, 0, 0)$. For all function f in the surface M we get

$$\begin{aligned} f_{uu} &= D_u u(f) \\ &= (\nabla_u u + h(u,u) + g(u,u)N - P)(f) \\ &= (a\varphi_u + (K - P) - P)(f) \\ &= \; P \times u(f) - 2P(f) + K \\ &= f \times f_u - 2f + K. \end{aligned}$$

with a same computing, we obtain

$$\begin{aligned} f_{uv} &= -f \times f - v, \\ f_{vv} &= -f \times f_u - ef + K, \end{aligned}$$

and

$$f_{uuu} = -3f_u + f \times k, \qquad f_{vvv} = -3f_v - f \times k.$$

By a direct calculation we obtain that our surface satisfies all these equations, then it is a totally real surface.

Chapter 3

Surfaces in the nearly cosymplectic 5-sphere

3.1 Nearly cosymplectic structure on S^5

In [Bla71] the author show how to induce a nearly cosymplectic structure on S^5. In the same way as last chapter, in order to do so, they look at S^5 as a hypersurface in S^6 equipped with its nearly Kaehler structure. We have

$$S^5 \hookrightarrow S^6 \hookrightarrow \mathbb{R}^7, \tag{3.1}$$

where S^6 is the unit sphere in $\mathbb{R}^7$ with its cross product $\times$ induced by the Cayley algebra. We denote by P the unit outer normal. It is well known that S^6 has a nearly Kaehler structure with respect to the induced metric, we now consider S^5 as a totally geodesic hypersurface of S^6 with the above nearly Kaehler structure, and with normal unit N such that the second fundamental form $\tilde{h}$ is vanished. Then we see that the induced structure on S^5 inherited from the nearly Kaehler structure is the nearly cosymplectic. In this case we have: for $P \in S^5$, V tangent vector to S^5 and N the normal vector of S^5 in S^6

$$\begin{aligned} P &= (x_1, x_2, x_3, 0, x_5, x_6, x_7), \\ N &= (0, 0, 0, 1, 0, 0, 0), \\ V &= (v_1, v_2, v_3, 0, v_5, v_6, v_7), \end{aligned}$$

and ξ and φ from the nearly Sasakian structure are respectively given by

$$\begin{aligned} \xi &= -P \times N, \\ \varphi(V) &= P \times V - \eta(V)N, \\ \eta(V) &= \langle \xi, V \rangle. \end{aligned}$$

With the cross product, we obtain that ξ is given by

$$\xi = (-x_5, -x_6, -x_7, 0, x_1, x_2, x_3). \tag{3.2}$$

3.1.1 Totally real surfaces

Definition 3.1.1 *Let M a surface of S^5 with nearly cosymplectic structure, we say that M is totally real submanifold of S^5 if for all $P \in M$ we have*

$$\xi \in N_pM \quad \text{and} \quad \varphi(T_pM) \subset N_pM, \tag{3.3}$$

where N_PM and T_PM denote respectively the normal space and the tangent space to M at the point P.

Let D be the standard Riemannian connection in $\mathbb{R}^7$. We denote the induced connections in S^6, S^5 and M by the previously mentioned immersions, respectively by $\tilde{\nabla}$, $\bar{\nabla}$ and ∇. Using the Gauss formula, we have

$$\begin{aligned} D_XY &= \tilde{\nabla}_XY - \langle X, Y\rangle P, \\ \tilde{\nabla}_XY &= \bar{\nabla}_XY, \\ \bar{\nabla}_XY &= \nabla_XY + h(X,Y), \end{aligned}$$

where P denotes the position vector of the immersion of M into $\mathbb{R}^7$ and h is the second fundamental forms of M, and X, Y are tangent vectors fields on M.

Remark 3.1.1 *There is not a normal component of S^5 on S^6.*

It then follows that

$$D_XY = \underbrace{\underbrace{\underbrace{\nabla_XY}_{T_pM} + \underbrace{h(X,Y)}_{N_pM}}_{T_pS^5=T_pS^6} - \underbrace{\langle X, Y\rangle P}_{N_pS^6}}_{T_p\mathbb{R}^7}. \tag{3.4}$$

Remarks : As the same as the nearly Sasakian case, if ξ is normal then the surface M is not always anti-invariant, i.e. $\varphi(T_PM) \subset N_PM$. In the following subsection we show it.

3.1.2 2-sphere in the nearly cosymplectic 5-sphere

[BS16]
We look at S^2, which we parametrize in the usual way by

$$(cos\theta\cos\psi,\ \sin\theta\cos\psi,\ \sin\psi).$$

We define a 1-parameter family of immersions in the nearly cosymplectic $S^5 \subset S^6(1) \subset \mathbb{R}^7$ by

$$\begin{aligned}P = (&\cos a\cos\theta\cos\psi,\ \cos a\sin\theta\cos\psi,\ \cos a\sin\psi, 0,\\ &\sin a\cos\theta\cos\psi,\ \sin a\sin\theta\cos\psi,\ \sin a\sin\psi),\end{aligned}$$

where a is an arbitrary constant. We also get that

$$\begin{aligned}N &= (0,0,0,1,0,0,0),\\ \xi &= (\sin a\cos\theta\cos\psi,\ \sin a\sin\theta\cos\psi,\ \sin a\sin\psi, 0,\\ &\quad -\cos a\cos\theta\cos\psi,\ -\cos a\sin\theta\cos\psi,\ -\cos a\sin\psi).\end{aligned}$$

In the case that ξ is normal, we get that for all $V \in T_P M$

$$\eta(V) = \langle \xi, V\rangle = 0 \quad \text{and} \quad \varphi(V) = P \times V.$$

The tangent vectors of S^2 are

$$\frac{\partial P}{\partial\theta} = \begin{pmatrix} -\cos a\sin\theta\cos\psi \\ \cos a\cos\theta a\cos\psi \\ 0 \\ 0 \\ -\sin a\sin\theta\cos\psi \\ \sin a\cos\theta\cos\psi \\ 0 \end{pmatrix}, \quad \frac{\partial P}{\partial\psi} = \begin{pmatrix} -\cos a\cos\theta\sin\psi \\ -\cos a\sin\theta\sin\psi \\ \cos a\cos\psi \\ 0 \\ -\sin a\cos\theta\sin\psi \\ -\sin a\sin\theta\sin\psi \\ \sin a\cos\psi \end{pmatrix}.$$

With the cross product computations, we get that :

$$\varphi(\frac{\partial P}{\partial\theta}) = P \times \frac{\partial P}{\partial\theta} = \frac{1}{2}\begin{pmatrix} -2\sin a\sin\theta\cos\psi - \cos 2a\cos\theta\sin 2\psi \\ 2\cos\psi(\sin a\cos\theta - \cos 2a\sin\theta\sin\psi) \\ 2\cos 2a\cos^2\psi \\ 0 \\ 2\cos a\cos\psi(2\sin a\cos\theta\sin\psi + \sin\theta) \\ -2\cos a\cos\psi(\cos\theta - 2\sin a\sin\theta\sin\psi) \\ -4\sin a\cos a\cos^2\psi \end{pmatrix}$$

$$\varphi(\frac{\partial P}{\partial\psi}) = P\times\frac{\partial P}{\partial\psi} = \frac{1}{2}\begin{pmatrix} 2\cos 2a\sin\theta - 2\sin a\cos\theta\sin\psi \\ -2\cos 2a\cos\theta - 2\sin a\sin\theta\sin\psi \\ 2\sin a\cos\psi \\ 0 \\ 2\cos a\,(\cos\theta\sin\psi - 2\sin a\sin\theta) \\ 2\cos a\,(2\sin a\cos\theta + \sin\theta\sin\psi) \\ -2\cos a\cos\psi \end{pmatrix}.$$

from which it follows that

$$\begin{aligned}
&\langle\varphi(\frac{\partial P}{\partial\theta}),\xi\rangle = 0, \quad \langle\varphi(\frac{\partial P}{\partial\psi}),\xi\rangle = 0, \quad \langle\varphi(\frac{\partial P}{\partial\theta}),\varphi(\frac{\partial P}{\partial\psi})\rangle = 0,\\
&\langle\varphi(\frac{\partial P}{\partial\theta}),\frac{\partial P}{\partial\theta}\rangle = 0, \quad \langle\varphi(\frac{\partial P}{\partial\psi}),\frac{\partial P}{\partial\psi}\rangle = 0,\\
&\langle\varphi(\frac{\partial P}{\partial\theta}),\frac{\partial P}{\partial\psi}\rangle = \frac{1}{2}\cos a(2\cos 2a - 1)\cos\psi,\\
&\langle\varphi(\frac{\partial P}{\partial\psi}),\frac{\partial P}{\partial\theta}\rangle = -\frac{1}{2}\cos a(2\cos 2a - 1)\cos\psi.
\end{aligned}$$

So we see that for all of these examples ξ is a normal vector to the immersion. However $\varphi(T_pM) \nsubseteq N_pM$, unless $a = \frac{\pi}{2} + k\pi$, or $a = \pm\frac{\pi}{3} + k\pi$, where $k \in \mathbb{Z}$.

Therefore, in the nearly cosymplectic case we define the notion of a totally real submanifold by demanding that ξ is normal and $\varphi(T_PM) \subset N_PM$. So for the immersions in our family which are totally real, i.e. when $a = \frac{\pi}{2} + k\pi$, or $a = \pm\frac{\pi}{3} + k\pi$, we will now compute the second fundamental form. Note that in this case, $\{\varphi(\frac{\partial P}{\partial\theta}), \varphi(\frac{\partial P}{\partial\psi}), \xi\}$ is a frame of normal space consisting of the mutually orthogonal vectors.

Proposition 3.1.1 *A totally real 2-sphere in an nearly cosymplectic 5-sphere is always totally geodesic.*

Proof
Therefore the second fundamental form of the immersion is given by

$$\begin{aligned}
h(\frac{\partial P}{\partial\theta},\frac{\partial P}{\partial\theta}) &= \frac{\langle\frac{\partial^2 P}{\partial\theta^2},\varphi(\frac{\partial P}{\partial\theta})\rangle}{\left|\frac{\partial P}{\partial\theta}\right|^2}\varphi(\frac{\partial P}{\partial\theta}) + \frac{\langle\frac{\partial^2 P}{\partial\theta^2},\varphi(\frac{\partial P}{\partial\psi})\rangle}{\left|\frac{\partial P}{\partial\psi}\right|^2}\varphi(\frac{\partial P}{\partial\psi}) + \langle\frac{\partial^2 P}{\partial\theta^2},\xi\rangle\xi,\\
h(\frac{\partial P}{\partial\psi},\frac{\partial P}{\partial\psi}) &= \frac{\langle\frac{\partial^2 P}{\partial\psi^2},\varphi(\frac{\partial P}{\partial\theta})\rangle}{\left|\frac{\partial P}{\partial\theta}\right|^2}\varphi(\frac{\partial P}{\partial\theta}) + \frac{\langle\frac{\partial^2 P}{\partial\psi^2},\varphi(\frac{\partial P}{\partial\psi})\rangle}{\left|\frac{\partial P}{\partial\psi}\right|^2}\varphi(\frac{\partial P}{\partial\psi}) + \langle\frac{\partial^2 P}{\partial\psi^2},\xi\rangle\xi,\\
h(\frac{\partial P}{\partial\theta},\frac{\partial P}{\partial\psi}) &= \frac{\langle\frac{\partial^2 P}{\partial\theta\partial\psi},\varphi(\frac{\partial P}{\partial\theta})\rangle}{\left|\frac{\partial P}{\partial\theta}\right|^2}\varphi(\frac{\partial P}{\partial\theta}) + \frac{\langle\frac{\partial^2 P}{\partial\theta\partial\psi},\varphi(\frac{\partial P}{\partial\psi})\rangle}{\left|\frac{\partial P}{\partial\psi}\right|^2}\varphi(\frac{\partial P}{\partial\psi}) + \langle\frac{\partial^2 P}{\partial\theta\partial\psi},\xi\rangle\xi.
\end{aligned}$$

Before fixing a, straightforward computations, show that

$$\langle \frac{\partial^2 P}{\partial \theta^2}, \varphi(\frac{\partial P}{\partial \psi})\rangle = \langle \frac{\partial^2 P}{\partial \theta^2}, \xi\rangle = 0,$$
$$\langle \frac{\partial^2 P}{\partial \psi^2}, \varphi(\frac{\partial P}{\partial \theta})\rangle = \langle \frac{\partial^2 P}{\partial \psi^2}, \varphi(\frac{\partial P}{\partial \psi})\rangle = \langle \frac{\partial^2 P}{\partial \psi^2}, \xi\rangle = 0,$$
$$\langle \frac{\partial^2 P}{\partial \psi \partial \theta}, \varphi(\frac{\partial P}{\partial \theta})\rangle = \langle \frac{\partial^2 P}{\partial \psi \partial \theta}, \xi\rangle = 0,$$

and

$$\langle \frac{\partial^2 P}{\partial \theta^2}, \varphi(\frac{\partial P}{\partial \theta})\rangle = \frac{1}{2}\cos a(2\cos 2a - 1)\cos^2\psi \sin\psi,$$
$$\langle \frac{\partial^2 P}{\partial \psi \partial \theta}, \varphi(\frac{\partial P}{\partial \psi})\rangle = -\frac{1}{2}\cos a(2\cos 2a - 1)\sin\psi.$$

Then, if M is totally real we find

$$h(\frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial \theta}) = h(\frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial \psi}) = h(\frac{\partial P}{\partial \psi}, \frac{\partial P}{\partial \psi}) = 0.$$

Therefore we obtain our totally real surfaces S^2 which are totally geodesic in the nearly cosymplectic S^5. □

3.2 Surfaces in the nearly cosymplectic 5-sphere

[BS16]
In this subsection, M will always denote a totally real surface of the 5-dimension nearly cosymplectic sphere S^5 which we consider as a subset of $\mathbb{R}^7$. The structure of M is, as previously, built with the immersions :

$$M \hookrightarrow S^5 \hookrightarrow S^6 \hookrightarrow \mathbb{R}^7. \quad (3.5)$$

We now will give the important theorem of this section and his proof.

Theorem 3.2.1 *A totally real surface of the nearly cosymplectic S^5 is always minimal.*

To prove this theorem, we divide it into three lemmas.

Lemma 3.2.1 *Let M be a totally real surface of the 5-dimensional nearly cosymplectic sphere and let $\{u, v\}$ be a local orthonormal basis of tangent vector fields on*

M. Then $\{\xi, \varphi u, \varphi v\}$ is an orthonormal basis of the normal space N_pM. Moreover a basis of $\mathbb{R}^7$ is given by

$$\{\underbrace{u, \quad v}_{T_pM}, \underbrace{\xi \quad , \quad \varphi u \quad , \quad \varphi v}_{N_pM}, \underbrace{-N}_{N_pS^5}, \underbrace{P}_{N_pS^6}\},$$

where we denote $\varphi V := \varphi(V)$.

Proof : We have $\xi = -P \times N$ and $\varphi u = P \times u - \eta(u)N$. As $\eta(u) = \langle u, \xi \rangle = 0$, it follows

$$\xi = -P \times N, \quad \varphi u = P \times u, \quad \varphi v = P \times v.$$

Using the properties of the cross product we get :

$$\begin{aligned}
\langle \xi, \varphi u \rangle &= \langle -P \times N, P \times u \rangle = -\langle P, P \rangle \langle N, u \rangle + \langle P, N \rangle \langle P, u \rangle, \\
\langle \xi, \varphi v \rangle &= \langle -P \times N, P \times v \rangle = -\langle P, P \rangle \langle N, v \rangle + \langle P, N \rangle \langle P, v \rangle, \\
\langle \varphi u, \varphi v \rangle &= \langle P \times u, P \times v \rangle = \langle P, P \rangle \langle u, v \rangle - \langle P, u \rangle \langle P, v \rangle.
\end{aligned}$$

or $u, v \in T_PM$ and $N \in N_PM$ in S^5, then $\langle N, u \rangle = 0$ and $\langle N, v \rangle = 0$, with the same reasoning we get $\langle P, u \rangle = \langle P, v \rangle = 0$. Then

$$\begin{aligned}
\langle \xi, \varphi u \rangle &= 0, \\
\langle \xi, \varphi v \rangle &= 0, \\
\langle \varphi u, \varphi v \rangle &= 0.
\end{aligned}$$

thus $\xi \perp \varphi u$, $\xi \perp \varphi v$ and $\varphi u \perp \varphi v$ and $\{\xi, \varphi u, \varphi v\}$ is an orthonormal basis of the normal space N_pM.
In order to be able to use the multiplication table for the cross product in $\mathbb{R}^7$ (1.15), we first remark that $\xi = -P \times N = u \times v$ and therefore

$$\{u = e_2,\ v = e_4, \quad P \times u = e_3, \quad P \times v = e_5, \quad \xi = e_6, \quad -N = e_7, \quad P = e_1\}$$

is a G_2 basis along the surface. □

Lemma 3.2.2 *On the surface M, for any tangent vector fields X, Y we have*

$$h(X, Y) \perp \xi.$$

Moreover, we get in each point P in M:

$$\begin{aligned}
\langle h(u, v), \varphi v \rangle &= \langle h(v, v), \varphi u \rangle, \\
\langle h(u, v), \varphi u \rangle &= \langle h(u, u), \varphi v \rangle.
\end{aligned}$$

Proof We know that ξ is in the normal space of M in S^5, from the lemma 3.2.1, we have that $\{u, v, \xi, \varphi u, \varphi v, -N, P\}$ is an orthonormal basis of $\mathbb{R}^7$, then $\langle u, \xi\rangle = \langle v, \xi\rangle = 0$ and we have

$$\begin{aligned}
D_v\langle u, \xi\rangle &= 0\\
&= \langle D_v u, P \times N\rangle + \langle u, D_v P \times N\rangle + \langle u, P \times D_v N\rangle = 0\\
&= \langle \nabla_v u, P \times N\rangle + \langle h(u, v), P \times N\rangle + \langle u, v \times N\rangle + \langle u, P \times v\rangle = 0\\
&= \langle h(u, v), P \times N\rangle + \langle u, v \times N\rangle = 0\\
&= \underbrace{\langle h(u, v), P \times N\rangle}_{Symmetric} + \underbrace{\langle u \times v, N\rangle}_{antisymmetric} = 0\\
&\Rightarrow \langle h(u, v), P \times N\rangle = 0 \quad and \quad \langle u \times v, N\rangle = 0,\\
&\Rightarrow h(u, v) \perp \xi.\\
D_u\langle u, \xi\rangle &= 0\\
&= \langle D_u u, P \times N\rangle + \langle u, D_u P \times N\rangle + \langle u, P \times D_u N\rangle = 0\\
&= \langle \nabla_u u + h(u, u) - P, P \times N\rangle + \langle u, u \times N\rangle + \langle u, P \times u\rangle = 0\\
&= \langle h(u, u), P \times N\rangle,\\
&\Rightarrow h(u, u) \perp \xi.\\
D_v\langle v, \xi\rangle &= 0 \quad \text{in the same}\\
&\Rightarrow h(v, v) \perp \xi.
\end{aligned}$$

Therefore $h(X, Y) \perp \xi$, for any tangent vector fields X and Y on M. Next using that the immersion is anti-invariant, i.e. $\langle u, \varphi v\rangle = 0$ and $\langle v, \varphi u\rangle = 0$, we deduce that:

$$\begin{aligned}
D_v\langle u, \varphi v\rangle &= 0\\
&= \langle D_v u, P \times v\rangle + \langle u, D_v P \times v\rangle + \langle u, P \times D_v v\rangle = 0\\
&= \langle \nabla_v u, P \times v\rangle + \langle h(v, u), P \times v\rangle + \langle u, v \times v\rangle + \langle u \times P, D_v v\rangle = 0\\
&= \langle h(u, v), P \times v\rangle - \langle h(v, v) - P, P \times u\rangle = 0,\\
&\Rightarrow \langle h(u, v), P \times v\rangle = \langle h(v, v), P \times u\rangle.\\
D_u\langle v, \varphi u\rangle &= 0\\
&= \langle D_u v, P \times u\rangle + \langle v, D_u P \times u\rangle + \langle v, P \times D_u u\rangle = 0\\
&= \langle \nabla_u v, P \times u\rangle + \langle h(v, u), P \times u\rangle + \langle v, u \times u\rangle + \langle v \times P, D_u u\rangle = 0\\
&= \langle h(u, v), P \times u\rangle - \langle h(u, u) - P, P \times v\rangle = 0,\\
&\Rightarrow \langle h(u, v), P \times u\rangle = \langle h(u, u), P \times v\rangle.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.2.3 *We have*

$$\nabla_u u = \alpha\, v, \quad \nabla_v u = \beta\, v, \quad \nabla_v v = -\beta\, u, \quad \nabla_u v = -\alpha\, u,$$

where α and β are local functions on M.

Proof We have $\langle u, u\rangle = \langle v, v\rangle = 1$ and $\langle u, v\rangle = 0$. From

$$D_u\langle u, u\rangle = 2\langle D_u u, u\rangle = 0$$

we have

$$\langle \nabla_u u + h(u, u) - P, u\rangle = 0$$

i.e $\langle \nabla_u u, u\rangle = 0$. Then

$$D_u\langle u, u\rangle = 0 \Rightarrow \nabla_u u = \alpha\, v, \qquad \text{where } \alpha \in \mathcal{F}(M).$$

In the same way, we get that there exist local functions α', β and β' on M such that :

$$\begin{aligned} D_v\langle u, u\rangle = 0 &\Rightarrow \nabla_v u = \beta\, v,\\ D_v\langle v, v\rangle = 0, &\Rightarrow \nabla_v v = \alpha'\, u,\\ D_u\langle v, v\rangle = 0, &\Rightarrow \nabla_u v = \beta'\, u.\end{aligned}$$

Finally from $D_v\langle u, v\rangle = 0$

$$\begin{aligned} D_v\langle u, v\rangle &= 0\\ &= \langle \nabla_v u + h(u, v), v\rangle + \langle u, \nabla_v v + h(v, v) - P\rangle = 0\\ &= \langle \nabla_v u, v\rangle + \langle u, \nabla_v v\rangle = 0\\ &= \langle \beta v, v\rangle + \langle u, \alpha' u\rangle = 0,\end{aligned}$$

we obtain that $\alpha' = -\beta$. In same from $D_u\langle u, v\rangle = 0$, we get $\beta' = -\alpha$.
This completes the proof of the lemma. □

Now we give the proof of the theorem 3.2.1.
Proof From lemmas 3.2.2 and 3.2.3, we have that $h \perp \xi$, therefore we get :

$$h(u, u) = a_1\varphi u + a_2\varphi v, \quad h(u, v) = b_1\varphi u + b_2\varphi v, \quad h(v, v) = c_1\varphi u + c_2\varphi v, \quad (3.6)$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are local functions on M.
As $\xi = -P \times N = u \times v$, we can compute the covariant derivative of ξ in two different ways. Indeed, we have that

$$D_u\xi = D_u(-P \times N) = D_u(u \times v).$$

So, we see that

$$\begin{aligned} -D_u(P \times N) &= -D_u P \times N = -u \times N \\ &= -P \times v, \end{aligned}$$

and

$$\begin{aligned} D_u(u \times v) &= D_u u \times v + u \times D_u v \\ &= (\nabla_u u + h(u,u) - P) \times v + u \times (\nabla_u v + h(u,v)) \\ &= (\alpha v + a_1 P \times u + a_2 P \times v - P) \times v + u \times (-\alpha u + b_1 P \times u + b_2 P \times v) \\ &= (-a_1 - b_2)N + (-a_2 + b_1)P - P \times v. \end{aligned}$$

From these equalities, we get $(-a_1 - b_2)N + (-a_2 + b_1)P = 0$, so

$$\begin{cases} b_1 = a_2, \\ b_2 = -a_1. \end{cases}$$

In a similar way, we obtain :

$$D_v \xi = D_v(-P \times N) = D_v(u \times v) \Rightarrow \begin{cases} c_1 = b_2, \\ c_2 = -b_1. \end{cases}$$

Finally we get

$$\begin{cases} c_1 = b_2 = -a_1, \\ c_2 = -b_1 = -a_2. \end{cases}$$

Therefore we have $h(u,u) = -h(v,v)$ i.e M is a minimal surface. □

3.2.1 Minimal surfaces in the nearly cosymplectic 5-sphere and minimal Lagrangian submanifolds

In this section we give the relation of our minimal surface proved before, and the minimal Lagrangian surface, for this we must go through several changes of variables [BS16].

As indicated in the following proposition, we can further improve our choice of basis.

Proposition 3.2.1 *Let M be a totally real surface of the nearly cosymplectic sphere S^5. Then if necessary by restricting to an open dense subset there exists a local orthonormal frame $\{u, v\}$ of M at each point P of M such that*

$$\nabla_u u = \alpha\, v, \quad \nabla_v u = \beta\, v, \quad \nabla_v v = -\beta\, u, \quad \nabla_u v = -\alpha\, u,$$

$$h(u,u) = aP \times u, \quad h(u,v) = -aP \times v,$$

where α, β are the functions defined before and a is a function on this open dense subset of M satisfying :

$$\begin{cases} v(\alpha) - u(\beta) + 2a^2 - \alpha^2 - \beta^2 - 1 = 0, \\ v(\beta) + u(\alpha) = 0, \\ v(a) = 3\alpha a, \\ u(a) = -3\beta a. \end{cases} \tag{3.7}$$

First, note that if the immersion is totally geodesic on an open part, then we can take $a_1 = a_2 = 0$ (where a_1, a_2 are defined in (3.6)) and it follows from the Gauss equation that the first equations in (3.7) is satisfied.

Else, by restricting to an open dense subset, we may assume that $a_1^2 + a_2^2 \neq 0$.

Lemma 3.2.4 *Let M be a totally real surface. In a neighborhood of a non totally geodesic point of M, there exist a local function a such that the second fundamental form h can be written as*

$$h(u,u) = a\varphi u, \quad h(u,v) = -a\varphi v, \quad h(v,v) = -a\varphi u,$$

where $\{u, v\}$ is the frame of M at each point P of the neighborhood.

Proof Using a rotation of the orthonormal frame, we write

$$\begin{cases} U = \cos\theta\, u - \sin\theta\, v, \\ V = \sin\theta\, u + \cos\theta\, v, \end{cases}$$

where θ is a differentiable function. Then we have

$$h(U,U) = A_1 P \times U + A_2 P \times V.$$

We now want to find a function θ such that A_2 vanishes.
First we compute $h(U,U)$:

$$\begin{aligned} h(U,U) &= h(\cos\theta\, u - \sin\theta\, v, \cos\theta\, u - \sin\theta\, v) \\ &= \cos^2\theta\, h(u,u) + \sin^2\theta\, h(v,v) - 2\cos\theta\, \sin\theta\, h(u,v) \end{aligned}$$

As
$h(u,u) = a_1\varphi u + a_2\varphi v, \quad h(u,v) = a_2\varphi u - a_1\varphi v, \quad h(v,v) = -a_1\varphi u - a_2\varphi v,$
we get :

$$h(U,U) = (a_1(\cos^2\theta - \sin^2\theta) - 2a_2\cos\theta\sin\theta)P \times u$$

$$+ (a_2(\cos^2\theta - \sin^2\theta) + 2a_1 \cos\theta \sin\theta)P \times v.$$

We have

$$P \times V = \sin\theta\, P \times u + \cos\theta\, P \times v$$

and $A_2 = \langle h(U,U), P \times V\rangle = 0$, or

$$\begin{aligned} A_2 =& \langle (a_1(\cos^2\theta - \sin^2\theta) - 2a_2 \cos\theta \sin\theta)P \times u \\ & + (a_2(\cos^2\theta - \sin^2\theta) + 2a_1 \cos\theta \sin\theta)P \times v, \sin\theta\, P \times u + \cos\theta\, P \times v\rangle. \\ =& (a_1(\cos^2\theta - \sin^2\theta) - 2a_2 \cos\theta \sin\theta) \sin\theta \\ & + (a_2(\cos^2\theta - \sin^2\theta) + 2a_1 \cos\theta \sin\theta) \cos\theta \\ =& a_1 \sin\theta(3\cos^2\theta - \sin^2\theta) + a_2 \cos\theta(\cos^2\theta - 3\sin^2\theta), \end{aligned}$$

then

$$A_2 = a_1 \sin 3\theta + a_2 \cos 3\theta = 0.$$

As by assumption a_1 and a_2 do not both vanish, we see that it is sufficient to take θ such that

$$\cos 3\theta = -\frac{a_1}{a_1^2 + a_2^2} \quad \text{and} \quad \sin 3\theta = \frac{a_2}{a_1^2 + a_2^2}$$

and take $a = A_1$. □

Using the previous lemma, we complete the proof of proposition 3.2.1.

Proof Using the frame vectors along our surface, we can compute the curvature tensor of $\mathbb{R}^7$. So, we take X and Y tangent vector fields to the surface and for Z we take any vector field belonging to our frame of $\mathbb{R}^7$. As the connection on $\mathbb{R}^7$ is flat, we have that

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z = 0.$$

Using lemma 3.2.3 and 3.2.4 and Gauss formula, straightforward computations in the nearly Sasakian case show that

$$\begin{aligned} R(u,v)P &= D_u D_v P - D_v D_u P - D_{[u,v]}P \\ &= D_u v - D_v u - D_{-\alpha u - \beta v}P \\ &= \nabla_u v + h(u,v) - (\nabla_v u + h(u,v)) - (\alpha u - \beta v) \\ &= -\alpha u - aP \times v - \beta v + aP \times +\alpha u + \beta v \\ &= 0. \end{aligned}$$

At the same we get :

$$R(u,v)u = (u(\beta) - v(\alpha) - 2a^2 + \alpha^2 + \beta^2 + 1)v$$

$$\begin{aligned}
&\qquad + (3\alpha a - v(a))P \times u + (-3\beta a - u(a))P \times v,\\
R(u,v)v &= (-u(\beta) + v(\alpha) + 2a^2 - \alpha^2 - \beta^2 - 1)u\\
&\qquad + (-3\beta a - u(a))P \times u + (-3\alpha a + v(a))P \times v,\\
R(u,v)(P \times u) &= (-3\alpha a + v(a))u + (3\beta a + u(a))v\\
&\qquad + (u(\beta) - v(\alpha) - 2a^2 + \alpha^2 + \beta^2 + 1)P \times v,\\
R(u,v)(P \times v) &= (3\beta a + u(a))u + (3\alpha a - v(a))v\\
&\qquad + (-u(\beta) + v(a) + 2a^2 - \alpha^2 - \beta^2 - 1)P \times u,\\
R(u,v)(P \times N) &= 0.
\end{aligned}$$

We deduce that :

$$\begin{cases} v(\alpha) - u(\beta) + 2a^2 - \alpha^2 - \beta^2 - 1 = 0,\\ v(a) = 3\alpha a,\\ u(a) = -3\beta a.\end{cases}$$

Note that, as we are working with the Levi Civita connection, we have that

$$[v,u](f) = v(u(f)) - u(v(f)) = (\nabla_v u - \nabla_u v)(f),$$

where f is an arbitrary function. With these two ways of computing the Lie bracket for the function a, we deduce that

$$-4\beta v(a) - 4\alpha u(a) - 3v(\beta)a - 3u(\alpha)a = 0.$$

From this equation, we obtain $v(\beta) + u(\alpha) = 0$.
This completes the proof of the proposition. □

In order to relate our surfaces to minimal Lagrangian surfaces in the complex projective space, we will use this proposition in order to introduce suitable coordinates on the surface.

Theorem 3.2.2 *Let M be totally real surface of the 5-dimension sphere S^5 with nearly cosymplectic structure, then around each non totally geodesic point, M locally corresponds to a minimal Lagrangian in the complex projective space $\mathbb{CP}^2$ with the group $SU(3)$.*

Proof As in the nearly Sasakian case, we have the differential system (3.7) :

$$\begin{cases} v(\alpha) - u(\beta) + 2a^2 - \alpha^2 - \beta^2 - 1 = 0,\\ v(\beta) + u(\alpha) = 0,\\ v(a) = 3\alpha a,\\ u(a) = -3\beta a.\end{cases}$$

As we are working on a neighborhood of a non totally geodesic point, we have that $a \neq 0$. We define a function ρ on M by $\rho = a^{-\frac{1}{3}}$. It then follows that

$$\begin{cases} u(\ln \rho) = \beta, \\ v(\ln \rho) = -\alpha, \end{cases} \quad \text{or equivalently} \quad \begin{cases} u(\rho) = \rho\beta, \\ v(\rho) = -\rho\alpha. \end{cases}$$

Computing now $\nabla_{\rho v}\rho u$ and $\nabla_{\rho u}\rho v$, we get :

$$\begin{aligned} \nabla_{\rho v}\rho u &= \rho\nabla_v \rho u \\ &= \rho(v(\rho)u + \rho\nabla_v u) \\ &= \rho\, v(\rho)u + \rho^2\beta v \\ &= -\rho^2\alpha u + \rho^2\beta v, \\ \nabla_{\rho u}\rho v &= \rho\, u(\rho)v - \rho^2\alpha u \\ &= \rho^2\beta v - \rho^2\alpha u. \end{aligned}$$

Hence

$$[\rho u, \rho v] = 0.$$

This implies that there exist local coordinates x and y on M such that

$$\frac{\partial}{\partial x} = \rho u \quad \text{and} \quad \frac{\partial}{\partial y} = \rho v.$$

It now follows that

$$u = \frac{1}{\rho}\frac{\partial}{\partial x} \quad \text{and} \quad v = \frac{1}{\rho}\frac{\partial}{\partial y}, \tag{3.8}$$

$$\alpha = -\frac{1}{\rho^2}\frac{\partial}{\partial y}(\rho) \quad \text{and} \quad \beta = \frac{1}{\rho^2}\frac{\partial}{\partial x}(\rho). \tag{3.9}$$

We compute $v(\alpha)$ and $u(\beta)$:

$$\begin{aligned} v(\alpha) &= \frac{1}{\rho}\frac{\partial}{\partial y}(\alpha) = \frac{2}{\rho^4}\left(\frac{\partial}{\partial y}(\rho)\right)^2 - \frac{1}{\rho^3}\frac{\partial^2}{\partial y^2}(\rho), \\ u(\beta) &= \frac{1}{\rho}\frac{\partial}{\partial x}(\beta) = \frac{-2}{\rho^4}\left(\frac{\partial}{\partial x}(\rho)\right)^2 + \frac{1}{\rho^3}\frac{\partial^2}{\partial x^2}(\rho). \end{aligned}$$

Replacing it all in the equation $v(\alpha) - u(\beta) - \alpha^2 - \beta^2 + 2a^2 - 1 = 0$,

$$\begin{aligned} v(\alpha) - u(\beta) &- \alpha^2 - \beta^2 + 2a^2 - 1 \\ &= \frac{2}{\rho^4}\left(\frac{\partial}{\partial y}(\rho)\right)^2 - \frac{1}{\rho^3}\frac{\partial^2}{\partial y^2}(\rho) + \frac{2}{\rho^4}\left(\frac{\partial}{\partial x}(\rho)\right)^2 - \frac{1}{\rho^3}\frac{\partial^2}{\partial x^2}(\rho) \end{aligned}$$

$$-\left(-\frac{1}{\rho^2}\frac{\partial}{\partial y}(\rho)\right)^2-\left(\frac{1}{\rho^2}\frac{\partial}{\partial x}(\rho)\right)^2-2\left(\rho^3\right)^2-1,$$

it reduces to the following differential equation :

$$-\rho\Delta\rho+\left(\frac{\partial}{\partial x}(\rho)\right)^2+\left(\frac{\partial}{\partial y}(\rho)\right)^2+2\rho^{-2}-\rho^4=0. \tag{3.10}$$

where Δ is the Laplace operator (i.e. $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$).

If we write now $\rho=e^{\psi}$, we get :

$$\begin{cases}\frac{\partial}{\partial x}(\rho)=\frac{\partial}{\partial x}(\psi)e^{\psi},\\ \frac{\partial}{\partial y}(\rho)=\frac{\partial}{\partial y}(\psi)e^{\psi},\end{cases}\quad\text{and}\quad\begin{cases}\frac{\partial^2}{\partial x^2}(\rho)=\left(\frac{\partial}{\partial x}(\psi)\right)^2e^{\psi}+\frac{\partial^2}{\partial x^2}(\psi)e^{\psi},\\ \frac{\partial^2}{\partial y^2}(\rho)=\left(\frac{\partial}{\partial y}(\psi)\right)^2e^{\psi}+\frac{\partial^2}{\partial y^2}(\psi)e^{\psi},\end{cases}$$

and the differential equation (3.10) become :

$$\begin{aligned}-\rho\Delta\rho+\left(\frac{\partial}{\partial x}(\rho)\right)^2+\left(\frac{\partial}{\partial y}(\rho)\right)^2+2\rho^{-2}-2\rho^4&\\ =-e^{\psi}\Delta(e^{\psi})+\left(\frac{\partial}{\partial x}(e^{\psi})\right)^2+\left(\frac{\partial}{\partial y}(e^{\psi})\right)^2+2e^{-2\psi}-e^{4\psi},&\end{aligned}$$

it reduces to :

$$2\Delta\psi-4e^{-4\psi}+2e^{2\psi}=0.$$

In this last equation, applying the change of coordinates $\psi=\gamma+d$, where d is constant gives :

$$2\Delta\gamma-4e^{-4d}e^{-4\gamma}+2e^{2d}e^{2\gamma}=0.$$

For $d=-\frac{1}{2}\ln 2$, we get the final equation :

$$2\Delta\gamma-16\,e^{-4\gamma}+e^{2\gamma}=0, \tag{3.11}$$

with $\gamma=-\frac{1}{3}\ln a+\frac{1}{2}\ln 2$.

The differential equations (3.11) is elliptic equations of Ţiţeica type of the form :

$$2\Delta\psi+\epsilon Q^2e^{-4\psi}+\lambda e^{2\psi}=0.$$

Using the classification in [LM13] for both cases, we obtain that our surface M is **minimal Lagrangian in** $\mathbb{CP}^2$ with $SU(3)$ group.
This completes the proof of the theorem. □

We give an example of totally real surfaces which are not totally geodesic in S^5, equipped with the nearly cosymplectic structure .

Example 3 :
We consider the sphere S^5 equipped with the nearly cosymplectic structure constructed before, and the surface M defined by the position vector :

$$P = \begin{pmatrix} \frac{1}{3}\left(2\cos\left(\sqrt{\frac{3}{2}}x\right)\cos\left(\frac{y}{\sqrt{2}}\right)+\cos\left(\sqrt{2}y\right)\right) \\ \sqrt{\frac{2}{3}}\sin\left(\sqrt{\frac{3}{2}}x\right)\cos\left(\frac{y}{\sqrt{2}}\right) \\ \frac{1}{3}\sqrt{2}\left(\cos\left(\sqrt{2}y\right)-\cos\left(\sqrt{\frac{3}{2}}x\right)\cos\left(\frac{y}{\sqrt{2}}\right)\right) \\ 0 \\ \frac{1}{3}\left(\sin\left(\sqrt{2}y\right)-2\cos\left(\sqrt{\frac{3}{2}}x\right)\sin\left(\frac{y}{\sqrt{2}}\right)\right) \\ -\sqrt{\frac{2}{3}}\sin\left(\sqrt{\frac{3}{2}}x\right)\sin\left(\frac{y}{\sqrt{2}}\right) \\ \frac{1}{3}\sqrt{2}\sin\left(\frac{y}{\sqrt{2}}\right)\left(\cos\left(\sqrt{\frac{3}{2}}x\right)+2\cos\left(\frac{y}{\sqrt{2}}\right)\right) \end{pmatrix}.$$

We prove that this surface is totally real in S^5 but not totally geodesic. In fact, we find from the construction of this example that $a = \frac{1}{\sqrt{2}}$, $\rho = 2^{1/6}$, and the second fundamental form h is given by

$$h(u,u) = \frac{1}{\sqrt{2}}\varphi u, \quad h(u,v) = -\frac{1}{\sqrt{2}}\varphi v, \quad h(v,v) = -\frac{1}{\sqrt{2}}\varphi u.$$

then M is minimal and no totally geodesique.
For all totally real surface M we have that

$$P = (x_1, x_2, x_3, 0, x_5, x_6, x_7), \quad N = (0,0,0,1,0,0,0),$$

For all function f in the surface M we have

$$\begin{aligned} f_{uu} &= D_u u(f) \\ &= (\nabla_u u + h(u,u) - P)(f) \\ &= (a\varphi_u - P)(f) \\ &= \frac{1}{\sqrt{2}}P \times u(f) - P(f) \\ &= \frac{1}{\sqrt{2}}f \times f_u - f. \end{aligned}$$

with a same computing, we obtain

$$f_{uv} = -\frac{1}{\sqrt{2}}f \times f_v,$$

$$f_{vv} = -\frac{1}{\sqrt{2}} f \times f_u - f,$$

and

$$f_{uuu} = -\frac{3}{2} f_u, \qquad f_{vvv} = -\frac{1}{\sqrt{2}} f_u \times f_v - \frac{1}{2} f_v.$$

By a direct calculation we obtain that our surface satisfies all these equations, then it is a totally real surface.

Part II

Four-dimensional locally strongly convex homogeneous affine hypersurfaces

Chapter 4

Preliminaries

We begin by recalling some fundamental definitions and proprieties of affine geometry, and affine hypersurface. We cant refer the reader for this chapter to the Nomizu-Sasaki books [NS94], and Li, Simon, Zhao, Hu books [LSZH15].
Most of the materiel here can be recalled in the first chapter - part I of this thesis, or from the standard references on geometry and on differential geometry; basically we shall suppose that the reader is familiar whit the terminology in manifold theory, and also the affine geometry in the euclidean spaces. [Bla76, GHL80, KN69, YK84].

4.1 Affine space

Definition 4.1.1 *Let V be a real n-dimension vector space. A non-empty set Ω it said to be an* ***affine space*** *associated to V if there exist a mapping*

$$\Omega \times \Omega \to V$$

denoted by

$$(p, q) \in \Omega \times \Omega \mapsto \overrightarrow{pq} \in V$$

satisfying the following axioms :

1. *for any $p, q, r \in \Omega$, we have $\overrightarrow{pr} = \overrightarrow{pq} + \overrightarrow{qr}$;*
2. *for any $p \in \Omega$ and for any $x \in V$ there is one and only one $q \in \Omega$ such that $x = \overrightarrow{pq}$.*

Example 4.1.1 *Let V be a real vector space of dimension n. Consider V as a set and, for $(p, q) \in V \times V$, define $x = \overrightarrow{pq}$ to be the vector $q - p \in V$. In this way, V becomes an n-dimensional affine space.*

Example 4.1.2 *In particular, let V be the standard real n- dimensional vector space $\mathbb{R}^n$. Then regard it as an affine space in the manner of the last example. We call it the standard n-dimensional affine space.*

Definition 4.1.2 *An affine coordinate system with origin $o \in \Omega$ can be defined as follows. Let $\{e_1, \ldots, e_n\}$ be the basis of V. For any point $p \in \Omega$ we write*

$$\overrightarrow{oq} = \sum_{i=1}^{n} x^i(p) e_i,$$

*where $(x^1(p), \ldots, x^n(p))$ is the uniquely determined n-tuple real numbers, called the **coordinates** of p.*
*The set of functions $\{x^1, \ldots, x^n\}$ is called an **affine coordinate system**.*

If we have two affine coordinate systems $\{x^1, \ldots, x^n\}$ and $\{y^1, \ldots, y^n\}$, then they are related by

$$y^i = \sum_{j=1}^{n} a^i_j + c^i, \qquad 1 \leq i \leq n,$$

where $A = [a^i_j]$ is a non-singular $n \times n$ and $c = [c^i]$ is a vector.
This relation may be expressed by the equation

$$y = Ax + c,$$

or in its expanded matrix form

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

We now define the notion of affine transformation. Let $f : \Omega \to \Omega$ be a one-to-one mapping of Ω onto itself. Foe each $p \in \Omega$ we deffine a mapping $F_p : V \to V$ as follows. for each $x \in V$, let $r \in \Omega$ be uniquely determined point in Ω such that $\overrightarrow{pr} = x$. Then we set

$$F_p(x) = \overrightarrow{f(p)f(r)}.$$

Definition 4.1.3 *We say that f is an **affine transformation** if, for certain $p \in \Omega$, the map F_p is a linear transformation of V onto itself (and it is non-singular, since it is one-to-one together with f).*
In this case, if follows that for any point $q \in \Omega$ the map F_q coincides with F_p.
*We can therefore call this map the **associated linear transformation** and denote it simply by F.*

Let $\{x^1, \ldots, x^n\}$ be an affine coordinate system with origin o and based on a basis $\{e_1, \ldots, e_n\}$.
Let $\{y^1, \ldots, y^n\}$ be the affine coordinate system with origin $f(o)$ and based on the basis $\{F(e_1), \ldots, F(e_n)\}$. Then we have :

$$y_i(f(p)) = x_i(p) \quad \text{for} \quad p \in \Omega.$$

We can write the relationship between the coordinate system $\{x^1, \ldots, x^n\}$ and $\{y^1, \ldots, y^n\}$ in the form :

$$x^i = \sum_{j=1}^{n} b^i_j y^j + d^i, \qquad 1 \leq i \leq n.$$

Definition 4.1.4 *A non-empty subset Ω' of Ω is called an* ***affine subspace*** *if, for a certain point $p \in \Omega'$, the set of vector $\{\overrightarrow{pq}\ ,\ q \in \Omega'\}$ of Ω form a vector subset W of V. In this case, p in the condition can be replaced by any point of Ω' with same vector subspace W resulting.*
The dimension of an affine subspace is, by definition, the dimension of the associated vector space.

4.2 Affine differential geometry

In this subsection we give the foundation of the study of the affine geometry, we begin by this example in the euclidean space (see[NS94])

Example 4.2.1 *Let an affine space Ω, with corresponding vector space V.*
Given two points $p, q \in \Omega$, the line pq in a 1-dimensional affine subspace

$$\{r \in \Omega : \overrightarrow{pr} = t\overrightarrow{pq}\ ,\ \ for\ t \in \mathbb{R}\}$$

We now consider the standard real affine space $\Omega = \mathbb{R}^n$ as an n-dimensional differential manifold. For each point $p \in \mathbb{R}^n$ we may identify the tangent space $T_p\Omega = T_p\mathbb{R}^n$ with a vector space $V = \mathbb{R}^n$.
This means that we consider each $x \in V$ as a geometric vector placed at p, that is, $\overrightarrow{pq}$ interpreted as the pair of initial point p and end point q.
Furthermore, we consider $x \in V$ as a vector field that assigns to each $p \in \Omega$ a tangent vector $\overrightarrow{pq}$ determined by x.
Geometrically, all these vectors determined by x are parallel. From the construction of an affine coordinate system $\{x^1, \ldots, x^n\}$it follows that $\partial/\partial x^i$ as a vector field corresponds to e_i, where $\{e_1, \ldots, e_n\}$ is a basis of V on which the affine coordinate

system is based.
We now consider the notion of parallel volume element in $\Omega = \mathbb{R}^n$. First we fix the volume element ω in the vector space $V = \mathbb{R}^n$. This is nothing but a non-zero alternating n-form; once an orientation of V is fixed, it is determined up to positive constant factor.that is, for avry oriented basis $\{e_1, \ldots, e_n\}$, the value $\omega(e_1, \ldots, e_n)$ can be assigned to be an arbitrary positive number c, with determines ω uniquely.
A volume element ω in V determines a volume element on the manifold Ω, that is, a non-vanishing differential n-form denoted by the same letter ω, such that :

$$\omega\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) = c,$$

where the vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ correspond to $e_1, \ldots, e_n$ as explained above.
It obvious that $\omega(X_1, \ldots, X_n) = \omega(Y_1, \ldots, Y_n)$ if each $Y_i \in T_y\mathbb{R}^{n+1}$ is parallel to $X_i \in T_x\mathbb{R}^{n+1}$. Hence ω is said to be ***parallel****.*

Definition 4.2.1 *Once a parallel volume element is fixed in affine space Ω, an affine transformation f is said to be* ***equiaffine (or unimodular)*** *if it preserves the volume element, that is the associated linear transformation F preserves the corresponding volume element in V.*

The geometry of submanifolds of an affine space is called **affine differential geometry**. We study the properties that is invariant under the group of affine transformations, just as Euclidean differential geometry is the geometry of submanifolds of Euclidean space in which we study de properties invariant under Euclidean isometries.
In affine differential geometry, particularly important in the study of properties invariant under equiaffine transformations.

4.2.1 Affine connections

In this section, we summarize the basic notions concerning affine connections (see [NS94]).
Let M be a differentiable manifold of class C^∞. When we want to emphasize its dimension, we write M^n. the mast convenient to define the notion of an affine (or linear) connection is through covariant differentiation of a vector field with respect to another.
We shell denoted by $\mathcal{F}(M)$ the set of all differentiable functions and by $\mathfrak{X}(M)$ the set of all smooth vector fields on M.

Definition 4.2.2 *By a rule of* ***covariant differentiation*** *on M we mean a mapping*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$(X, Y) \quad \mapsto \nabla_X Y := \nabla(X, Y)$$

satisfying the following conditions :

$$\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y, \tag{4.1}$$
$$\nabla_{\phi X} Y = \phi \nabla_X Y, \tag{4.2}$$
$$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2, \tag{4.3}$$
$$\nabla_X (\phi Y) = (X\phi) Y + \phi \nabla_X Y, \tag{4.4}$$

*where $X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(M)$ and $\phi \in \mathcal{F}(M)$. An **affine connection** on M is nothing but a rule of covariant differentiation on M and we denoted it by ∇.*

An affine connection on M induces an affine connection on any open submanifolds U of M in the natural way. In particular, if U is the coordinate neighborhood with local coordinates $\{x^1, \dots, x^n\}$ then we may write

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k}, \tag{4.5}$$

where the system of functions Γ_{ij}^k $(i, j, k = 1, \dots, n)$ are called **Christoffel symbols** for the affine connection relative to the local coordinate system at hand.

4.2.2 Differential forms and tensor fields

In this subsection, we will give a brief introduction to the tensor fields, for more details, we send the reader to Yano-Con books [YK84], Kobayashi-Nomizu books [KN69] and Nomizu-Sasaki books [NS94]. Let M be a differentiable manifold, and T_pM a tangent space of M at each point $p \in M$.

Definition 4.2.3 *Let T_p^*M be the dual space of the tangent space T_pM at p.*
*An element of T_p^*M is called a **covector** at p. An assignment of covector at each point p is called a **1-form** or **differential form of degree 1** .*
*For each function f for M, the **total differential** of f at p (denoted df_p) is defined by :*

$$< df_p, X >= Xf \quad \text{for} \quad X \in T_pM,$$

where $<\ ,\ >$ denotes the value of the first entry on the second entry as a linear functional on T_pM.

Let $\{x^1, \dots, x^n\}$ be a local coordinate system at $p \in M$. Then $\{dx_p^1, \dots, dx_p^n\}$ form a basis for T_p^*M. They form the dual basis od the basis $\{\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^n}|_p\}$ for T_pM. In an neighborhood U of p, every 1-form ω can be uniquely written as

$$\omega = \sum_{j=1}^{n} f_j dx^j,$$

where f_j are functions in U and are called the **components** of ω with respect to $\{x^1, \ldots, x^n\}$.
The 1-form ω is called **differentiable form** if all functions f_j are differentiable. This condition in independent of the choice of a local coordinate system. We shall only consider differentiable 1-forms.

A 1-form ω can be also as an $\mathcal{F}(M)$-linear mapping of $\mathcal{F}(M)$-module (we can take the example of set of the vector field) $\mathfrak{X}(M)$ into $\mathcal{F}(M)$. The two definitions are related by

$$\omega(X)_p =< \omega_p, X_p >, \qquad X \in \mathfrak{X}(M)$$

If p is a point in M, we define $T^r_s(p)$ as a set of all $\mathbb{R}$-multilinear mappings of

$$T : \underbrace{T^*_pM \times \ldots \times T^*_pM}_{r-times} \times \underbrace{T_pM \times \ldots \times T_pM}_{s-times} \to \mathbb{R}$$

An element T of $T^r_s(p)$ is said a **tensor field** on M at p of type (r, s), this tensor field K is said to be **contravariant** of degree r and **covariant** of degree s. In particular, the tensor fields of type $(0, 0)$, $(1, 0)$ and $(0, 1)$ on M are just the differentiable functions, the vector fields and the 1-forms on M, respectively. A tensor fields are **alternate**.

The set of all tensor fields is denoted $\Omega(M)$:

$$\Omega(M) = \sum_{r=0,\, s=0}^{\infty} T^r_s \quad \text{and} \quad T(p) = \sum_{r=0,\, s=0}^{\infty} T^r_s(p)$$

Let a tensor field T of type (r, s) and $X \in \mathfrak{X}(M)$. Then $\nabla_X T$ is a tensor field of same type (r, s). We may also regard ∇T as a tensor field of type $(r, s + 1)$, i.e : a linear mapping

$$X \in \mathfrak{X}(M) \mapsto \nabla_X T.$$

Proposition 4.2.1 *Let M differentiable manifold and T a tensor field of M, then*

1. *If K is an a covariant tensor field of degree s, that is a s-linear map*

$$T : \mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M) \to \mathcal{F}(M)$$

then for all $X, X_1, \ldots, X_s \in \mathfrak{X}(M)$ we have :

$$(\nabla_X T)(X_1, \ldots, X_s) = X(T(X_1, \ldots, X_s)) - \sum_{i=1}^{s} T(X_1, \ldots, \nabla_X X_i, \ldots, X_s).$$

2. If K is a tensor field of type $(1,s)$, that is an s-linear map

$$T : \mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$(\nabla_X T)(X_1, \ldots, X_s) = \nabla_X (T(X_1, \ldots, X_s)) - \sum_{i=1}^{s} T(X_1, \ldots, \nabla_X X_i, \ldots, X_s).$$

Definition 4.2.4 *Let a differentiable manifold M. We define a few tensor fields associated to a given affine connection ∇. The* ***torsion field*** *T is defined by :*

$$\begin{array}{llll} T: & \mathfrak{X}(M) \times \mathfrak{X}(M) & \to & \mathfrak{X}(M) \\ & (X,Y) & \mapsto & T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y], \end{array}$$

it is a tensor field of type $(1,2)$.

The torsion fields T induces for each point $p \in M$ a skew-symmetric bilinear mapping $T_pM \times T_pM \to T_pM$. The components of the torsion tensor T in the local coordinates are :

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

We say that ∇ has **zero torsion** or that ∇ is **torsion-free** if we have the torsion tensor of the given connection ∇ is 0.

Definition 4.2.5 *The* ***curvature tensor*** *field R, which is of type $(1,3)$ is defined by:*

$$\begin{array}{llll} R: & \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) & \to & \mathfrak{X}(M) \\ & (X,Y,Z) & \mapsto & R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{array}$$

Given $X, Y \in T_pM$, $p \in M$, $R(X,Y)$ is a linear transformation of T_pM. The components il local coordinates :

$$R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^j} = \sum_{i=1}^{n} R_{jkl}^i \frac{\partial}{\partial x^i},$$

are given by

$$R_{jkl}^i = \left(\frac{\partial \Gamma_{lj}^i}{\partial x^k} - \frac{\partial \Gamma_{kj}^i}{\partial x^l}\right) + \sum_{m=1}^{n} (\Gamma_{lj}^m \Gamma_{km}^i - \Gamma_{kj}^m \Gamma_{lm}^i).$$

Theorem 4.2.1 *Let differentiable manifold M, with an affine connection ∇, such that ∇ is torsion-free. Then we have the first ans de second Bianchi identities :*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$
$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0,$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Remark 4.2.1 .

1. *If R is identically 0 on M, we say that ∇ is an* ***flat affine connection****.*
2. *Thus an affine connection is flat if $T = 0$ and $R = 0$.*
3. *It is know that ∇ is flat if and only if around each point there exists a local coordinate system such that $\Gamma^k_{ij} = 0$ for all i, j, k.*
4. *If $T = 0$ and $\nabla R = 0$, then we say that ∇ is a* ***symmetric affine connection****.*

Definition 4.2.6 *We define the Ricci tensor Ric of type $(0,2)$, by*

$$Ric(Y,Z) = trace\,\{X \mapsto R(X,Y)Z\},$$

where X, Y, Z are a vector fields on M.

The components in local coordinates are given by :

$$R_{jk} = Ric\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \sum_{i=1}^{n} R^i_{kij}\ .$$

Definition 4.2.7 *By an* ***equiaffine connection*** *∇ on M we mean a torion-free affine connection that admits a parallel volume element ω on M.*
If ω is an volume element on M such that $\nabla\omega = 0$, we say that (∇, ω) is an ***equiaffine structure*** *on M.*

4.2.3 Metrics and inner product

We give a breve about inner products on real vector spaces. (see all books of differential geometry)
Let V be an n-dimensional real vector space. For a given bilinear function f:

$$\begin{array}{rccl} f: & V \times V & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) \end{array}$$

there existe a basis of V : $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}, e_{p+q+1}, \ldots, e_n\}$ such that

$$\begin{array}{ll} f(e_i, e_j) = 0 & \text{for all } i \neq j, \\ f(e_i, e_i) = 1 & \text{for all } i,\ 1 \leq i \leq p, \\ f(e_j, e_j) = -1 & \text{for all } j,\ p+1 \leq j \leq p+q, \\ f(e_k, e_k) = 1 & \text{for all } k,\ p+q+1 \leq k \leq n. \end{array}$$

Definition 4.2.8 *For a given f,*

1. *the subspace $V_0 = \{x \in V : f(x, y) = x \text{ for all } y \in V\}$ is called the* ***null space*** *or* ***kernel*** *denoted ker . It's dimension is equal to $n - (p+q)$.*
2. *We say that f is* ***nondegenerate*** *if the kernel is : $ker(f) = \{0\}$, thus f nondegenerate if and only if $p + q = n$.*
3. *A nondegenerate, symmetric bilinear function is called an* ***inner product*** *on V.*
4. *The pair (p, q) is called its* ***signature****.*
5. *If $p = n$, then $f(x, x) \geq 0$ for all x, and the equality holds if and only if $x = 0$, in this case, f is said to be* ***positive-definite****. If $q = n$, then f is* ***negative-definite****.*
   If f is positive-definite or negative-definite we say that f is ***definite****. Otherwise, we say f is* ***indefinite****.*

Let a n-dimensional differentiable manifold M,

Definition 4.2.9 *Let g a symmetric tensor field on M, of type $(0, 2)$, called* ***metric*** *on M*

$$\begin{array}{rccl} g: & \mathfrak{X}(M) \times \mathfrak{X}(M) & \to & \mathbb{R} \\ & (X, Y) & \mapsto & g(X, Y) \end{array}$$

By a ***nondegenerate metric****, say a metric g on M that is : If $g(X, Y) = 0$ for all $Y \in \mathfrak{X}(M)$, then $X = 0$. We also call it a* ***pseudo-Riemannian*** *(or* ***semi-Riemannian)*** ***metric****.*

If g is positive-definite, we say that g is a ***Riemannian metric****.*

Theorem 4.2.2 *Given a nondegenerate metric g on M, there is an affine connection with torsion-free that is : $\nabla g = 0$. Such a connection is unique.*
It is called the ***Levi-Civita connection*** *for g.*

Now we consider a more general situation. Suppose an n-dimensional manifold M is provided with a nondegenerate metric, say h, and an affine connection ∇. We define a new affine connection $\bar{\nabla}$ by requiring that the following holds for all $X, Y, Z \in \mathfrak{X}(M)$:

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \bar{\nabla}_X Z).$$

This equation determines a unique affine connection $\bar{\nabla}$, which we call the **conjugate connection** of ∇.

We are thus interested in the case where ∇ is not necessarily metric relative to h. if we consider the conjugate connection of $\bar{\nabla}$ relative to h, then, obviously, we get back ∇. We have

Proposition 4.2.2 *The torsion tensor T and $\bar{T}$ of ∇ and $\bar{\nabla}$, respectively, satisfy :*

$$(\nabla_X h)(Y,Z) + h(Y, T(X,Z)) = (\nabla_Y h)(X,Z) + h(Y, \bar{T}(X,Z)),$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Corollary 4.2.1 *Assume that ∇ is torsion-free.*
Then $\bar{\nabla}$ is torsion-free if and only if (∇, h) satisfies Codazzi's equation :

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(Y,X), \qquad \text{for all } X, Y, Z \in \mathfrak{X}(M).$$

If (∇, h) satisfies Codazzi's equation, then the function defined by :

$$\begin{array}{rlcl} C: & \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) & \to & \mathbb{R} \\ & (X,Y,Z) & \mapsto & C(X,Y,Z) := (\nabla_X h)(Y,Z) \end{array}$$

is a symmetric 3-linear function. Wi call it the **cubic form** for (∇, h).
In this case we shell also say that (∇, h) is **compatible**.

Corollary 4.2.2 *If ∇ is torsion-free and (∇, h) is compatible, then*

1. *$(\bar{\nabla}, h)$ satisfies Codazzi's equation.*
2. *$\hat{\nabla} = \frac{1}{2}(\nabla + \bar{\nabla})$ is the Levi-Civita connection for h.*

Proposition 4.2.3 *The curvature tensors R and $\bar{R}$ of ∇ and $\bar{\nabla}$, respectively, are related by*

$$h(R(X,Y)Z, U) = -h(Z, \bar{R}(X,Y)U).$$

for all $X, Y, Z, U \in \mathfrak{X}(M)$.

4.3 Affine immersions

We introduce the notion of affine immersion, special cases. The fundamental concepts and equations are established. (see [NS94]). We shell always assume that the given affine connection have torsion-free.

Consider two differentiable manifolds with affine connections $(\tilde{M}^m, \tilde{\nabla})$ and (M^n, ∇). Let $k = m - n$.

Definition 4.3.1 *A differentiable immersion $f : M \to \tilde{M}$ is said* ***affine immersion*** *if the following condition is satisfied:*
There is a k-dimensional differentiable distribution N along $f : p \in M \mapsto N_p$, a subspace of $T_{f(p)}\tilde{M}$, such that

$$T_{f(p)}\tilde{M} = f_*(T_pM) \oplus N_p \qquad \textit{direct sum,}$$

and such that for all $X, Y \in \mathfrak{X}(M)$, we have at each point $p \in M$:

$$(\tilde{\nabla}_X f_*(Y))_p = (f_*(\nabla_X Y))_p + (\alpha(X,Y))_p, \qquad \textit{where } \alpha(X,Y)_p \in N_p.$$

This distribution N^k may be regarded as a bundle of **transversal** k-subspaces. In the case where $f : M \to \tilde{M}$ is an immersion of a manifold M into a Riemannian manifold $\tilde{M}$ with positive-definite metric g, we can certainly choose the normal space at each point, namely

$$N_p = \{\xi \in T_{f(p)}\tilde{M} \ : \ g(\xi, X) = 0 \quad \text{for all} \quad X \in T_pM\}.$$

In this situation, it is easy to show that the map $(X,Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \to \alpha(X,Y)$ acually defines for each point $p \in M$ asymmetric bilinear map

$$T_pM \times T_pM \to N_p.$$

Definition 4.3.2 *In the case where $f : M \to \tilde{M}$ is an immersion of a manifold M into a Riemannian manifold $\tilde{M}$, this map α is called the* ***second fundamental form****.*
Whereas the ***first fundamental form*** *refers to the induced Riemannian metric in M.*
In the geometry of affine immersions, we shell call α simply the ***affine fundamental form****.*

For the remainder of this section, we shall concentrate on the case of codimension $k = 1$ and also assume that the manifold $(\tilde{M}, \tilde{\nabla})$ is an affine space $\mathbb{R}^{n+1}$ with its usual flat affine connection D. We shall use a fixed parallel volume element $\tilde{\omega}$ on $\mathbb{R}^{n+1}$ whenever necessary, but with a word of caution.

4.3.1 Affine hypersurfaces

Definition 4.3.3 *We consider an n-dimensional differentiable manifold M together with an immersion $f : M \to \mathbb{R}^{n+1}$.*
We call M a ***hypersurface*** *and f a* ***hypersurface immersion****. We may also call M an immersed hypersurface.*

For each point $p \in M$ we choose a local field of **transversal vectors** $\xi : q \in U \mapsto \xi_q$, where U is a neighborhood of p. **Transversality** means in this case that:

$$T_{f(q)}\mathbb{R}^{n+1} = f_*(T_qM) + Span\{\xi_q\},$$

where $Span\{\xi_q\}$ means the 1-dimensional subspace spanned by ξ_q.

Proposition 4.3.1 *For a hypersurface immersion $f : M \to \mathbb{R}^{n+1}$, suppose we have a transversal vector field ξ on M. then we have a torsion-free induced connection ∇ satisfying:*

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \qquad \text{the } \textbf{\textit{Gauss formula}}, \tag{4.6}$$

where $X, Y \in \mathfrak{X}(M)$ and at each point $p \in M$, h is a symmetric bilinear function on the tangent space T_pM.

The symmetric bilinear function h is called the ***affine fundamental form*** *(relative to the transversal vector ξ).*

Definition 4.3.4 *Let (M, ∇) be an n-dimensional manifold with an affine connection ∇. An immersion $f : M \to \mathbb{R}^{n+1}$ is called an* ***affine immersion*** *if there is an vector field ξ on M such that the Gauss formula holds.*

Thus for an immersed hypersurface $f : M \to \mathbb{R}^{n+1}$ a choice of transversal vector field ξ provides an induced connection ∇ in such a way that f becomes an affine immersion. We also state

Proposition 4.3.2 *For all $X \in \mathfrak{X}(M)$ we have*

$$D_X\xi = -f_*(SX) + \tau(X)\xi, \qquad \textbf{\textit{Weingarten formula}}, \tag{4.7}$$

where S is a $(1,1)$-tensor, called the (affine) ***shape operator****,*
τ is a 1-form, called the ***transversal connection form****.*

Proposition 4.3.3 *Let (M, ∇) be an n-dimensional manifold equipped with an affine connection ∇ and Let $f : (M, \nabla) \to \mathbb{R}^{n+1}$ and $\bar{f} : (M, \nabla) \to \mathbb{R}^{n+1}$ be affine immersions relative to transversal vector fields ξ and $\bar{\xi}$, respectively. The objects h, S, and τ for ξ and for $\bar{\xi}$ are denoted $\bar{h}, \bar{S}$ and $\bar{\tau}$. Assume that*

$$h = \bar{h}, \qquad S = \bar{S}, \qquad \tau = \bar{\tau}.$$

Then there is an affine transformation A such that $\bar{f} = A\,f$.

In these situations we are also interested in equiaffine connections. So we take a fixed parallel volume element $\tilde{\omega}$ in $\mathbb{R}^{n+1}$. For a hypersurface immersion $f : M \to \mathbb{R}^{n+1}$, let ξ be a transversal vector field. In addition to the induced connection ∇ and the affine fundamental form h, we consider the following volume element θ on M :

$$\theta(X_1, \ldots, X_n) = \tilde{\omega}(X_1, \ldots, X_n, \xi), \qquad \text{for } X_1, \ldots, X_n \in \mathfrak{X}(M). \tag{4.8}$$

Clearly θ is a volume element on M, called the **induced volume element**.

We are interested in the question whether (∇, θ) defines an equiaffine structure, that is, whether $\nabla\theta = 0$ holds. This question is answered as follows (for the proof see [NS94]).

Proposition 4.3.4 *We have*

$$\nabla_X \theta = \tau(X)\theta \qquad \textit{for all } X \in T_pM,\ p \in M. \tag{4.9}$$

Consequently, the following two conditions are equivalent:

1. $\nabla\theta = 0$;
2. $\tau = 0$, *that is,* $D_X\xi$ *tangential for evry* $X \in \mathfrak{X}(M)$.

In view of this Proposition, it is convenient to make the following definition.

Definition 4.3.5 .

1. *For the hypersurface immersion* $f : M \to \mathbb{R}^{n+1}$, *a transversal vector field* ξ *is said* ***equiaffine*** *if* $D_X\xi$ *is tangent to* M *for each* $X \in T_pM$, $p \in M$.
2. *With an equiaffine transversal vector field* ξ, *we have an* ***equiaffine structure*** (∇, θ) *on* M.
3. *Thus we may call* $f : (M, \nabla, \theta) \to \mathbb{R}^{n+1}$ *an* ***equiaffine immersion***.

We give now more fundamental equation for the hypersurface immersion $f : M \to \mathbb{R}^{n+1}$. first, we consider the case where the given transversal vector field ξ is arbitrary. We have

Theorem 4.3.1 *For an arbitrary transversal vector field* ξ *the induced connection* ∇, *the affine fundamental form* h, *the shape operator* S, *and the transversal connection form* τ *satisfy the following equations:*

$$\textit{Gauss}: \quad R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY; \tag{4.10}$$

$$\textit{Codazzi-h}: (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z); \tag{4.11}$$

$$\textit{Codazzi-S}: (\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX; \tag{4.12}$$

$$\textit{Ricci}: \quad h(X,SY) - h(SX,Y) = d\tau(D,Y). \tag{4.13}$$

The reader can see the proof in [NS94].

Corollary 4.3.1 *The Ricci tensor of the induced connection is given by*

$$Ric(Y, Z) = h(Y, Z) tr\, S - h(SY, Z). \tag{4.14}$$

The equation in Theorem 4.3.1 are not all independent. We have

Proposition 4.3.5 *Let M be a differentiable manifold with a torsion-free affine connection ∇, a symmetric covariant tensor field h of degree 2, a $(1,1)$-tensor field S, and a 1-form τ that together satisfy the equation of Gauss (4.10) and the equation of Codazzi for h (4.11). If rank $h \geq 3$, then the equation of Codazzi for S (4.12) is satisfied.*

Remark 4.3.1 .

1. *From (4.14) and (4.13) it follows that Ric is symmetric if and only if $d\tau = 0$.*
2. *In the equation of Codazzi (4.11) we see that the left-hand side is symmetric in X and Y as well as in Y and Z. Therefore if we set*

$$C(X, Y, Z) = (\nabla_X h)(Y, Z) + \tau(X)(Y, Z), \tag{4.15}$$

*we see that it is symmetric in all three variables. We call C the **cubic form** for the affine immersion.*

We shall now consider the change of a transversal vector field for a given immersion f.

Proposition 4.3.6 *Suppose we change a transversal vector field ξ to*

$$\bar{\xi} = \phi\xi + f_*(Z),$$

where Z is a tangent vector field on M and ϕ is a nonvanishing function. Then the affine fundamental form, the induced connection, the transversal connection form, and the affine shape operator change as follows:

$$\bar{h} = \frac{1}{\phi}; \tag{4.16}$$

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{\phi} h(X, Y) Z; \tag{4.17}$$

$$\bar{\tau} = \tau + \frac{1}{\phi} h(Z, .) + d \log |\phi|; \tag{4.18}$$

$$\bar{S} = \phi S - \nabla_. Z + \bar{\tau}(.) Z; \tag{4.19}$$

$h(Z, .), \nabla_. Z$ and $\bar{\tau}(.)$ are the 1-forms whose on X are $h(Z, X), \nabla_X Z$ and $\bar{\tau}(X)$, respectively.

Definition 4.3.6 .

The equation (4.16) *shows that the rank of the affine fundamental form is independent of the choice of transversal vector field. We define it as the* ***rank*** *of the hypersurface or the hypersurface immersion.*

In particular, if the rank is n*, that is, if* h *is nondegenerate, then we say that the hypersurface or the hypersurface immersion is* ***nondegenerate****.*

4.4 Blaschke immersions

Let $f : M \to \mathbb{R}^{n+1}$ be a nondegenerate hypersurface immersion. We know that no matter which transversal field ξ we may choose, the affine fundamental form h has rank n, and can be treated as a nondegenerate metric on M. This is the basic assumption on which Blaschke developed affine differential geometry of hypersurfaces. In this section, we shall give a rigorous foundation from a structural point of view. We pick a fixed volume element on $\mathbb{R}^{n+1}$ (given by the determinant function, say). The reader cas see the book of Nomizu-Sasaki [NS94] for more details and for also the proves.

If we choose an arbitrary transversal vector field ξ, then we obtain on M the affine fundamental form h, the induced connection ∇, and the induced volume element θ. We want to achieve, by an appropriate choice of ξ, the following two goals:

I. (∇, θ) is an equiaffine structure, that is, $\nabla\theta = 0$;

II. θ coincides with the volume element ω_h of the nondegenerate metric h;

where $\omega_h(X_1, \ldots, X_n) = |\det[h_{ij}]|^{1/2}$, and $h_{ij} = h(X_i, X_j)$ for an unimodular basis $\{X_1, \ldots, X_n\}$ for θ, i.e: $\theta(X_1, \ldots, X_n) = 1$.

Theorem 4.4.1 *Let* $f : M \to \mathbb{R}^{n+1}$ *be a nondegenerate hypersurface immersion. For each point* $p_0 \in M$*, there is a transversal vector field defined in a neighborhood of* p_0 *satisfying the conditions I. and II. above. Such a transversal vector field is unique up to sign.*

We find it convenient to formulate the following concepts:
Suppose θ is an arbitrary volume element and h a nondegenerate metric, both defined in a neighborhood of a point. We define the determinant $\det_\theta h$ of a symmetric covariant tensor h of degree 2 relative to θ as follows. Let $\{X_1, \ldots, X_n\}$, be a unimodular basis for θ, that is, a basis in $T_X M$ such that $\theta(X_1, \ldots, X_n) = 1$. If we set $h_{ij} := h(X_i, X_j)$, then the determinant of the matrix $[h_{ij}]$ is independent of the choice of unimodular basis $\{X_1, \ldots, X_n\}$. We denote this number by $\det_\theta h$.

Definition 4.4.1 *A transversal vector field satisfying I. and II. is called the* ***affine normal field*** *or* ***Blaschke normal field****. Locally, it is uniquely determined up to sign. For each point $p \in M$ we take the line through p in the direction of the affine normal vector ξ_p. This line, which is independent of the choice of sign for ξ, is called the* ***affine normal*** *through p.*

Definition 4.4.2 *By fixing an affine normal field ξ we have the induced connection ∇, the affine fundamental form h, which is traditionally called the* ***affine metric****, and the affine shape operator S determined by the formulas of Gauss and Weingarten.*
We shall call (∇, h, S) the ***Blaschke structure*** *on the hypersurface M.*
The affine immersion $f : (M, \nabla) \to (\mathbb{R}^{n+1}, D)$ with affine normal field ξ is called a ***Blaschke immersion****.*
We shall also speak of M as a ***Blaschke hypersurface****.*
The induced connection ∇ is independent of the choice of sign for ξ and is called the ***Blaschke connection****.*

Remark 4.4.1 *From the uniqueness in Theorem 4.4.1 it follows that the Blaschke structure is invariant under every equiaffine transformation of the ambient space $\mathbb{R}^{n+1}$. If a nondegenerate hypersurface M is orientable, then there is a globally defined affine normal field, which is unique up to sign. In fact, we may orient ξ using an orientation of M.*

As a special case of Theorem 4.3.1, we have

Theorem 4.4.2 *For a Blaschke hypersurface M , we have the following fundamental equations :*

Gauss : $$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY; \tag{4.20}$$
Codazzi equation for h : $$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z); \tag{4.21}$$
Codazzi equation for S : $$(\nabla_X S)(Y) = (\nabla_Y S)(X); \tag{4.22}$$
Ricci : $$h(X,SY) = h(SX,Y); \tag{4.23}$$
Equiaffine condition : $$\nabla\theta = 0; \tag{4.24}$$
Volume condition : $$\theta = \omega_h; \tag{4.25}$$
Apolarity condition : $$\nabla\omega_h = 0. \tag{4.26}$$

Proposition 4.4.1 *For a nondegenerate hypersurface $(n \geq 2)$ with Blaschke structure, we have :*

1. *$S = 0$ if and only if $R = 0$.*

2. *The Ricci tensor is given by* $Ric(X,Y) = Tr\,S\,h(X,Y) - h(SY,Z)$, *as before, And Ric is* 0 *if and only if* $S = 0$.

3. *If* $S = \lambda I$ *where* λ *is a scalar function, then* λ *is constant.*

Definition 4.4.3 .
A Blaschke hypersurface M *is called an* ***improper affine hypersphere*** *if* S *is identically* 0.
If $S = \lambda I$ *where* λ *is a nonzero constant, then* M *is called a* ***proper affine hypersphere***.

Proposition 4.4.2 *Let* $f : M^n \to \mathbb{R}^{N+1}$ *be a nondegenerate immersion with Blaschke structure. Then* M *is an affine hypersphere if and only if every* ∇-*geodesic on* M^n *lies on a certain* 2-*plane in* $\mathbb{R}^{n+1}$.

We give now a definitions of the locally homogeneous hypersurfaces, and the locally strongly convex hypersurfaces [DV93b].

Definition 4.4.4 *Let* $f : M \to \mathbb{R}^{n+1}$ *be a non degenerate hypersurface with Blaschke structure. We call* M ***locally homogeneous*** *if for all point s* p *and* q *in* M, *there exists a neighbourhood* U_p *of* p *in* M, *and an equiaffine transformation* A *of* $\mathbb{R}^{n+1}$, *i.e* $A \in SL(n+1, \mathbb{R}) \ltimes \mathbb{R}^{n+1}$, *such that* $A(p) = q$, *and* $A(U_p) \subset M$.
If $U_p = M$ *for all* $p \in M$, *then* M *is called* ***homogeneous***.

Proposition 4.4.3 .

1. *Every equiaffine transformation leaving* M *locally invariant, preserves the affine metric* h *and the induced connection* ∇.

2. *if for all points* p *and* q *of* M, *there exist neighbourhoods* U_p *of* p *and* U_q *of* q *in* M, *and a diffeomorphism* $f : U_p \to U_q$, *with* $f(p) = f(q)$ *such that* f *preserves both* h *and* ∇, *then* M *is locally homogeneous.*

3. *Let* G *the pseudogroup defined by*

$$G = \{A \in SL(n+1, \mathbb{R}) \ltimes \mathbb{R}^{n+1} \mid \exists U \text{ open in } M : A(U) \subset M\}.$$

then M *is homogeneous if and only if* G *"acts" transitively on* M.
If M *is homogeneous, then* G *is a group and every element of* G *maps the whole of* M *into* M.

Definition 4.4.5 *Let* $f : M \to \mathbb{R}^{n+1}$ *be a non degenerate hypersurface with Blaschke structure. If* h *is positive-definite, then we call* M *is* ***locally strongly convex*** *hypersurface.*

Procedure for finding the affine normal field

1. Choose a tentative transversal vector field ξ. Compute τ.
2. Determine the affine fundamental form h for ξ by using the formula of Gauss : $D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi$. Verify that h is nondegenerate.
3. Determine the induced volume element θ for ξ from
$$\theta(X_1, \ldots, \mathfrak{X}()n) = \det[f_* X_1, \ldots, f_* X_n, \xi].$$
4. Choose a unimodular basis $\{X_1, \ldots, X_n\}$ with
$$\theta(X_1, \ldots, X_n) = 1.$$

Set $h_{ij} = h(X_i, X_j)$ and compute $\det_\theta h = \det[h_{ij}]$.

5. Take $\phi = |\det_\theta h|^{1/(n+2)}$ and set $\bar{\xi} = \phi\xi + Z$, where Z is to be determined by
$$\tau + \frac{1}{\phi} h(Z, .) + d\log\phi = 0.$$
If $tau = 0$ this equation is simplify $h(Z,X) = -X\phi$ for every X.
6. Once we get the affine normal field $\bar{\xi}$, it is easy to compute the affine metric $\bar{h} = h/\phi$, the affine shape operator S, and the induced connection ∇.

The reader can fend many examples in the book of Nomizu-Sasaki [NS94].

4.5 Cubic form

We continue with a nondegenerate hypersurface $f : M \to \mathbb{R}^{n+1}$ with its Blaschke structure. Our discussions here are mainly based on Theorems 4.4.1 and 4.4.2 in the preceding section. From the Codazzi equation for h we see that the cubic form :
$$C(X,Y,Z) = (\nabla_X h)(Y,Z), \tag{4.27}$$
is symmetric in X, Y, and Z.

Definition 4.5.1 *Let a nondegenerate hypersurface $f : M \to \mathbb{R}^{n+1}$ with its Blaschke structure, with his cubuc form C, addition to the induced connection ∇ on M, we my consider the Levi-civita connection $\hat{\nabla}$ for the affine metric h. We consider the* ***difference tensor*** *of type $(1,2)$:*
$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y. \tag{4.28}$$

Since both ∇ and $\hat{\nabla}$have torsion-free, we have $K(X,Y) = K(Y,X)$. We shall also write

$$K_X Y = K(X,Y) \qquad \text{and} \qquad K_X = \nabla_X - \hat{\nabla}_X;$$

thus for each $X \in \mathfrak{X}(M)$, K_X is a tensor of type $(1,1)$. We can now relate the cubic form to the difference tensor.

Proposition 4.5.1 *We have*

$$C(X,Y,Z) = -2\,h(K_X Y, Z). \tag{4.29}$$

Corollary 4.5.1 *The induced connection ∇ and the Levi-Civita connection $\hat{\nabla}$ coincide with each other if and only if $K = 0$, that is, if and only if the cubic form C vanishes identically.*

Theorem 4.5.1 *We have the apolarity condition :*

$$tr\ K_X = 0 \quad \textit{for all} \quad X \in T_pM; \tag{4.30}$$

in index notation,

$$\sum_{j=1}^{n} K_{ij}^{j} \quad \textit{for each fixed } i.$$

Theorem 4.5.2 *The apolarity condition* (4.30) *is equivalent to each of the following conditions :*

$$tr_h\{(Y,Z) \mapsto C(X,Y,Z)\} = 0 \quad \textit{for all} \quad X \in T_pM; \tag{4.31}$$

in index notation, $\sum_{j,k=1}^{n} h^{jk} C_{ijk} = 0$ for each i;

$$tr_h(\nabla_x h) = 0 \quad \textit{for all} \quad X \in T_pM; \tag{4.32}$$

$$tr_h K = 0; \tag{4.33}$$

in index notation, $\sum_{j,k=1}^{n} h^{jk} K_{jk}^{i} = 0$ for each i;

The following is an important classical theorem due to Blaschke (for analytic surfaces), Pick (for surfaces) and Berwald (for hypersurfaces).

Theorem 4.5.3 *Let $f : M \to \mathbb{R}^{n+1}$, $n > 2$, be a nondegenerate hypersurface with Blaschke structure. If the cubic form C vanishes identically, then $f(M)$ is a hyperquadric in $\mathbb{R}^{n+1}$.*

Lemma 4.5.1 *If the cubic form of a Blaschke hypersurface M is identically* 0*, then M is an affine hypersphere.*

Remark 4.5.1 *The basic equations can also be expressed in terms of the Levi Civita connection and the difference tensor. In particular, we have the Codazzi equation for K which states that*

$$\begin{aligned}&\hat{\nabla}K(X,Y,Z)-\hat{\nabla}K(Y,X,Z)\\&=\frac{1}{2}h(Y,Z)SX-\frac{1}{2}h(X,Z)SY+\frac{1}{2}h(SX,Z)Y-\frac{1}{2}h(SY,Z)X.\end{aligned}\qquad(4.34)$$

For the prove of this Lemma, and two equivalent proves of the Theorem 4.5.3, the reader can find them in[NS94].

Proposition 4.5.2 *Let $f : M \to \mathbb{R}^{n+1}$, $n > 2$, be a locally nondegenerate, locally strongly convex hypersurface with Blaschke structure. Then :*

1. *It follow from Ricci equation that the affine shape operator is diagonalisable.*
2. *So M is homogeneous, the eigenvalues of the shape operator are constant it follow from [Nom68].*

Chapter 5

The initial results

Let the immersion $f : M \to \mathbb{R}^5$, and M is the 4-dimensional differentiable submanifold. For the remainder of this part, M will always denote a locally strongly convex, locally homogeneous Blaschke hypersurface of $\mathbb{R}^5$. And we will omit to write the immersion f in the equations.

Since M is locally strongly convex, using Proposition 4.5.2, it follows from the Ricci equation that the affine shape operator is diagonalizable.

In view of the previous results we will restrict ourselves here to the case that M has tow distinct eigenvalues both of multiplicity 2.

Let a point $p \in M$. We construct a tangent basis $\{e_1, e_2, e_3, e_4\}$ at the point p and λ_1, λ_2 by the eigenvalues of the shape operator S, such that

$$Se_1 = \lambda_1 e_1, \quad Se_2 = \lambda_1 e_2, \quad Se_3 = \lambda_2 e_3, \quad Se_4 = \lambda_2 e_4. \tag{5.1}$$

Then, since λ_1 and λ_2, are different numbers, and as M is homogeneous the eigenvalues of the shape operator are constant it follows from Proposition 4.5.2, that we can extend these vectors to local vector fields $\{E_1, E_2, E_3, E_4\}$, such that

$$SE_1 = \lambda_1 E_1, \quad SE_2 = \lambda_1 E_2, \quad SE_3 = \lambda_2 E_3, \quad SE_4 = \lambda_2 E_4. \tag{5.2}$$

We define functions (Christoffel symbols) Γ_{ij}^k such that the connection ∇ is given by

$$\nabla_{E_i} E_j = \sum_{k=1}^{4} \Gamma_{ij}^k E_k \qquad for\ i = 1, \ldots, 4, \quad j = 1, \ldots, 4. \tag{5.3}$$

Our job is to find all these functions, in this case, the number of Christoffel symbols is $4 \times 4 \times 4 = 64$.

5.1 First step: Cadazzi and apolarity equations

In this section we give the values of 40 Gammas, using the Codazzi, for h and S equations, and the apolarity equation.

5.1.1 Codazzi for S equations

In the Theorem 4.4.2 we have the Codazzi for S equation (4.22) si given by : for all $X, Y \in \mathfrak{X}(M)$ we have

$$(\nabla_X S)(Y) = (\nabla_Y S)(X).$$

So in the frame $\{E_1, E_2, E_3, E_4\}$, and the fact that $SE_i = \lambda_{1,2}E_i$ (equations (5.2)), it follow that

$$\begin{aligned}(\nabla_{E_i}S)(E_j) &= \nabla_{E_i}SE_j - S(\nabla_{E_i}E_j)\\ &= \begin{cases}\nabla_{E_i}\lambda_1 E_j - S(\sum_{k=1}^4 \Gamma_{ij}^k E_k) & \text{if } j = 1,2,\\ \nabla_{E_i}\lambda_2 E_j - S(\sum_{k=1}^4 \Gamma_{ij}^k E_k) & \text{if } j = 3,4,\end{cases}\\ &= \begin{cases}\lambda_1 \sum_{k=1}^4 \Gamma_{ij}^k E_k - \sum_{k=1}^4 \Gamma_{ij}^k SE_k & \text{if } j = 1,2,\\ \lambda_2 \sum_{k=1}^4 \Gamma_{ij}^k E_k - \sum_{k=1}^4 \Gamma_{ij}^k SE_k & \text{if } j = 3,4,\end{cases}\end{aligned}$$

then

$$(\nabla_{E_i}S)(E_j) = \sum_{k=1}^4 (\lambda_{\epsilon_1} - \lambda_{\epsilon_2})\Gamma_{ij}^k E_k, \tag{5.4}$$

where $\epsilon_1 = \begin{cases}1 & \text{if } j = 1,2\\ 2 & \text{if } j = 3,4\end{cases}$, and $\epsilon_2 = \begin{cases}1 & \text{if } k = 1,2\\ 2 & \text{if } k = 3,4\end{cases}$.

And the Codazzi equation for S (4.22) in this frame become :

$$\sum_{k=1}^4 \left((\lambda_{\epsilon_1} - \lambda_{\epsilon_2})\Gamma_{ij}^k - (\lambda_{\epsilon_1'} - \lambda_{\epsilon_2})\Gamma_{ji}^k\right) E_k = 0, \tag{5.5}$$

where $\epsilon_1 = \begin{cases}1 \text{ if } j = 1,2\\ 2 \text{ if } j = 3,4\end{cases}$, and $\epsilon_1' = \begin{cases}1 \text{ if } i = 1,2\\ 2 \text{ if } i = 3,4\end{cases}$, and $\epsilon_2 = \begin{cases}1 \text{ if } k = 1,2\\ 2 \text{ if } k = 3,4\end{cases}$.
From all $i = 1, \ldots, 4$, $j = 1, \ldots, 4$ and $k = 1, \ldots, 4$ and the last equation of Codazzi for S we get

$$\Gamma_{13}^1, \Gamma_{13}^2, \Gamma_{14}^1, \Gamma_{14}^2, \Gamma_{23}^1, \Gamma_{23}^2, \Gamma_{24}^1, \Gamma_{24}^2, \Gamma_{31}^3, \Gamma_{31}^4, \Gamma_{32}^3, \Gamma_{32}^4, \Gamma_{41}^3, \Gamma_{41}^4, \Gamma_{42}^3, \Gamma_{42}^4$$

vanish and that

$$\Gamma_{21}^3 = \Gamma_{12}^3, \quad \Gamma_{21}^4 = \Gamma_{12}^4, \quad \Gamma_{43}^1 = \Gamma_{34}^1, \quad \Gamma_{43}^2 = \Gamma_{34}^2.$$

5.1.2 Codazzi for h equations

In the same way of the last subsection, we have the Codazzi for h equation (4.21) si given by : for all $X, Y, Z \in \mathfrak{X}(M)$ we have

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z).$$

in the same frame $\{E_1, E_2, E_3, E_4\}$, and the fact that $SE_i = \lambda_{1,2} E_i$ (equations (5.2)), it follow that

$$\begin{aligned}(\nabla_{E_i} h)(E_j, E_k) &= -h(\nabla_{E_i} E_j, E_k) - h(E_j, \nabla_{E_i} E_k) \\ &= -\sum_{l=1}^{4} \Gamma_{ij}^l h(E_k, E_l) - \sum_{l=1}^{4} \Gamma_{ik}^l h(E_j, E_l)\end{aligned}$$

And the Codazzi equation for h (4.21) in this frame become :

$$-\sum_{l=1}^{4} \big((\Gamma_{ij}^l - \Gamma_{ji}^l) h(E_k, E_l) + \Gamma_{ik}^l h(E_j, E_l) - \Gamma_{jk}^l h(E_i, E_l)\big) = 0. \tag{5.6}$$

If we write $h(E_i, E_j) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, then the last equation become

$$-\sum_{l=1}^{4} \big((\Gamma_{ij}^l - \Gamma_{ji}^l)\delta_{kl} + \Gamma_{ik}^l \delta_{jl} - \Gamma_{jk}^l \delta_{il}\big) = 0.$$

From all $i = 1, \ldots, 4$, $j = 1, \ldots, 4$ and $k = 1, \ldots, 4$ and the last equation of Codazzi for h we get

$$\begin{aligned}&\Gamma_{11}^3 = 2\Gamma_{31}^1, \quad \Gamma_{11}^4 = 2\Gamma_{41}^1, \quad \Gamma_{22}^3 = 2\Gamma_{32}^2, \quad \Gamma_{22}^4 = 2\Gamma_{42}^2, \\ &\Gamma_{33}^1 = 2\Gamma_{13}^3, \quad \Gamma_{33}^2 = 2\Gamma_{23}^3, \quad \Gamma_{44}^1 = 2\Gamma_{14}^4, \quad \Gamma_{44}^2 = 2\Gamma_{24}^4, \\ &\Gamma_{14}^3 = \Gamma_{43}^1 - \Gamma_{13}^4, \quad \Gamma_{24}^3 = \Gamma_{43}^2 - \Gamma_{23}^4, \quad \Gamma_{32}^1 = \Gamma_{12}^3 - \Gamma_{31}^2, \quad \Gamma_{42}^1 = \Gamma_{12}^4 - \Gamma_{41}^2, \\ &\Gamma_{12}^1 = 2\Gamma_{21}^1 - \Gamma_{11}^2, \quad \Gamma_{22}^1 = 2\Gamma_{12}^2 - \Gamma_{21}^2, \quad \Gamma_{34}^3 = 2\Gamma_{43}^3 - \Gamma_{33}^4, \quad \Gamma_{44}^3 = 2\Gamma_{34}^4 - \Gamma_{43}^4.\end{aligned}$$

5.1.3 Apolarity condition

Using the apolarity condition (4.26) of the Theorem 4.3.1

$$\nabla \omega_h = 0$$

This condition in same frame $\{E_1, E_2, E_3, E_4\}$ give by : for all $i = 1, \ldots, 4$ we have:

$$\nabla_{E_i} \omega_h = -2 \sum_{j=1}^{4} h(\nabla_{E_i} E_j, Ej) = 0.$$

then the apolarity condition in the frame become

$$\sum_{j,k=1}^{4} \Gamma_{ij}^{k}\delta_{jk} = 0. \tag{5.7}$$

So we get :

$$\begin{aligned} \Gamma_{13}^{3} &= -\Gamma_{11}^{1} - \Gamma_{12}^{2} - \Gamma_{14}^{4}, & \Gamma_{23}^{3} &= -\Gamma_{21}^{1} - \Gamma_{22}^{2} - \Gamma_{24}^{4}, \\ \Gamma_{32}^{2} &= -\Gamma_{31}^{1} - \Gamma_{33}^{3} - \Gamma_{34}^{4}, & \Gamma_{42}^{2} &= -\Gamma_{41}^{1} - \Gamma_{43}^{3} - \Gamma_{44}^{4}. \end{aligned}$$

5.2 Second step : Gauss equations

Hence the only remaining equations are those obtained from the Gauss equation. Here, in case the frame is completely defined in an equiaffine invariant way we can use the fact that M is affine homogeneous implies that all connection coefficients are constant. This is of course not the case when the frame is not uniquely defined. Therefore before exploiting the Gauss equations we first look at the remaining degrees of freedom.

The fact that we have, the eigenvalues have multiplicity two, means that there are rotations possible in $\mathcal{E}_1 = span\{E_1, E_2\}$ and $\mathcal{E}_2 = span\{E_3, E_4\}$ which are the two eigenspaces relative to λ_1 and λ_2 respectively. These rotations are given by

$$\begin{cases} \tilde{E}_1 = \cos t\, E_1 + \sin t\, E_2, \\ \tilde{E}_2 = -\sin t\, E_1 + \cos t\, E_2, \end{cases} \quad and \quad \begin{cases} \tilde{E}_3 = \cos s\, E_3 + \sin s\, E_4, \\ \tilde{E}_4 = -\sin s\, E_3 + \cos s\, E_4, \end{cases} . \tag{5.8}$$

However we see that the vectors T_1, T_2 defined by

$$\begin{cases} T_1 = K(\tilde{E}_1, \tilde{E}_1) + K(\tilde{E}_2, \tilde{E}_2) = K(E_1, E_1) + K(E_2, E_2), \\ T_2 = K(\tilde{E}_3, \tilde{E}_3) + K(\tilde{E}_4, \tilde{E}_4) = K(E_3, E_3) + K(E_4, E_4), \end{cases} \tag{5.9}$$

are independent under such rotations and therefore defined in an equiaffine invariant way. Moreover if we write $T_i = V_i + W_i$, where $V_i \in \mathcal{E}_1$ and $W_i \in \mathcal{E}_2$ for $i = 1, 2$ both components are defined in an affine invariant way. Of course the apolarity condition gives

$$\begin{cases} V_1 + V_2 = 0, \\ W_1 + W_2 = 0. \end{cases}$$

If necessary up to interchanging the two eigenspaces, we have to investigate the following 3 cases.

$$\text{Type 1 :} \quad \begin{cases} V_1 \neq 0, \\ W_2 \neq 0, \end{cases} \tag{5.10}$$

$$\text{Type 2 :} \qquad \begin{cases} V_1 \neq 0, \\ W_2 = 0, \end{cases} \tag{5.11}$$

$$\text{Type 3 :} \qquad \begin{cases} V_1 = 0, \\ W_2 = 0. \end{cases} \tag{5.12}$$

We will study all types, one after the other, in different chapters. Note that in the first case, the frame is uniquely determined, as we may assume that E_1 is in the direction of V_1 and E_3 is in the direction of W_2. In the second case we can still choose E_1 in the direction of V_1 and we retain the rotation freedom in the bundle $\mathcal{E}_2$. In the third case we retain the rotation freedom in both bundles. In the next chapters we will show how to introduce also in those cases a unique frame, in order to be able to apply the next lemma.

Note that in all the cases, by (5.9) we still have that

$$\begin{aligned} \Gamma_{22}^2 &= -\Gamma_{21}^1, \\ \Gamma_{44}^4 &= -\Gamma_{43}^3. \end{aligned}$$

We recall that the Gauss equation (4.20) in Theorem 4.4.2 is given by

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY$$

Then by evaluating the Gauss equation in our frame, we get by a straightforward calculation that

Lemma 5.2.1 *Let M be a locally strongly convex, affine homogeneous hypersurface. Assume that M has two distinct eigenvalues both of multiplicity 2. Suppose that the frame is chosen in a unique affine invariant way such that all the previous formulas remain valid. Then all the connection coefficients are constant and therefore the Gauss equations imply that*

$$\begin{aligned}
&G1211 \quad \Gamma_{11}^1(\Gamma_{11}^2 - \Gamma_{21}^1) - \Gamma_{12}^2(2\Gamma_{11}^2 + \Gamma_{21}^1) + 3\Gamma_{21}^1\Gamma_{21}^2 = 0,\\
&G1212 \quad (\Gamma_{21}^2)^2 - \Gamma_{11}^1\Gamma_{21}^2 + \Gamma_{11}^2(\Gamma_{11}^2 + \Gamma_{21}^1) + \lambda_1 = 0,\\
&G1213 \quad -\Gamma_{12}^3(2\Gamma_{11}^1 + 2\Gamma_{12}^2 + \Gamma_{14}^4 - 2\Gamma_{21}^2) + \Gamma_{12}^4(\Gamma_{34}^1 - \Gamma_{13}^4) - 2(\Gamma_{24}^4(\Gamma_{32}^2 + \Gamma_{33}^3 + \Gamma_{34}^4)\\
&\qquad\qquad + \Gamma_{11}^2(2\Gamma_{32}^2 + \Gamma_{33}^3 + \Gamma_{34}^4) + (\Gamma_{23}^4 - \Gamma_{34}^2)\Gamma_{42}^2) = 0,\\
&G1214 \quad \Gamma_{12}^3\Gamma_{13}^4 - \Gamma_{12}^4(\Gamma_{11}^1 + \Gamma_{12}^2 - \Gamma_{14}^4 - 2\Gamma_{21}^2) + 2\Gamma_{23}^4(\Gamma_{32}^2 + \Gamma_{33}^3 + \Gamma_{34}^4) \quad + 2(\Gamma_{24}^4 - 2\Gamma_{11}^2)\Gamma_{42}^2 = 0,\\
&G1221 \quad -(\Gamma_{11}^2 - 3\Gamma_{21}^1)(\Gamma_{11}^2 - 2\Gamma_{21}^1) + (2\Gamma_{12}^2 - \Gamma_{21}^2)(\Gamma_{11}^1 - 2\Gamma_{12}^2 + \Gamma_{21}^2) - \lambda_1 = 0,\\
&G1222 \quad 3\Gamma_{11}^2\Gamma_{12}^2 - 3\Gamma_{21}^1\Gamma_{21}^2 = 0,\\
&G1223 \quad \Gamma_{12}^3(2\Gamma_{11}^2 - 4\Gamma_{21}^1 + \Gamma_{24}^4) + \Gamma_{12}^4(\Gamma_{23}^4 - \Gamma_{34}^2) - 2(\Gamma_{11}^1\Gamma_{32}^2 + \Gamma_{14}^4\Gamma_{32}^2 - 2\Gamma_{21}^2\Gamma_{32}^2 - \Gamma_{21}^2\Gamma_{33}^3\\
&\qquad\qquad - \Gamma_{21}^2\Gamma_{34}^4 + \Gamma_{12}^2(5\Gamma_{32}^2 + 2(\Gamma_{33}^3 + \Gamma_{34}^4)) + (\Gamma_{13}^4 - \Gamma_{34}^1)\Gamma_{42}^2) = 0,\\
&G1224 \quad -\Gamma_{12}^3\Gamma_{23}^4 + \Gamma_{12}^4(2\Gamma_{11}^2 - 4\Gamma_{21}^1 - \Gamma_{24}^4) + 2\Gamma_{13}^4\Gamma_{32}^2 + 2(-4\Gamma_{12}^2 + \Gamma_{14}^4 + 2\Gamma_{21}^2)\Gamma_{42}^2 = 0,\\
&G1234 \quad (\Gamma_{11}^1 + 2\Gamma_{14}^4 + \Gamma_{21}^2)\Gamma_{23}^4 + \Gamma_{13}^4(\Gamma_{11}^2 - \Gamma_{21}^1 - 2\Gamma_{24}^4) = 0,\\
&G1243 \quad (\Gamma_{11}^1 + 2(\Gamma_{12}^2 + \Gamma_{14}^4) - \Gamma_{21}^2)(\Gamma_{23}^4 - \Gamma_{34}^2) - (\Gamma_{11}^2 - \Gamma_{21}^1 + 2\Gamma_{24}^4)(\Gamma_{13}^4 - \Gamma_{34}^1) = 0,\\
&G1244 \quad \Gamma_{14}^4(\Gamma_{11}^2 - \Gamma_{21}^1) + (\Gamma_{21}^2 - \Gamma_{12}^2)\Gamma_{24}^4 - \Gamma_{23}^4\Gamma_{34}^1 + \Gamma_{13}^4\Gamma_{34}^2 = 0,\\
&G1311 \quad -\Gamma_{11}^2\Gamma_{12}^3 + 3\Gamma_{21}^1\Gamma_{31}^2 - (6\Gamma_{11}^1 + 5(\Gamma_{12}^2 + \Gamma_{14}^4))(\Gamma_{32}^2 + \Gamma_{33}^3 + \Gamma_{34}^4) + (\Gamma_{13}^4 + 2\Gamma_{34}^1)\Gamma_{42}^2 = 0,\\
&G1312 \quad (2\Gamma_{12}^2 + \Gamma_{14}^4 + \Gamma_{21}^2)\Gamma_{31}^2 - 4\Gamma_{24}^4(\Gamma_{32}^2 + \Gamma_{33}^3 + \Gamma_{34}^4) - \Gamma_{11}^2(3\Gamma_{32}^2 + 2(\Gamma_{33}^3 + \Gamma_{34}^4)) - \Gamma_{13}^4\Gamma_{41}^2
\end{aligned}$$

$+ 2\Gamma^2_{34}\Gamma^2_{42} = 0,$

$G1313$ $2(\Gamma^3_{12}\Gamma^2_{31} + (\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34})(2\Gamma^2_{32} + 3\Gamma^3_{33} + 2\Gamma^4_{34}) - \Gamma^2_{42}(\Gamma^4_{33} - 2\Gamma^3_{43})) + \lambda_2 = 0,$

$G1314$ $2(\Gamma^4_{12}\Gamma^2_{31} + \Gamma^4_{33}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) + (2(\Gamma^2_{32} + \Gamma^3_{33}) + 3\Gamma^4_{34})\Gamma^2_{42}) = 0,$

$G1321$ $\Gamma^3_{12}(4\Gamma^1_{11} + 2\Gamma^2_{12} + 3\Gamma^4_{14}) - (2\Gamma^1_{11} - 2\Gamma^2_{12} + \Gamma^4_{14} + \Gamma^2_{21})\Gamma^2_{31} - (\Gamma^2_{11} - 2\Gamma^1_{21})\Gamma^2_{32}$
$- \Gamma^4_{12}(\Gamma^4_{13} + \Gamma^1_{34}) + \Gamma^4_{13}\Gamma^2_{41} = 0,$

$G1322$ $\Gamma^3_{12}(\Gamma^2_{11} + 2\Gamma^4_{24}) - 3\Gamma^1_{21}\Gamma^2_{31} + (\Gamma^1_{11} + \Gamma^4_{14})\Gamma^2_{32} - \Gamma^4_{12}\Gamma^2_{34} - \Gamma^2_{12}(\Gamma^3_{33} + \Gamma^4_{34}) - \Gamma^4_{13}\Gamma^2_{42} = 0,$

$G1323$ $2\Gamma^2_{31}(2\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) - \Gamma^3_{12}(2\Gamma^2_{32} + 4\Gamma^3_{33} + 3\Gamma^4_{34}) + \Gamma^4_{12}(\Gamma^4_{33} - 2\Gamma^3_{43}) = 0,$

$G1324$ $- \Gamma^4_{12}(\Gamma^3_{33} + 2\Gamma^4_{34}) + 4\Gamma^2_{31}\Gamma^2_{42} - \Gamma^3_{12}(\Gamma^4_{33} + 2\Gamma^2_{42}) = 0,$

$G1331$ $- 2((\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})(3\Gamma^1_{11} + 2(\Gamma^2_{12} + \Gamma^4_{14})) - (\Gamma^2_{11} - 2\Gamma^1_{21})\Gamma^4_{24} + \Gamma^4_{13}\Gamma^1_{34}) - \lambda_1 = 0,$

$G1332$ $- 2(\Gamma^2_{11}(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14}) + (2\Gamma^1_{11} + 3\Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^4_{24} + \Gamma^4_{13}\Gamma^2_{34}) = 0,$

$G1333$ $- \Gamma^4_{24}(2\Gamma^3_{12} + \Gamma^2_{31}) + \Gamma^4_{33}\Gamma^1_{34} + (\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})(5\Gamma^2_{32} + 6\Gamma^3_{33} + 5\Gamma^4_{34}) - 3\Gamma^4_{13}\Gamma^3_{43} = 0,$

$G1341$ $- 2\Gamma^4_{13}(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14}) + (4\Gamma^1_{11} + 3\Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^1_{34} - (\Gamma^2_{11} - 2\Gamma^1_{21})\Gamma^2_{34} = 0,$

$G1342$ $- 4\Gamma^4_{13}\Gamma^4_{24} + (\Gamma^2_{11} + 2\Gamma^4_{24})\Gamma^1_{34} + (\Gamma^1_{11} + 2\Gamma^2_{12})\Gamma^2_{34} = 0,$

$G1344$ $\Gamma^4_{24}\Gamma^2_{31} - \Gamma^4_{14}(\Gamma^2_{32} + \Gamma^3_{33}) + \Gamma^4_{12}\Gamma^2_{34} + (\Gamma^1_{11} + \Gamma^2_{12})\Gamma^4_{34} - \Gamma^1_{34}(\Gamma^4_{33} + 2\Gamma^2_{42}) + 3\Gamma^4_{13}\Gamma^3_{43} = 0,$

$G1411$ $- \Gamma^2_{11}\Gamma^4_{12} - (\Gamma^4_{13} - 3\Gamma^1_{34})(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) + 3\Gamma^1_{21}\Gamma^2_{41} - (\Gamma^1_{11} - 5\Gamma^4_{14})\Gamma^2_{42} = 0,$

$G1412$ $\Gamma^2_{31}(\Gamma^4_{13} - \Gamma^1_{34}) + 2\Gamma^2_{34}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) + (-\Gamma^1_{11} + \Gamma^2_{12} - \Gamma^4_{14} + \Gamma^2_{21})\Gamma^2_{41}$
$+ (4\Gamma^4_{24} - 3\Gamma^2_{11})\Gamma^2_{42} = 0,$

$G1413$ $2(\Gamma^3_{12}\Gamma^2_{41} + \Gamma^2_{32}(2\Gamma^2_{42} + \Gamma^3_{43}) + \Gamma^3_{33}(2\Gamma^2_{42} + \Gamma^3_{43}) + \Gamma^4_{34}(4\Gamma^2_{42} + \Gamma^3_{43}) - \Gamma^2_{42}\Gamma^4_{43}) = 0,$

$G1414$ $4(\Gamma^2_{42})^2 - 2\Gamma^3_{43}\Gamma^2_{42} + 2\Gamma^4_{12}\Gamma^2_{41} + 2(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34})\Gamma^4_{43} + \lambda_2 = 0,$

$G1422$ $\Gamma^4_{12}(\Gamma^2_{11} - 2\Gamma^4_{24}) + \Gamma^2_{32}(\Gamma^4_{13} - \Gamma^1_{34}) - \Gamma^3_{12}\Gamma^2_{34} - 3\Gamma^1_{21}\Gamma^2_{41} - (\Gamma^2_{12} + \Gamma^4_{14})\Gamma^2_{42} = 0,$

$G1423$ $2(2\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34})\Gamma^2_{41} - \Gamma^3_{12}\Gamma^3_{43} + \Gamma^4_{12}(\Gamma^4_{43} - 2(\Gamma^2_{32} + \Gamma^3_{33} + 2\Gamma^4_{34})) = 0,$

$G1424$ $4\Gamma^2_{41}\Gamma^2_{42} + \Gamma^4_{12}(\Gamma^3_{43} - 2\Gamma^2_{42}) - \Gamma^3_{12}\Gamma^4_{43} = 0$

$G1431$ $- 2\Gamma^4_{13}(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14}) + (4\Gamma^1_{11} + 3\Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^1_{34} - (\Gamma^2_{11} - 2\Gamma^1_{21})\Gamma^2_{34} = 0,$

$G1433$ $\Gamma^3_{12}\Gamma^2_{34} + \Gamma^4_{13}(\Gamma^3_{33} - 2\Gamma^4_{34}) - \Gamma^4_{24}\Gamma^2_{41} + (\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})\Gamma^2_{42} - \Gamma^4_{14}\Gamma^3_{43}$
$+ \Gamma^1_{34}(-2\Gamma^2_{32} - 3\Gamma^3_{33} - 2\Gamma^4_{34} + \Gamma^4_{43}) = 0,$

$G1441$ $2((\Gamma^1_{11} - 2\Gamma^4_{14})\Gamma^4_{14} - (\Gamma^2_{11} - 2\Gamma^1_{21})\Gamma^4_{24} + (\Gamma^4_{13} - \Gamma^1_{34})\Gamma^1_{34}) - \lambda_1 = 0,$

$G1442$ $2(\Gamma^2_{11}\Gamma^4_{14} + (\Gamma^2_{12} - 2\Gamma^4_{14})\Gamma^4_{24} + (\Gamma^4_{13} - \Gamma^1_{34})\Gamma^2_{34}) = 0,$

$G1444$ $3\Gamma^4_{13}\Gamma^4_{34} + \Gamma^4_{24}(2\Gamma^4_{12} + \Gamma^2_{41}) + \Gamma^4_{14}(\Gamma^3_{43} - 5\Gamma^2_{42}) - \Gamma^1_{34}(\Gamma^4_{34} + \Gamma^4_{43}) = 0,$

$G2311$ $\Gamma^3_{12}(3\Gamma^1_{11} + 2(\Gamma^2_{12} + \Gamma^4_{14}) - \Gamma^2_{21}) - (\Gamma^1_{11} - 2\Gamma^2_{12})\Gamma^2_{31} + \Gamma^1_{21}\Gamma^2_{32} - \Gamma^4_{12}\Gamma^1_{34}$
$- \Gamma^4_{24}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) + \Gamma^4_{23}\Gamma^2_{42} = 0,$

$G2312$ $\Gamma^3_{12}(\Gamma^2_{11} + 2\Gamma^4_{24}) + (-\Gamma^2_{11} - 2\Gamma^1_{21} + \Gamma^4_{24})\Gamma^2_{31} - \Gamma^4_{12}\Gamma^2_{34} - \Gamma^2_{21}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) - \Gamma^4_{23}\Gamma^2_{41} = 0,$

$G2313$ $2\Gamma^2_{31}(2\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) - \Gamma^3_{12}(2\Gamma^2_{32} + 4\Gamma^3_{33} + 3\Gamma^4_{34}) + \Gamma^4_{12}(\Gamma^4_{33} - 2\Gamma^3_{43}) = 0,$

$G2314$ $- \Gamma^4_{12}(\Gamma^3_{33} + 2\Gamma^4_{34}) + 4\Gamma^2_{31}\Gamma^2_{42} - \Gamma^3_{12}(\Gamma^4_{33} + 2\Gamma^2_{42}) = 0,$

$G2322$ $\Gamma^3_{12}(\Gamma^2_{12} + \Gamma^2_{21}) - 3\Gamma^2_{12}\Gamma^2_{31} - (\Gamma^1_{21} - 5\Gamma^4_{24})\Gamma^2_{32} - (\Gamma^4_{23} + 2\Gamma^2_{34})\Gamma^2_{42} = 0,$

$G2323$ $2((\Gamma^3_{12})^2) - \Gamma^2_{31}\Gamma^3_{12} + \Gamma^2_{32}(2\Gamma^2_{32} - \Gamma^3_{33}) + \Gamma^2_{42}(\Gamma^4_{33} - 2\Gamma^3_{43})) + \lambda_2 = 0,$

$G2331$ $- 2((\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})\Gamma^1_{21} + (2(\Gamma^1_{11} + 2\Gamma^2_{12} + \Gamma^4_{14}) - \Gamma^2_{21})\Gamma^4_{24} + \Gamma^4_{23}\Gamma^1_{34}) = 0,$

$G2332$ $- 2(2(\Gamma^4_{24})^2 - \Gamma^1_{21}\Gamma^4_{24} + (\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})\Gamma^2_{21} + \Gamma^4_{23}\Gamma^2_{34}) - \lambda_1 = 0,$

$G2333$ $- (\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})(3\Gamma^3_{12} - \Gamma^2_{31}) + \Gamma^4_{24}(\Gamma^3_{33} - 5\Gamma^2_{32}) + \Gamma^4_{33}\Gamma^2_{34} - 3\Gamma^4_{23}\Gamma^3_{43} = 0,$

$G2342$ $\Gamma^2_{21}\Gamma^1_{34} - \Gamma^1_{21}\Gamma^2_{34} + 2\Gamma^4_{24}(\Gamma^2_{34} - 2\Gamma^4_{23}) = 0,$

$G2343$ $- \Gamma^3_{12}(\Gamma^4_{13} - 2\Gamma^1_{34}) + \Gamma^2_{31}(\Gamma^4_{13} - \Gamma^1_{34}) + \Gamma^2_{34}(3\Gamma^2_{32} - \Gamma^3_{33} + \Gamma^4_{34}) + \Gamma^4_{24}(\Gamma^4_{33} - 2\Gamma^3_{43})$
$+ \Gamma^4_{23}(-\Gamma^2_{32} + \Gamma^3_{33} - 3\Gamma^4_{34} + \Gamma^4_{43}) = 0,$

$G2411$ $\Gamma^4_{12}(\Gamma^1_{11} - 2\Gamma^4_{14} - \Gamma^2_{21}) - \Gamma^3_{12}\Gamma^1_{34} - (\Gamma^4_{23} - \Gamma^2_{34})(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) - (\Gamma^1_{11} - 2\Gamma^2_{12})\Gamma^2_{41}$
$+ (\Gamma^1_{21} + \Gamma^4_{24})\Gamma^2_{42} = 0,$

$G2422$ $\Gamma^4_{12}(\Gamma^2_{12} + \Gamma^2_{21}) + \Gamma^2_{32}(\Gamma^4_{23} - 3\Gamma^2_{34}) - 3\Gamma^2_{12}\Gamma^2_{41} - (\Gamma^1_{21} + 5\Gamma^4_{24})\Gamma^2_{42} = 0,$

$G2424$ $2((\Gamma^4_{12})^2) - \Gamma^2_{41}\Gamma^4_{12} + \Gamma^2_{42}(2\Gamma^2_{42} + \Gamma^3_{43}) - \Gamma^2_{32}\Gamma^4_{43}) + \lambda_2 = 0,$

$G2431$ $- 2(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^4_{23} + \Gamma^1_{21}\Gamma^1_{34} + (2(\Gamma^1_{11} + 2\Gamma^2_{12} + \Gamma^4_{14}) - \Gamma^2_{21})\Gamma^2_{34} = 0,$

$G2441$ $2(\Gamma^4_{14}(\Gamma^1_{21} - 2\Gamma^4_{24}) + (2\Gamma^2_{12} - \Gamma^2_{21})\Gamma^4_{24} + \Gamma^1_{34}(\Gamma^4_{23} - \Gamma^2_{34})) = 0,$

$$
\begin{aligned}
&G2442 \quad 2(\Gamma^4_{14}\Gamma^2_{21} - \Gamma^4_{24}(\Gamma^1_{21} + 2\Gamma^4_{24}) + (\Gamma^4_{23} - \Gamma^2_{34})\Gamma^2_{34}) - \lambda_1 = 0,\\
&G2444 \quad 3\Gamma^4_{23}\Gamma^4_{34} + \Gamma^4_{14}(3\Gamma^4_{12} - \Gamma^2_{41}) + \Gamma^4_{24}(5\Gamma^2_{42} + \Gamma^3_{43}) - \Gamma^2_{34}(\Gamma^4_{34} + \Gamma^4_{43}) = 0,\\
&G3411 \quad (\Gamma^3_{12} - \Gamma^2_{31})\Gamma^2_{41} + \Gamma^2_{31}(\Gamma^2_{41} - \Gamma^4_{12}) - (\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34})(\Gamma^4_{33} - \Gamma^3_{43}) + \Gamma^2_{42}(\Gamma^4_{34} - \Gamma^4_{43}) = 0,\\
&G3431 \quad -2(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^4_{33} + (\Gamma^3_{12} - \Gamma^2_{31})\Gamma^2_{34} + 2\Gamma^4_{24}(\Gamma^4_{12} - \Gamma^2_{41}) - 2(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})\Gamma^2_{42}\\
&\qquad\qquad - \Gamma^1_{34}(\Gamma^2_{32} + 2(\Gamma^3_{33} + \Gamma^4_{34} - \Gamma^4_{43})) = 0,\\
&G3433 \quad \Gamma^3_{33}(\Gamma^4_{33} - \Gamma^3_{43}) - \Gamma^4_{34}(2\Gamma^4_{33} + \Gamma^3_{43}) + 3\Gamma^3_{43}\Gamma^4_{43} = 0,\\
&G3434 \quad (\Gamma^4_{43})^2 - \Gamma^3_{33}\Gamma^4_{43} + \Gamma^4_{33}(\Gamma^4_{33} + \Gamma^3_{43}) + \lambda_2 = 0,\\
&G3441 \quad 2\Gamma^4_{24}(\Gamma^3_{12} - \Gamma^2_{31}) + \Gamma^2_{34}(\Gamma^2_{41} - \Gamma^4_{12}) + \Gamma^1_{34}(2\Gamma^4_{33} + \Gamma^2_{42} - 4\Gamma^3_{43}) - 2\Gamma^4_{14}(\Gamma^2_{32} + \Gamma^3_{33} + 5\Gamma^4_{34}\\
&\qquad\qquad - 2\Gamma^4_{43}) - 2(\Gamma^1_{11} + \Gamma^2_{12})(2\Gamma^4_{34} - \Gamma^4_{43}) = 0,\\
&G3442 \quad 2\Gamma^4_{14}\Gamma^2_{31} - \Gamma^1_{34}\Gamma^2_{41} + \Gamma^2_{34}(2\Gamma^4_{33} - \Gamma^2_{42} - 4\Gamma^3_{43}) + 2\Gamma^4_{24}(\Gamma^2_{32} - 4\Gamma^4_{34} + 2\Gamma^4_{43}) = 0,\\
&G3443 \quad -(\Gamma^4_{33} - 3\Gamma^3_{43})(\Gamma^4_{33} - 2\Gamma^3_{43}) + (2\Gamma^4_{34} - \Gamma^4_{43})(\Gamma^3_{33} - 2\Gamma^4_{34} + \Gamma^4_{43}) - \lambda_2 = 0.
\end{aligned}
$$

Note that in the previous lemma $G\,ijk\ell$ means the component of Gauss equation for $X = E_i, Y = E_j, Z = E_k$ in the direction of E_ℓ.

Chapter 6

Hypersurfaces of type 1

The only reference to this chapter is the paper of A. Chikh Salah and L. Vrancken [CSV16].
This chapter is consecrated to the studies of type 1 given by the system (5.10)

$$\begin{cases} V_1 \neq 0, \\ W_2 \neq 0, \end{cases}$$

As mentioned before the frame is completely determined and therefore we can apply the previous lemma. We will show that this case leads to a contradiction. As M is of Type 1, we have that

$$\Gamma_{11}^1 + \Gamma_{12}^2 \neq 0, \qquad \Gamma_{33}^3 + \Gamma_{34}^4 \neq 0. \tag{6.1}$$

Adding (G1211) and (G1222), we get that

$$(\Gamma_{11}^1 + \Gamma_{12}^2)(\Gamma_{11}^2 - \Gamma_{21}^1) = 0.$$

Using (6.1) we deduce that

$$\Gamma_{11}^2 = \Gamma_{21}^1.$$

Similarly we obtain from adding (G3433) and (G3444) that

$$\Gamma_{33}^4 = \Gamma_{43}^3.$$

Adding (G1221) and (G1212) we see that

$$2(\Gamma_{11}^1 - 2\Gamma_{12}^2)(\Gamma_{12}^2 - \Gamma_{21}^2) = 0. \tag{6.2}$$

Similarly from (G3443) and (G3434) we get that

$$2(\Gamma_{33}^3 - 2\Gamma_{34}^4)(\Gamma_{34}^4 - \Gamma_{43}^4) = 0. \tag{6.3}$$

So if necessary upto interchanging the vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ there are 3 sub-cases to consider.

$$\begin{cases} \textbf{Case I}: & \Gamma^2_{12} \neq \Gamma^2_{21}, & \Gamma^4_{34} \neq \Gamma^4_{43} \\ \textbf{Case II}: & \Gamma^2_{12} = \Gamma^2_{21}, & \Gamma^4_{34} = \Gamma^4_{43}. \\ \textbf{Case III}: & \Gamma^2_{12} \neq \Gamma^2_{21}, & \Gamma^4_{34} = \Gamma^4_{43}. \end{cases}$$

Case I : $\Gamma^2_{12} \neq \Gamma^2_{21}$ and $\Gamma^4_{34} \neq \Gamma^4_{43}$.
Hence, from the equations (6.2) and (6.3), we must have that

$$\Gamma^2_{12} = \tfrac{1}{2}\Gamma^1_{11} \quad \text{and} \quad \Gamma^4_{34} = \tfrac{1}{2}\Gamma^3_{33}.$$

From (6.1), we see that the condition of this chapter is replaced by

$$\Gamma^1_{11} \neq 0 \quad \text{and} \quad \Gamma^3_{33} \neq 0.$$

Moreover (G1211) and (G3433) become, respectively,

$$-\frac{3}{2}\Gamma^1_{21}(\Gamma^1_{11} - 2\Gamma^2_{21}) = 0, \qquad -\frac{3}{2}\Gamma^3_{43}(\Gamma^3_{33} - 2\Gamma^4_{43}) = 0$$

it follows that $\Gamma^1_{21} = \Gamma^3_{43} = 0$.

Next we use (G1341) and the difference of (G1441) and (G1331). This gives us

$$\begin{aligned} 2\Gamma^4_{14}(\Gamma^1_{34} - 2\Gamma^4_{13}) + \Gamma^1_{11}(\frac{11}{2}\Gamma^1_{34} - 3\Gamma^4_{13}) = 0, \\ -2(\Gamma^1_{34})^2 + 4\Gamma^4_{13}\Gamma^1_{34} + 4\Gamma^1_{11}(3\Gamma^1_{11} + 4\Gamma^4_{14}) = 0. \end{aligned}$$

Therefore it follows from the previous equations that supposing that $\Gamma^1_{11} \neq -\frac{4}{3}\Gamma^4_{14}$, solving the first equations for Γ^4_{13} and substituting this result in the second equation, it give

$$\frac{4\Gamma^1_{11}\left((3\Gamma^1_{11} + 4\Gamma^4_{14})^2 + 4(\Gamma^1_{34})^2\right)}{3\Gamma^1_{11} + 4\Gamma^4_{14}} = 0,$$

leads to a contradiction. Therefore we must have that

$$\Gamma^4_{14} = -\tfrac{3}{4}\Gamma^1_{11} \quad \text{and} \quad \Gamma^1_{34} = 0.$$

Similarly from the symmetry of the problem in this case we deduce that also

$$\Gamma^3_{12} = 0 \quad \text{and} \quad \Gamma^2_{32} = -\tfrac{3}{4}\Gamma^3_{33}.$$

Using now (G1331) and (G1313) we immediately see that

$$\lambda_1 = -\tfrac{15}{4}(\Gamma^1_{11})^2,$$

$$\lambda_2 = -\tfrac{15}{4}(\Gamma_{33}^3)^2.$$

The equations (G1311) and (G1424) now become

$$\Gamma_{13}^4\Gamma_{42}^2 - \frac{57}{16}\Gamma_{11}^1\Gamma_{33}^3 = 0,$$
$$-2(\Gamma_{12}^4 - 2\Gamma_{41}^2)\Gamma_{42}^2 = 0,$$

which implies that

$$\Gamma_{42}^2 \neq 0 \quad \text{and} \quad \Gamma_{12}^4 = 2\Gamma_{41}^2.$$

Similarly we get interchanging the two vector bundles that also

$$\Gamma_{24}^4 \neq 0 \quad \text{and} \quad \Gamma_{34}^2 = 2\Gamma_{23}^4.$$

The equations (G4113 and (G2331)) now immediately imply that

$$\Gamma_{43}^4 = \tfrac{5}{2}\Gamma_{33}^3, \qquad \Gamma_{21}^2 = \tfrac{5}{2}\Gamma_{11}^1.$$

A contradiction now follows from :

$$\text{G1414} \;:\quad 4\left((\Gamma_{41}^2)^2 + (\Gamma_{42}^2)^2\right) = 0.$$

Then, there is no-solutions in the **case I**.

Case II : $\Gamma_{12}^2 = \Gamma_{21}^2, \qquad \Gamma_{34}^4 = \Gamma_{43}^4$.
The Gauss equation (G3422) is given by

$$\Gamma_{12}^4\Gamma_{31}^2 - \Gamma_{12}^3\Gamma_{41}^2 = 0;$$

If we suppose that Γ_{31}^2 and Γ_{41}^2 are nonzero concurrently, than, there exist θ_1 such that :

$$\Gamma_{12}^4 = \theta_1\Gamma_{41}^2 \quad \text{and} \quad \Gamma_{12}^3 = \theta_1\Gamma_{31}^2.$$

So $\mathcal{E}_1$ and $\mathcal{E}_2$ are symmetric, we get the same result in second subspace :

$$\text{G1244} \;:\; -\Gamma_{23}^4\Gamma_{34}^1 + \Gamma_{13}^4\Gamma_{34}^2 = 0;$$

If Γ_{13}^4 and Γ_{23}^4 are nonzero concurrently, than, there exist ϑ_1 such that

$$\Gamma_{34}^2 = \vartheta_1\Gamma_{23}^4 \quad \text{and} \quad \Gamma_{34}^1 = \vartheta_1\Gamma_{13}^4.$$

We have the Guass equations G3412 and G1234

$$-2\Gamma_{31}^2\Gamma_{42}^2 + \Gamma_{41}^2(2\Gamma_{32}^2 + \Gamma_{33}^3 + \Gamma_{43}^4) = 0,$$
$$(\Gamma_{11}^1 + 2\Gamma_{14}^4 + \Gamma_{21}^2)\Gamma_{23}^4 - 2\Gamma_{13}^4\Gamma_{24}^4 = 0.$$

With the same conditions, we get that there exist θ_2 and ϑ_2 such that :

$$\Gamma_{32}^2 = \theta_2\Gamma_{31}^2 - \frac{1}{2}(\Gamma_{33}^3 - 1\Gamma_{43}^4), \quad \Gamma_{42}^2 = \theta_2\Gamma_{41}^2,$$
$$\Gamma_{14}^4 = \vartheta_2\Gamma_{13}^4 - \frac{1}{2}(\Gamma_{11}^1 - \Gamma_{21}^2), \quad \Gamma_{24}^4 = \vartheta_2\Gamma_{23}^4.$$

See the equations G2342 and G4124, we notice two possible sub-cases, (we give just the proof of the first case, in the same way we can prove the second)
If $\vartheta_1 = 0$ than G2341+G1341 give

$$-4\vartheta_2\left((\Gamma_{23}^4)^2 + (\Gamma_{12}^3)^2\right) = 0,$$

using the conditions before we get : $\vartheta_2 = 0$. Let the system

$$\begin{cases} \text{G1331 :} & -\lambda_1 - (\Gamma_{11}^1 + \Gamma_{21}^2)(2\Gamma_{11}^1 + \Gamma_{21}^2) = 0, \\ \text{G2332 :} & -\Gamma_{21}^1(\Gamma_{11}^1 + \Gamma_{21}^2) = 0, \\ \text{G1221 :} & -\lambda_1 - 2(\Gamma_{21}^1)^2 + (\Gamma_{11}^1 - \Gamma_{21}^2)\Gamma_{21}^2 = 0, \end{cases}$$

solving this system we get complex result to the Gammas, who is a contradiction.

Remark 6.0.1 *In the same way, we find the contradiction in the case where* $\Gamma_{31}^2 = \Gamma_{41}^2 = 0$ *and* $\Gamma_{13}^4 = \Gamma_{23}^4 = 0$.

Then, there is no-solutions in the **case II**.

With similar computations, we can also prove the non-existence of solutions in the **cases III**, which is given by :

$$\textbf{Case III} : \Gamma_{12}^2 \neq \Gamma_{21}^2, \qquad \Gamma_{12}^2 = \frac{1}{2}\Gamma_{11}^1, \qquad \Gamma_{34}^4 = \Gamma_{43}^4.$$

Hence there is not hypersurface in $\mathbb{R}^5$ of **type 1**.

Chapter 7

Hypersurfaces of type 2

The only reference to this chapter is the paper of A. Chikh Salah and L. Vrancken [CSV16].

In this case we have that $V_1 \neq 0$ and $W_2 = 0$. We choose E_1 in the direction of V_1. This fixes the rotation freedom in $\mathcal{E}_1$. As W_2 vanishes, we have that $\Gamma_{34}^4 = -\Gamma_{33}^3$. In order to determine a frame in $\mathcal{E}_2$ we consider the following subcases:

1. The restriction of the difference tensor K to the space $\mathcal{E}_2$ does not vanish identically. Applying a rotation we can choose in this case the vector field E_3 such that $h(K(E_3, E_3), E_4) = 0$. So we have that $\Gamma_{43}^3 = 0$ and $\Gamma_{33}^3 \neq 0$.

2. The restriction of the difference tensor K to the space $\mathcal{E}_2$ vanishes identically. However $K(\mathcal{E}_1, \mathcal{E}_1) \not\subset \mathcal{E}_1$. In that case we can fix the vector field E_3 such that one of the following holds:

   (a) $h(K(E_1, E_1), E_3) \neq 0$ and $h(K(E_1, E_1), E_4) = 0$,

   (b) $h(K(E_1, E_2), E_3) \neq 0$ and
   $h(K(E_1, E_1), E_3) = h(K(E_1, E_1), E_4) = h(K(E_1, E_2), E_4) = 0$,

   (c) $h(K(E_2, E_2), E_3) \neq 0$ and
   $h(K(E_1, E_1), E_3) = h(K(E_1, E_1), E_4) = h(K(E_1, E_2), E_3) = 0$
   and $h(K(E_1, E_2), E_4) = h(K(E_2, E_2), E_4) = 0$,
   which leads to an immediate contradiction as $W_1 = 0$.

3. We have that $h(K(\mathcal{E}_1, \mathcal{E}_1), \mathcal{E}_2) = h(K(\mathcal{E}_2, \mathcal{E}_2), \mathcal{E}_2) = 0$. In this case we can consider the following subcases:

   (a) the symmetric operator K_{E_1} restricted to $\mathcal{E}_2$ has two distinct eigenvalues. In this case we can pick E_3 and E_4 in the direction of the corresponding eigenspaces,

(b) the symmetric operator K_{E_1} restricted to $\mathcal{E}_2$ is a multiple of the identity, but the symmetric operator K_{E_2} restricted to $\mathcal{E}_2$ has two distinct eigenvalues. In this case we can again pick E_3 and E_4 in the direction of the corresponding eigenspaces,

(c) both the symmetric operators K_{E_1} and K_{E_2} restricted to $\mathcal{E}_2$ are a multiple of the identity,

So we see that except in the last case, the frame is uniquely determined and we can apply the Lemma 5.2.1.

7.1 Hypersurfaces of type 2.1

We have that $\Gamma^3_{43} = 0$ and as a conditions of this case :

$$\Gamma^3_{33} \neq 0 \quad \text{and} \quad \Gamma^1_{11} + \Gamma^2_{12} \neq 0. \tag{7.1}$$

From the Gauss equation

$$\text{G3433}: \quad 3\Gamma^3_{33}\Gamma^4_{33} = 0,$$

it immediately follows that $\Gamma^4_{33} = 0$. Adding now $(G3443)$ and $(G3434)$ we get

$$-6\Gamma^3_{33}(\Gamma^3_{33} + \Gamma^4_{43}) = 0,$$

which implies that $\Gamma^4_{43} = -\Gamma^3_{33}$. Now $(G1424)$ becomes

$$\Gamma^3_{12}\Gamma^3_{33} + 2\Gamma^1_{41}(\Gamma^4_{12} - 2\Gamma^2_{41}) = 0.$$

So there exist a constant β_1 such that

$$\begin{aligned} \Gamma^4_{12} &= \beta_1\Gamma^3_{33} + 2\Gamma^2_{41}, \\ \Gamma^3_{12} &= -2\beta_1\Gamma^1_{41}. \end{aligned}$$

Moreover from the equation

$$\text{G3434}: \qquad \lambda_2 + 2(\Gamma^3_{33})^2 = 0,$$

we deduce that $\lambda_2 = -2(\Gamma^3_{33})^2$. And from

$$\text{G1423}: \qquad \Gamma^3_{33}\left(\beta_1(2\Gamma^1_{31} + \Gamma^3_{33}) + 2\Gamma^2_{41}\right) = 0,$$

we get that $\Gamma^2_{41} = -\frac{\beta_1}{2}(2\Gamma^1_{31} + \Gamma^3_{33})$. Seeing the equation

$$G1413 \qquad 2(1+\beta_1^2)(2\Gamma^1_{31} + \Gamma^3_{33})\Gamma^1_{41} = 0,$$

we have two small subcases to consider
1. if $\Gamma_{41}^{1}=0$ then from solving

$$\begin{cases} G1313 & 2(\Gamma_{31}^{1}-\Gamma_{33}^{3})(2\Gamma_{31}^{1}+\Gamma_{33}^{3})=0,\\ G2323 & 2(2\Gamma_{31}^{1}-\Gamma_{33}^{3})(\Gamma_{31}^{1}+\Gamma_{33}^{3})=0,\end{cases}$$

we have that $\Gamma_{31}^{1}=0$ and $\Gamma_{33}^{3}=0$, which is a contradiction with condition of this case.
2. if $\Gamma_{41}^{1}\neq 0$ then $\Gamma_{31}^{1}=-\frac{1}{2}\Gamma_{33}^{3}$. From the equation

$$G2423 \qquad -4\left(\beta_1^2+1\right)\Gamma_{33}^{3}\Gamma_{41}^{1}=0,$$

we again find that $\Gamma_{33}^{3}=0$, which is a contradiction with our assumptions.
Then there is not hypersurfaces of **type 2.1** in $\mathbb{R}^5$.

7.2 Hypersurfaces of type 2.2

Type 2.2.a We have that $\Gamma_{43}^{3}=\Gamma_{33}^{3}=\Gamma_{41}^{1}=0$ and as condition of this case

$$\Gamma_{31}^{1}\neq 0 \quad \text{and} \quad \Gamma_{11}^{1}+\Gamma_{12}^{2}\neq 0. \tag{7.2}$$

By adding $G1211$ and $G1222$, we have

$$\text{G1211+G1222}: \quad (\Gamma_{11}^{1}+\Gamma_{12}^{2})(\Gamma_{11}^{2}-\Gamma_{21}^{1})=0,$$

it follows that $\Gamma_{11}^{2}=\Gamma_{21}^{1}$. From $G1423$ we get that

$$\Gamma_{12}^{4}(2\Gamma_{31}^{1}+\Gamma_{43}^{4})-4\Gamma_{31}^{1}\Gamma_{41}^{2}=0.$$

As $\Gamma_{31}^{1}\neq 0$ (condition (7.2)), there exists a constant β_1 such that

$$\Gamma_{43}^{4}=(4\beta_1-2)\Gamma_{31}^{1},$$
$$\Gamma_{41}^{2}=\beta_1\Gamma_{12}^{4}.$$

Similarly it follows from $G1323$ that we can write

$$\Gamma_{12}^{3}=\beta_2\Gamma_{33}^{4}+2\Gamma_{31}^{2},$$
$$\Gamma_{12}^{4}=-2\beta_2\Gamma_{31}^{1}.$$

The equations ($G1314$) and ($G1414$) are given by

$$-2\Gamma_{31}^{1}(2\beta_2\Gamma_{31}^{2}+\Gamma_{33}^{4})=0 \quad \text{and} \quad \lambda_2+4(1+2\beta_1(-1+\beta_2^2))(\Gamma_{31}^{1})^2=0,$$

now immediately imply respectively that

$$\Gamma_{33}^4 = -2\beta_2\Gamma_{31}^2, \qquad \lambda_2 = -4(1+2\beta_1(-1+\beta_2^2))\Gamma_{31}^1.$$

Now the equations ($G1313$) and ($G2424$) reduces to

$$\begin{cases} -4\,(\beta_2^2-1)\,(2\beta_1(\Gamma_{31}^1)^2+(\Gamma_{31}^2)^2)=0, \\ -8(-1+2\beta_1)\,(\beta_2^2-1)\,(\Gamma_{31}^1)^2=0. \end{cases}$$

Hence $\beta_2^2 = 1$. Note that if necessary by replacing E_3 with $-E_3$ we may assume that $\beta_2 = -1$. The equation ($G2322$) then become

$$(\Gamma_{21}^1 - 5\Gamma_{24}^4)\Gamma_{31}^1 - 3\Gamma_{12}^2\Gamma_{31}^2 = 0.$$

So there exists a constant β_3 such that

$$\Gamma_{31}^2 = \beta_3\Gamma_{31}^1, \quad \Gamma_{21}^1 = 3\beta_3\Gamma_{12}^2 + 5\Gamma_{24}^4.$$

The equation of ($G1433$) becomes

$$-2\beta_1\Gamma_{31}^1(\Gamma_{24}^4 - 2\Gamma_{43}^1) = 0.$$

If $\Gamma_{24}^4 \neq 2\Gamma_{43}^1$, then $\beta_1 = 0$, and the equation

$$G3421: \quad -2\beta_3^2(\Gamma_{31}^1)^2 = 0,$$

gives $\beta_3 = 0$. The equations ($G3442$) and ($G2311$)

$$-10\Gamma_{24}^4\Gamma_{31}^1 = 0 \quad \text{and} \quad -2\Gamma_{31}^1(2\Gamma_{24}^4 + \Gamma_{43}^1) = 0$$

now immediately imply that $\Gamma_{24}^4 = \Gamma_{43}^1 = 0$ which is a contradiction.

Therefore we always have $\Gamma_{24}^4 = 2\Gamma_{43}^1$.
The equations (($G1444$), ($G1344$), ($G2444$), ($G2333$), ($G1322$) and ($G1441$)) are given by

$$\begin{aligned}
&G1444 \quad 10\Gamma_{31}^1\Gamma_{34}^1 = 0,\\
&G1344 \quad \Gamma_{31}^1(\Gamma_{14}^4 + 2\Gamma_{34}^2) = 0,\\
&G2444 \quad -10\Gamma_{31}^1\Gamma_{43}^2 = 0,\\
&G2333 \quad \beta_3(\Gamma_{11}^1 + \Gamma_{12}^2)\Gamma_{31}^1 = 0,\\
&G1322 \quad -\Gamma_{11}^1\Gamma_{31}^1 = 0,\\
&G1441 \quad -\lambda_1 = 0,
\end{aligned}$$

immediately imply respectively

$$\Gamma^1_{34} = 0, \quad \Gamma^4_{14} = -2\Gamma^2_{34}, \quad \Gamma^2_{34} = 0, \quad \beta_3 = 0, \quad \Gamma^1_{11} = 0, \quad \lambda_1 = 0.$$

As $\Gamma^1_{11} = 0$ we must have that $\Gamma^2_{12} \neq 0$. A contradiction now follows from

$$G1331: \quad -4(\Gamma^2_{12})^2 = 0.$$

Type 2.2.b We have that $\Gamma^3_{43} = \Gamma^3_{33} = \Gamma^1_{31} = \Gamma^1_{41} = 0$.
The conditions in this case now imply that

$$\Gamma^3_{12} \neq 0 \quad \text{and} \quad \Gamma^4_{12} = 0.$$

Using

$$G1414: \quad \lambda_2 = 0,$$

we get that $\lambda_2 = 0$. And we have

$$\begin{aligned} G1313 &: \quad 2\Gamma^3_{12}\Gamma^2_{31} = 0, \\ G2323 &: \quad 2\Gamma^3_{12}(\Gamma^3_{12} - \Gamma^2_{31}) = 0, \end{aligned}$$

Adding this equations, we get $(\Gamma^3_{12})^2 = 0$, which is again a contradiction.

Then there is not hypersurfaces of **type 2.2** in $\mathbb{R}^5$.

7.3 Hypersurfaces of type 2.3

Type 2.3.a We have that $\Gamma^3_{43} = \Gamma^3_{33} = \Gamma^1_{31} = \Gamma^1_{41} = \Gamma^3_{12} = \Gamma^4_{12} = 0$. By the choice of E_3 and E_4 we also have that

$$\begin{cases} \Gamma^1_{34} = 0, \\ 2\Gamma^4_{14} + \Gamma^1_{11} + \Gamma^2_{12} \neq 0, \\ \Gamma^1_{11} + \Gamma^2_{12} \neq 0. \end{cases} \tag{7.3}$$

The equations ($G1313$) and ($G3434$) become

$$\lambda_2 = 0 \quad \text{and} \quad (\Gamma^4_{33})^2 + (\Gamma^4_{43})^2 = 0,$$

we immediately see that

$$\lambda_2 = 0, \quad \Gamma^4_{33} = 0, \quad \Gamma^4_{43} = 0.$$

Adding ($G2322$) and ($G2311$), we get that

$$-(\Gamma^1_{11} + \Gamma^2_{12})\Gamma^2_{31} = 0.$$

So we deduce from the condition (7.3) that $\Gamma^2_{31} = 0$.
Similarly we obtain from adding ($G2422$) and ($G2411$)

$$-(\Gamma^1_{11} + \Gamma^2_{12})\Gamma^2_{41} = 0, \quad \text{then } \Gamma^2_{41} = 0.$$

Similarly from adding ($G1211$) and ($G1222$) we obtain

$$(\Gamma^1_{11} + \Gamma^2_{12})(\Gamma^2_{11} - \Gamma^1_{21}) = 0, \quad \text{then } \Gamma^2_{11} = \Gamma^1_{21}.$$

From (G1212) we obtain that

$$\lambda_1 = -2(\Gamma^1_{21})^2 + (\Gamma^1_{11} - \Gamma^2_{21})\Gamma^2_{21}.$$

From the equation ($G1211$)

$$3\Gamma^1_{21}(-\Gamma^2_{12} + \Gamma^2_{21}) = 0,$$

we obtain two cases,

Case 1: We assume that $\Gamma^2_{21} \neq \Gamma^2_{12}$ then $\Gamma^1_{21} = 0$.
Moreover, adidding ($G1212$) and ($G1221$), it reduce to

$$2(\Gamma^1_{11} - 2\Gamma^2_{12})(\Gamma^2_{12} - \Gamma^2_{21}) = 0,$$

we deduce that $\Gamma^2_{12} = \frac{1}{2}\Gamma^1_{11}$.
Using the condition (7.3), the equation ($G1341$) is

$$-2\Gamma^4_{13}(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14}) = 0,$$

we immediately see that $\Gamma^4_{13} = 0$. The equation ($G1342$) is

$$2\Gamma^1_{11}\Gamma^2_{43} = 0,$$

it yields $\Gamma^2_{43} = 0$. Using

$$G2431 \; : \quad -(3\Gamma^1_{11} + 4\Gamma^4_{14})\Gamma^4_{23} = 0,$$

we obtain $\Gamma^4_{23} = 0$. By subtracting equations ($G1332$) and ($G1442$) we get

$$-8\Gamma^1_{11}\Gamma^4_{24} = 0,$$

then $\Gamma^4_{24} = 0$. We are now left with the equations ($G1331$), ($G1441$), ($G2332$) and ($G2442$) which state that

$$(\Gamma^1_{11} - 2\Gamma^4_{14} - \Gamma^2_{21})(2\Gamma^4_{14} - \Gamma^2_{21}) = 0,$$

$$\begin{aligned}
&(-4\Gamma^1_{11} - 2\Gamma^4_{14} + \Gamma^2_{21})(3\Gamma^1_{11} + 2\Gamma^4_{14} + \Gamma^2_{21}) = 0,\\
&\Gamma^2_{21}(-4\Gamma^1_{11} - 2\Gamma^4_{14} + \Gamma^2_{21}) = 0,\\
&\Gamma^2_{21}(-\Gamma^1_{11} + 2\Gamma^4_{14} + \Gamma^2_{21}) = 0.
\end{aligned}$$

Solving the above system we get a contradiction with

$$2\Gamma^4_{14} + \Gamma^1_{11} + \Gamma^2_{12} = 2\Gamma^4_{14} + \tfrac{3}{2}\Gamma^1_{11} \neq 0.$$

Case 2: We have that $\Gamma^2_{21} = \Gamma^2_{12}$.
The equation $(G1234)$ is given by

$$(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^4_{23} - 2\Gamma^4_{13}\Gamma^4_{24} = 0.$$

1. We assume that $\{\Gamma^4_{23}, \Gamma^4_{24}\}$ are not both vanishing. Then there exist a constant α such that

$$\Gamma^4_{13} = \alpha\Gamma^4_{23}, \qquad \Gamma^4_{14} = \alpha\Gamma^4_{24} - \frac{1}{2}(\Gamma^1_{11} + \Gamma^2_{12}).$$

The condition (7.3) becomes $2\alpha\Gamma^4_{24} \neq 0$.
The equation $(G1243)$ is

$$-2\alpha\Gamma^4_{24}\Gamma^2_{43} = 0$$

then $\Gamma^2_{43} = 0$. In the same $(G1431)$ is given by

$$-4\alpha^2\Gamma^4_{23}\Gamma^4_{24} = 0,$$

yield that $\Gamma^4_{23} = 0$. Subtracting $(G2332)$ and $(G2442)$ we see that

$$4\Gamma^4_{24}(\Gamma^1_{21} - \alpha\Gamma^2_{12}) = 0.$$

Using (7.3) we deduce that $\Gamma^1_{21} = \alpha_1\Gamma^2_{12}$. Similarly subtracting $(G2331)$ and $(G2441)$ we see that

$$-4\Gamma^4_{24}\left(\left(\alpha^2 + 2\right)\Gamma^2_{12} + \Gamma^1_{11}\right) = 0.$$

We obtain $\Gamma^1_{11} = -(2 + \alpha_1^2)\Gamma^2_{12}$. The equations $(G1442)$ and $(G2332)$ now become

$$\begin{aligned}
&\alpha\left((\alpha^2 + 1)(\Gamma^2_{12})^2 - 4(\Gamma^4_{24})^2\right) = 0,\\
&-\lambda_1 + \left(\alpha^2 + 1\right)(\Gamma^2_{12})^2 - 4(\Gamma^4_{24})^2 = 0,
\end{aligned}$$

Such that $\lambda_1 \neq 0$, than $\alpha = 0$, which leads to a contradiction.

2. Now if $\Gamma^4_{23} = \Gamma^4_{24} = 0$.
Seeing the equation

$$G1441 : \quad -\lambda_1 + 2(\Gamma^1_{11} - 2\Gamma^4_{14})\Gamma^4_{14} = 0,$$

the fact that $\lambda_1 \neq 0$, implies that $\Gamma_{14}^4 \neq 0$. So

$$G2441 : \quad 2\Gamma_{14}^4\Gamma_{21}^1 = 0,$$

immediately implies that $\Gamma_{21}^1 = 0$. The equations ($G1243$) and ($G1341$) are given by

$$\begin{aligned} &-(\Gamma_{11}^1 + \Gamma_{12}^2 + 2\Gamma_{14}^4)\Gamma_{43}^2 = 0 \\ &-2\Gamma_{13}^4(\Gamma_{11}^1 + \Gamma_{12}^2 + 2\Gamma_{14}^4) = 0, \end{aligned}$$

using (7.3), we deduce that $\Gamma_{43}^2 = 0$ and $\Gamma_{13}^4 = 0$ respectively.
The equations $((G2442), (G1441), (G1331), (G2332))$ now reduce to

$$\begin{aligned} &-(3\Gamma_{11}^1 + \Gamma_{12}^2 + 2\Gamma_{14}^4)(2\Gamma_{11}^1 + 3\Gamma_{12}^2 + 2\Gamma_{14}^4) = 0, \\ &-(\Gamma_{11}^1 - \Gamma_{12}^2 - 2\Gamma_{14}^4)(\Gamma_{12}^2 - 2\Gamma_{14}^4) = 0, \\ &-\Gamma_{12}^2(3\Gamma_{11}^1 + \Gamma_{12}^2 + 2\Gamma_{14}^4) = 0, \\ &\Gamma_{12}^2(-\Gamma_{11}^1 + \Gamma_{12}^2 + 2\Gamma_{14}^4) = 0. \end{aligned}$$

We immediately see that the above system admits no solution which satisfies the condition (7.3).
Then there is not hypersurfaces of **type 2.2.a** in $\mathbb{R}^5$.

Type 2.3.b We have that $\Gamma_{43}^3 = \Gamma_{33}^3 = \Gamma_{31}^1 = \Gamma_{41}^1 = \Gamma_{12}^3 = \Gamma_{12}^4 = \Gamma_{34}^1 = 0$ and $\Gamma_{14}^4 = -\frac{1}{2}(\Gamma_{11}^1 + \Gamma_{12}^2)$. By the choice of E_3 and E_4 we also have that

$$\begin{cases} \Gamma_{34}^2 = 0, \\ \Gamma_{24}^4 \neq 0, \\ \Gamma_{11}^1 + \Gamma_{12}^2 \neq 0. \end{cases} \tag{7.4}$$

In exactly the same way as in the previous case, the equations

$$\begin{aligned} G1313 &: \quad \lambda_2 = 0, \\ G3434 &: \quad \lambda_2 + (\Gamma_{33}^4)^2 + (\Gamma_{43}^4)^2 = 0, \\ G1333 &: \quad -\Gamma_{24}^4\Gamma_{31}^2 = 0, \\ G1433 &: \quad -\Gamma_{24}^4\Gamma_{41}^2 = 0, \\ G1432 &: \quad -4\Gamma_{13}^4\Gamma_{24}^4 = 0, \\ G2432 &: \quad -4\Gamma_{23}^4\Gamma_{24}^4 = 0, \end{aligned}$$

we get $\lambda_2 = \Gamma_{13}^4 = \Gamma_{23}^4 = \Gamma_{33}^4 = \Gamma_{43}^4 = \Gamma_{31}^2 = \Gamma_{41}^2 = 0$.
Adding ($G1332$) and ($G1442$), we get

$$-2\Gamma_{11}^2(\Gamma_{11}^1 + \Gamma_{12}^2) = 0,$$

it follows from the condition (7.4) that $\Gamma_{11}^2 = 0$. The equation $(G1332)$ is

$$-2(\Gamma_{11}^1 + 2\Gamma_{12}^2)\Gamma_{24}^4 = 0,$$

implies that $\Gamma_{12}^2 = -\frac{1}{2}\Gamma_{11}^1$.
The equations $(G1212)$ and $(G1222)$ are given by

$$\begin{aligned} &\lambda_1 + \Gamma_{21}^2(-\Gamma_{11}^1 + \Gamma_{21}^2) = 0, \\ &-3\Gamma_{21}^1\Gamma_{21}^2 = 0, \end{aligned}$$

so $\lambda_1 \neq 0$, then $\Gamma_{21}^1 = 0$. From $(G2331)$ we get

$$(\Gamma_{11}^1 + 2\Gamma_{21}^2)\Gamma_{24}^4 = 0,$$

then $\Gamma_{21}^2 = -\frac{1}{2}\Gamma_{11}^1$.
The equations $(G1221)$ and $(G2332)$ are

$$\begin{aligned} &-\lambda_1 - 3/4(\Gamma_{11}^1)^2 = 0, \\ &-\lambda_1 + 1/4(\Gamma_{11}^1)^2 - 4(\Gamma_{24}^4)^2 = 0, \end{aligned}$$

from the first equation, we now see that λ_1 is negative and if necessary by replacing E_1 buy $-E_1$ we see that $\Gamma_{11}^1 = \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}$. Similarly from the second, we get that $\Gamma_{24}^4 = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}$. Therefore we have shown the following lemma.

Lemma 7.3.1 *Let M be an affine homogeneous hypersurface which is of type 2.3.b. Then λ_1 is negative and $\lambda_2 = 0$. Moreover, we can choose a local frame $\{E_1, E_2, E_3, E_4\}$ such that*

$$\begin{aligned} &\Gamma_{11}^1 = \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}, && \Gamma_{12}^2 = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, && \Gamma_{13}^3 = -\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}, \\ &\Gamma_{14}^4 = -\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}, && \Gamma_{21}^2 = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, && \Gamma_{22}^1 = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, \\ &\Gamma_{23}^3 = \frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, && \Gamma_{24}^4 = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, && \Gamma_{33}^1 = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, \\ &\Gamma_{33}^2 = \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}, && \Gamma_{44}^1 = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, && \Gamma_{44}^2 = -\frac{2}{\sqrt{3}}\sqrt{-\lambda_1}, \end{aligned}$$

and all the other connection coefficients vanish.

From this we can now show the following.

Theorem 7.3.1 *[CSV16] Let M^4 be a locally strongly convex, locally homogeneous, affine hypersurface in $\mathbb{R}^5$. Assume the affine shape operator of M has two distinct real eigenvalues, both multiplicity 2 and that M is of Type 2.3.b. Then M is locally affine congruent to an open part of the hypersurface*

$$\left(x_1 - x_4^2\right)^3 \left(x_2 - x_5^2\right)^3 x_3^2 = 1. \tag{7.5}$$

proof Using the local frame of the previous Lemma 7.3.1, we have that the Lie brackets are given by

$$\begin{aligned}
&[E_1, E_2] = 0,\\
&[E_1, E_3] = -\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}E_3,\\
&[E_1, E_4] = -\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}E_4,\\
&[E_2, E_3] = \frac{1}{\sqrt{3}}\sqrt{-\lambda_1}E_3,\\
&[E_2, E_4] = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}E_4,\\
&[E_3, E_4] = 0.
\end{aligned}$$

So if we now take a new frame in M :

$$E_1^* = E_1, \quad E_2^* = E_2, \quad E_3^* = \rho_1 E_3, \quad E_4^* = \rho_2 E_4,$$

where ρ_1, ρ_2 are functions in M. We see that with respect to the new frame the Lie brackets become

$$\begin{aligned}
&[E_1^*, E_2^*] = 0,\\
&[E_1^*, E_3^*] = (E_1(\rho_1) - \rho_1\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1})E_3,\\
&[E_1^*, E_4^*] = (E_1(\rho_2) - \rho_2\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1})E_4,\\
&[E_2^*, E_3^*] = (E_2(\rho_1) + \rho_1\frac{1}{\sqrt{3}}\sqrt{-\lambda_1})E_3,\\
&[E_2^*, E_4^*] = (E_2(\rho_2) - \rho_1\frac{1}{\sqrt{3}}\sqrt{-\lambda_1})E_4,\\
&[E_3^*, E_4^*] = 0.
\end{aligned}$$

Now a straightforward computations shows that the one forms ω_1 and ω_2 respectively defined by

$$\omega_1(E_1) = \frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}, \qquad \omega_1(E_2) = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, \tag{7.6}$$
$$\omega_1(E_3) = 0, \qquad \omega_1(E_4) = 0,$$
$$\omega_2(E_1) = \frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}, \qquad \omega_2(E_2) = \frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, \tag{7.7}$$
$$\omega_2(E_3) = 0, \qquad \omega_2(E_4) = 0.$$

are closed one forms. Hence there exist local functions such that $\omega_1 = d\ln\rho_1$ and $\omega_2 = d\ln\rho_2$. Using these functions we now see that with respect to the new frame all connection coefficients vanish. Hence there exist local coordinates (u, v, x, y) such that

$$\frac{\partial}{\partial u} = E_1, \quad \frac{\partial}{\partial v} = E_2, \quad \frac{\partial}{\partial x}\rho_1 E_3, \quad \frac{\partial}{\partial y} = \rho_2 E_4. \tag{7.8}$$

By integrating (7.6) and (7.7), and (7.8), we get the functions ρ_1 and ρ_2

$$\rho_1 = \alpha_1\, e^{\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}(u-2v)}, \qquad \rho_2 = \alpha_2\, e^{\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}(u+2v)}, \tag{7.9}$$

where α_1 and α_2 are positive constants. Of course by translating the coordinates u and v we may assume that $\alpha_1 = \alpha_2 = 1$.

We now note the immersion $F : M \hookrightarrow \mathbb{R}^{n+1}$. From the definition of the affine shape operator and Lemma 7.3.1 we get that

$$\xi_u = -\lambda_1 F_u, \qquad \xi_v = -\lambda_1 F_v, \qquad \xi_x = \xi_y = 0,$$

and

$$\begin{aligned}
F_{uu} &= \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}F_u + \xi,\\
F_{uv} &= -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}F_v,\\
F_{ux} &= 0,\\
F_{uy} &= 0,\\
F_{vv} &= -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}F_u + \xi,\\
F_{vx} &= 0,\\
F_{vy} &= 0,\\
F_{xx} &= \rho_1^2(-\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}F_u + \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}\partial_v F_v + \xi),\\
F_{xy} &= 0,\\
F_{yy} &= \rho_2^2(-\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}F_u - \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}F_v + \xi).
\end{aligned} \tag{7.10}$$

Taking the derivative of the first equation of (7.10) we get

$$F_{uuu} - \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}F_{uu} + \lambda_1 F_u = 0.$$

Its solutions are given by

$$F(u,v,x,y) = A_1(v,x,y) + A_2(v,x,y)e^{\mu_1 u} + A_3(v,x,y)e^{\mu_2 u}, \tag{7.11}$$

where $\mu_1 = -\frac{\sqrt{-\lambda_1}}{\sqrt{3}}$, $\mu_2 = \frac{3\sqrt{-\lambda_1}}{\sqrt{3}}$ and A_1, A_2, A_3 are functions in M. It immediately follows from (7.10) that A_2 depends only on v, A_1 dpends only on x and y and A_3 is a constant vector. Therefore

$$F(u,v,x,y) = A_1(x,y) + A_2(v)e^{\mu_1 u} + A_3 e^{\mu_2 u}. \tag{7.12}$$

It now follows that

$$\xi = F_{uu} - \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}F_u = -\lambda_1(A_2(v)e^{\mu_1 u} + A_3 e^{\mu_2 u}).$$

It now follows that for A_2 we get the following differential equation:

$$A_2''(v) = -\tfrac{4}{3}A_2\lambda_1.$$

Its solution are

$$A_2(v) = c_1.e^{-2\mu_1 v} + c_2.e^{2\mu_1 v},$$

where c_1, c_2 are constant vectors. The remaining differential equations now become

$$\begin{aligned}(A_1)_{xx} &= -\frac{8}{3}c_1\lambda_1,\\ (A_1)_{xy} &= 0,\\ (A_1)_{yy} &= -\frac{8}{3}c_2\lambda_1.\end{aligned}$$

Therefore

$$A_1(x,y) = c_3 x + c_4 y - \frac{4}{3}c_1\lambda_1 x^2 - \frac{4}{3}c_2\lambda_1 y^2 + c_5.$$

and

$$F(u,v,x,y) = -\frac{4}{3}c_1\lambda_1 x^2 - \frac{4}{3}c_2\lambda_1 y^2 + c_3 x + c_4 y + c_1.e^{\mu_1(u-2v)} + c_2.e^{\mu_1(u+2v)} + A_3.e^{\mu_2 u} + c_5.$$

We can write it

$$\begin{aligned}F(u,v,x,y) = \quad & c_1(-\frac{4}{3}\lambda_1 x^2 + e^{\mu_1(u-2v)})\\ & + c_2(-\frac{4}{3}\lambda_1 y^2 + e^{\mu_1(u+2v)})\\ & + A_3 e^{\mu_2 u}\\ & + c_3 x\end{aligned}$$

$$+c_4 y$$
$$+c_5.$$

Of course by a translation, we may now suppose that c_5 vanishes and by an affine transformation we may assume that c_1, c_2, c_3, c_4, A_3 is the canonical basis of $\mathbb{R}^5$. This completes the proof of the theorem. $\square$

Type 2.3.c We have that

$$\Gamma_{43}^3 = \Gamma_{33}^3 = \Gamma_{31}^1 = \Gamma_{41}^1 = \Gamma_{12}^3 = \Gamma_{12}^4 = \Gamma_{34}^1 = \Gamma_{34}^2 = 0,$$
$$\Gamma_{14}^4 = -\tfrac{1}{2}(\Gamma_{11}^1 + \Gamma_{12}^2) \quad \text{and} \quad \Gamma_{24}^4 = 0.$$

The difference tensor is given by

$$\begin{aligned}
K(E_1, E_1) &= \Gamma_{11}^1 E_1 + \Gamma_{21}^1 E_2,\\
K(E_1, E_2) &= \Gamma_{21}^1 E_1 + \Gamma_{12}^2 E_2,\\
K(E_2, E_2) &= \Gamma_{21}^1 E_1 + \Gamma_{12}^2 E_2,\\
K(E_1, E_3) &= \tfrac{-1}{2}(\Gamma_{11}^1 + \Gamma_{12}^2) E_3,\\
K(E_1, E_4) &= \tfrac{-1}{2}(\Gamma_{11}^1 + \Gamma_{12}^2) E_4,\\
K(E_2, E_3) &= \tfrac{-1}{2}(\Gamma_{11}^1 + \Gamma_{12}^2) E_1,\\
K(E_4, E_4) &= \tfrac{-1}{2}(\Gamma_{11}^1 + \Gamma_{12}^2) E_1,\\
K(E_2, E_3) &= K(E_2, E_4) = K(E_3, E_4) = 0.
\end{aligned}$$

Note that as E_1, E_2 are determined canonically in a unique way, the Christoffel symbols appearing in the above difference tensor are constant.
For this type of hypersurface we want to prove the following theorem.

Theorem 7.3.2 *Let M^4 be a locally strongly convex, locally homogeneous, affine hypersurface in $\mathbb{R}^5$ whose affine shape operator has two distinct real eigenvalues with multiplicity two. Assume that M is of Type 2.3.c. Then λ_1 is negative and λ_2 vanishes. Moreover M is affine equivalent with an open part of*

$$x_2^3 \left(x_1 - (x_3^2 + x_4^2) - \frac{x_5^2}{x_2} \right)^5 = 1. \tag{7.13}$$

Before starting the proof of this theorem we first need the following lemma.

Lemma 7.3.2 *We can choose the frame $\{E_1, E_2, E_3, E_4\}$ such that*

$$\Gamma_{11}^1 = -\frac{2}{\sqrt{15}}\sqrt{-\lambda_1}, \qquad \Gamma_{12}^2 = -\frac{1}{\sqrt{15}}\sqrt{-\lambda_1}, \qquad \Gamma_{13}^3 = \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1},$$

$$\Gamma^4_{14} = \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}, \qquad \Gamma^2_{21} = -\sqrt{\frac{5}{3}}\sqrt{-\lambda_1}, \qquad \Gamma^1_{22} = \sqrt{\frac{3}{5}}\sqrt{-\lambda_1},$$
$$\Gamma^1_{33} = \sqrt{\frac{3}{5}}\sqrt{-\lambda_1}, \qquad \Gamma^1_{44} = \sqrt{\frac{3}{5}}\sqrt{-\lambda_1},$$

and all the other connection coefficients vanish.

Proof We will use some of the Codazzi equations for the difference tensor K, such that this equation is in first chapter (4.34) is given by

$$\begin{aligned} &\hat{\nabla}K(X,Y,Z) - \hat{\nabla}K(Y,X,Z) \\ &= \frac{1}{2}h(Y,Z)SX - \frac{1}{2}h(X,Z)SY + \frac{1}{2}h(SX,Z)Y - \frac{1}{2}h(SY,Z)X. \end{aligned}$$

A straightforward computation yield that

$$\begin{aligned} &Glc1332 \quad -\frac{1}{2}\Gamma^2_{11}(\Gamma^1_{11} + \Gamma^2_{12}) = 0, \\ &Glc2331 \quad -\frac{1}{2}(\Gamma^1_{11} + \Gamma^2_{12})\Gamma^1_{21} = 0, \\ &Glc3432 \quad \frac{1}{2}(\Gamma^1_{11} + \Gamma^2_{12})\Gamma^2_{41} = 0, \\ &Glc3442 \quad -\frac{1}{2}(\Gamma^1_{11} + \Gamma^2_{12})\Gamma^2_{31} = 0, \end{aligned}$$

where we denote $Glc\,ijk\ell$ the ℓ-th component of the Codazzi equations (4.34) with

$$X = E_i, \quad Y = E_j, \quad Z = E_k.$$

Note that of course there are other non vanishing components. These however will not be used in the proof of the lemma.

From the equations $((Glc2331), (Glc1332), (Glc3432), (Glc3442))$, and the fact that M is of Type 2.3.c, we get that

$$\Gamma^1_{21} = \Gamma^2_{11} = \Gamma^2_{41} = \Gamma^2_{31} = 0.$$

In view of the Type of M we also have that $\Gamma^1_{11} \neq 0$. See now the following equations

$$\begin{aligned} &Glc1221 \quad 3(\Gamma^2_{11} - \Gamma^1_{21})\Gamma^1_{21} + (\Gamma^1_{11} - 2\Gamma^2_{12})(\Gamma^2_{12} - \Gamma^2_{21}) = 0, \\ &Glc1331 \quad \frac{1}{2}(-(\Gamma^1_{11} + \Gamma^2_{12})(2\Gamma^1_{11} + \Gamma^2_{12}) - \lambda_1 + \lambda_2) = 0, \\ &Glc2442 \quad \frac{1}{2}(-(\Gamma^1_{11} + \Gamma^2_{12})\Gamma^2_{21} - \lambda_1 + \lambda_2) = 0. \end{aligned}$$

Solving this equations we get after if necessary changing the sign of E_1 that

$$\Gamma_{11}^1 = -\frac{2\sqrt{-\lambda_1+\lambda_2}}{\sqrt{15}}, \quad \Gamma_{12}^2 = -\frac{\sqrt{-\lambda_1+\lambda_2}}{\sqrt{15}}, \quad \Gamma_{21}^2 = -\sqrt{\frac{5}{3}}\sqrt{-\lambda_1+\lambda_2}.$$

The remaining unknown connection coefficients are $\{\Gamma_{13}^4, \Gamma_{23}^4, \Gamma_{33}^4, \Gamma_{43}^4\}$. Note however that these connection coefficients are not necessary constants.

The remaining Gauss equations become

$$\begin{aligned}
&G1221 && \lambda_2 = 0,\\
&G1234 && -\frac{4\Gamma_{23}^4\sqrt{-\lambda_1+\lambda_2}}{\sqrt{15}} - E_2(\Gamma_{13}^4) + E_1(\Gamma_{23}^4) = 0,\\
&G1343 && \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1+\lambda_2}\Gamma_{33}^4 + \Gamma_{13}^4\Gamma_{43}^4 + E_3(\Gamma_{13}^4) - E_1(\Gamma_{33}^4) = 0,\\
&G2343 && \Gamma_{23}^4\Gamma_{43}^4 + E_3(\Gamma_{23}^4) - E_2(\Gamma_{33}^4) = 0,\\
&G2443 && -\Gamma_{23}^4\Gamma_{33}^4 + E_4(\Gamma_{23}^4) - E_2(\Gamma_{43}^4) = 0,\\
&G3443 && -(\Gamma_{33}^4)^2 - (\Gamma_{43}^4)^2 - \lambda_2 + E_4(\Gamma_{33}^4) - E_3(\Gamma_{43}^4) = 0,\\
&G1434 && -\frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1+\lambda_2}\Gamma_{43}^4 + \Gamma_{13}^4\Gamma_{33}^4 - E_4(\Gamma_{13}^4) + E_1(\Gamma_{43}^4) = 0.
\end{aligned}$$

The equation ($G1221$) now immediately implies that $\lambda_2 = 0$. The other equations are correspond precisely to the conditions that the one form ω defined by

$$\omega(E_1) = -\Gamma_{13}^4, \quad \omega(E_2) = -\Gamma_{23}^4, \quad \omega(E_3) = -\Gamma_{33}^4, \quad \omega(E_4) = -\Gamma_{43}^4, \tag{7.14}$$

is a closed one form. Hence there exists a function φ such that $\omega = d\varphi$. We now consider the following change of frame

$$(E_1^*, E_2^*, E_3^*, E_4^*) = (E_1, E_2, \cos\varphi\, E_3 + \sin\varphi\, E_4, -\sin\varphi\, E_3 + \cos\varphi\, E_4),$$

This frame still satisfies all of the previous conditions. However we moreover have that

$$\begin{aligned}
\hat{\nabla}_{E_1^*}E_3^* &= \left(E_1(\varphi) + \Gamma_{13}^4\right) E_4^* = 0,\\
\hat{\nabla}_{E_2^*}E_3^* &= \left(E_2(\varphi) + \Gamma_{23}^4\right) E_4^* = 0,\\
\hat{\nabla}_{E_3^*}E_3^* &= \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}E_1^* + \left((E_3(\varphi) + \Gamma_{33}^4)\cos\varphi + (E_4(\varphi) + \Gamma_{43}^4)\sin\varphi\right) E_4^*\\
&= \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}E_1^*,\\
\hat{\nabla}_{E_4^*}E_3^* &= -\left((E_3(\varphi) + \Gamma_{33}^4)\sin\varphi - (E_4(\varphi) + \Gamma_{43}^4)\cos\varphi\right) E_4^* = 0.
\end{aligned}$$

The conclusion of Lemma 7.3.2 follows from these and the expression of K. □

proof Proof of Theorem 7.3.2. We use the local frame constructed in Lemma 7.3.2. The Lie brackets of the last frame are given by

$$[E_1, E_2] = \frac{4}{\sqrt{15}}\sqrt{-\lambda_1}E_2, \quad [E_1, E_3] = \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}E_3,$$
$$[E_1, E_4] = \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}E_4, \quad [E_2, E_3] = [E_2, E_4] = [E_3, E_4] = 0.$$

Therefore we immediately see that the one forms ω_1 and ω_2 defined by

$$\omega_1(E_1) = -\frac{4}{\sqrt{15}}\sqrt{-\lambda_1}, \qquad \omega_2(E_1) = -\frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1},$$
$$\omega_1(E_2) = 0, \qquad \omega_1(E_3) = 0, \qquad \omega_1(E_4) = 0,$$
$$\omega_2(E_2) = 0, \qquad \omega_2(E_3) = 0, \qquad \omega_2(E_4) = 0.$$

are closed one forms. Hence there exist positive functions ρ_i such that $\omega_i = d\ln\rho_i$. If we now look at the frame given by $E_1, \rho_1E_2, \rho_2E_3, \rho_2E_4$ we have that all the Lie brackets vanish. So there exist local coordinates (u, v, x, y), such that

$$(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = (E_1, \rho_1E_2, \rho_2E_3, \rho_2E_4).$$

Choosing the initial conditions for the functions ρ_i appropriately we get that

$$\rho_1 = e^{-\frac{4}{\sqrt{15}}\sqrt{-\lambda_1}.u}, \qquad \rho_2 = e^{-\frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}.u}.$$

In the same way as in the previous theorem we now determine the system of differential equations for the position vector F and the affine normal of the immersion. We get that

$$\xi_u = -\lambda_1F_u, \qquad \xi_v = -\lambda_1F_v, \qquad \xi_x = \xi_y = 0,$$

and

$$\begin{aligned}
F_{uu} &= -\frac{2}{\sqrt{15}}\sqrt{-\lambda_1}F_u + \xi,\\
F_{uv} &= -\frac{5}{\sqrt{15}}\sqrt{-\lambda_1}F_v,\\
F_{ux} &= 0,\\
F_{uy} &= 0,\\
F_{vv} &= \rho_1^2\left(\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}F_u + \xi\right),\\
F_{vx} &= 0,\\
F_{vy} &= 0,\\
F_{xx} &= \rho_2^2\left(\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}F_u + \xi\right),\\
F_{xy} &= 0,\\
F_{yy} &= \rho_2^2\left(\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}F_u + \xi\right).
\end{aligned} \tag{7.15}$$

Deriving the first equation with respect to u, we get

$$F_{uuu} + \frac{2}{\sqrt{15}}\sqrt{-\lambda_1}F_{uu} + \lambda_1 F_u = 0.$$

Its solutions are given by

$$F(u, v, x, y) = A_1(v, x, y) + A_2(v, x, y)e^{\mu_1 u} + A_3(v, x, y)e^{\mu_2 u}, \tag{7.16}$$

where $\mu_1 = -\frac{5}{\sqrt{15}}\sqrt{-\lambda_1}$ and $\mu_2 = \frac{3}{\sqrt{15}}\sqrt{-\lambda_1}$, and A_1, A_2, A_3 are functions in M. The other equations now immediately imply that A_3 is a constant vector, A_2 depends only on v and A_1 is the sum of a function depending only on x and a function depending only on y. The first equations also implies that

$$\xi = -\lambda_1(A_2(v, x, y)e^{\mu_1 u} + A_3(v, x, y)e^{\mu_2 u}).$$

The remaining differential equations now reduce to

$$\begin{aligned}
(A_1)_{xx} &= -\tfrac{8}{5}A_3\lambda_1,\\
(A_1)_{yy} &= -\tfrac{8}{5}A_3\lambda_1,\\
(A_2)_{vv} &= -\tfrac{8}{5}A_3\lambda_1.
\end{aligned}$$

So there exist constant vectors A_3, c_1, c_2, c_3, c_4, c_5 such that

$$F(u,v,x,y) = -\frac{4}{5}A_3\lambda_1(x^2+y^2) + c_1x + c_2y + c_5 \\ + (-\frac{4}{5}\lambda_1 A_3 v^2 + c_3 v + c_4)e^{\mu_1 u} + A_3 e^{\mu_2 u}.$$

As λ_1 is negative, by a homothetic transformation we may of course assume that $-\frac{4}{5}\lambda_1 = 1$. Next by applying an affine transformation we may that c_5 vanishes and c_1, c_2, c_3, c_4, A_5 are mapped to the standard basis. So we can write

$$F(u,v,x,y) = ((x^2+y^2) + v^2 e^{\mu_1 u} + e^{\mu_2 u},\ e^{\mu_1 u},\ x,\ y,\ v e^{\mu_1 u}).$$

which is contained in

$$x_2^3\left(x_1 - (x_3^2 + x_4^2) - \frac{x_5^2}{x_2}\right)^5 = 1.$$

□

Chapter 8

Hypersurfaces of type 3

reference to all this chapter is always the paper of A.Chikh Salah and L.Vrancken [CSV16].

In this case we have that both $V_1 = 0$ and $W_2 = 0$ than

$$V_1 = 0 \iff \begin{cases} h(K(e_1,e_1),e_1) + h(K(e_2,e_2),e_1) = 0, \\ h(K(e_1,e_1),e_2) + h(K(e_2,e_2),e_2) = 0, \end{cases} \implies \begin{cases} \Gamma_{11}^1 = -\Gamma_{12}^2, \\ \Gamma_{22}^2 = -\Gamma_{21}^1. \end{cases}$$

$$W_2 = 0 \iff \begin{cases} h(K(e_3,e_3),e_3) + h(K(e_4,e_4),e_3) = 0, \\ h(K(e_3,e_3),e_4) + h(K(e_4,e_4),e_4) = 0, \end{cases} \implies \begin{cases} \Gamma_{34}^4 = -\Gamma_{33}^3, \\ \Gamma_{44}^4 = -\Gamma_{43}^3. \end{cases}$$

We again will use properties of the difference tensor K in order to determine a unique frame. Note that by the classical Berwald theorem, we know that if K vanishes identically, M is congruent with a quadric (and therefore an affine sphere). So we may assume that K does not vanish identically. This leaves the following possibilities.

1. Suppose that the restriction of K to either $\mathcal{E}_1$ or $\mathcal{E}_2$ does not vanish identically. If necessary by interchanging both distributions we may assume that $h(K(\mathcal{E}_1),\mathcal{E}_1),\mathcal{E}_1)$ does not vanish identically. In that case we can pick E_1 such that $h(K(E_1,E_1),E_2) = 0$ and $h(K(E_1,E_1),E_1) \neq 0)$.
   (a) the symmetric operator K_{E_1} restricted to $\mathcal{E}_2$ has two distinct eigenvalues. In this case we can pick E_3 and E_4 in the direction of the corresponding eigenspaces,
   (b) the symmetric operator K_{E_1} restricted to $\mathcal{E}_2$ is a multiple of the identity. Note that because $V_1 = 0$ this implies that K_{E_1} restricted to $\mathcal{E}_2$ vanishes. However the symmetric operator K_{E_2} restricted to $\mathcal{E}_2$ has two distinct eigenvalues. In this case we can again pick E_3 and E_4 in the direction of the corresponding eigenspaces,

(c) both the symmetric operators K_{E_1} and K_{E_2} restricted to $\mathcal{E}_2$ vanish. However $h(K(\mathcal{E}_2,\mathcal{E}_2),\mathcal{E}_2)$ does not vanish. In that case we can pick E_3 such that $h(K(E_3,E_3),E_4)=0$ and $h(K(E_3,E_3),E_3)\neq 0)$

(d) We have $K(E_1,E_1)=\nu_1 E_1=-K(E_2,E_2)$, $K(E_1,E_2)=-\nu_1 E_2$ and all other components of the difference tensor vanish. In this case we still have a rotation freedom in $\mathcal{E}_2$ and the frame is not determined uniquely.

2. Suppose that the restriction of K to both $\mathcal{E}_1$ and $\mathcal{E}_2$ vanishes identically. We can now look at the restriction of K to $\mathcal{E}_1$ and $\mathcal{E}_2$. This gives us two maps

$$K^1=K|_{\mathcal{E}_1}:\mathcal{E}_1\times\mathcal{E}_1\to\mathcal{E}_2,\qquad K^2=K|_{\mathcal{E}_2}:\mathcal{E}_2\times\mathcal{E}_2\to\mathcal{E}_1.$$

By the apolarity and the symmetry of K we now that $K(E_2,E_2)=-K(E_1,E_1)$ and $K(E_4,E_4)=-K(E_3,E_3)$. Hence the maps map the unit circle in respectively $\mathcal{E}_1$ and $\mathcal{E}_2$ to an ellipse (or a line), a circle or a point. Using the fact that K can not vanish identically this leaves us with the following possibilities:

(a) The image of K^1 is an ellipse (or line). We can choose E_3 and E_4 as the axes of the ellipse and we can choose E_1 such that $K(E_1,E_1)$ is in the direction of E_3.

(b) The image of both K^1 and K^2 are circles. We can pick E_3 and E_4 such that

$$\begin{aligned}
&K(E_1,E_1)=\nu_1E_3=-K(E_2,E_2),\quad K(E_1,E_2)=\nu_1E_4,\\
&K(E_1,E_3)=\nu_1E_1+\nu_2E_3-\epsilon\nu_3E_4,\\
&K(E_1,E_4)=\nu_1E_2-\epsilon\nu_3E_3-\nu_2E_4,\\
&K(E_2,E_3)=-\nu_1E_2+\nu_3E_3+\epsilon\nu_2E_4,\\
&K(E_2,E_4)=\nu_1E_1+\epsilon\nu_2E_3-\nu_3E_4,\\
&K(E_3,E_3)=\nu_2E_1+\nu_3E_2=-K(E_4,E_4),\\
&K(E_3,E_4)=\epsilon(-\nu_3E_1+\nu_2E_2),
\end{aligned}$$

where $\epsilon=\pm1$. Making now the following change of basis $E_1^*=\cos\varphi.E_1+\sin\varphi.E_2$, $E_2^*=-\sin\varphi.E_1+\cos\varphi.E_2$, $E_3^*=\cos 2\varphi.E_3+\sin 2\varphi.E_4$ and $E_4^*=-\sin 2\varphi.E_3+\cos 2\varphi.E_4$, we see that $\nu_1^*=\nu_1$ and

$$\begin{aligned}
\nu_2^*=\tfrac{1}{2}(&\nu_2((1+\epsilon)\cos(3\varphi)+(\epsilon-1)\cos(5\varphi))\\
&+\nu_3(-(1+\epsilon)\sin(3\varphi)+(1-\epsilon)\sin(5\varphi)),\\
\nu_3^*=\tfrac{1}{2}(&\nu_2((1+\epsilon)\sin(3\varphi)+(1-\epsilon)\sin(5\varphi))\\
&+\nu_3((1+\epsilon)\cos(3\varphi)-(1-\epsilon)\cos(5\varphi)).
\end{aligned}$$

Hence by assuming that ν_3^* has to vanish we can fix the frame completely.

(c) The image of K_1 is a circle and $h(K(\mathcal{E}_2, \mathcal{E}_2), \mathcal{E}_1) = 0$. Given E_1 and E_2, we can choose E_3 and E_4 such that

$$\begin{aligned} K(E_1, E_1) &= \nu_1 E_3 = -K(E_2, E_2), & K(E_1, E_2) &= \nu_1 E_4, \\ K(E_1, E_3) &= \nu_1 E_1, & K(E_1, E_4) &= \nu_1 E_2, \\ K(E_2, E_3) &= -\nu_1 E_2, & K(E_2, E_4) &= \nu_1 E_1. \end{aligned}$$

Note that in this case making the following change of basis preserves the difference tensor

$$\begin{aligned} E_1^* &= \cos\varphi.E_1 + \sin\varphi.E_2, \\ E_2^* &= -\sin\varphi.E_1 + \cos\varphi.E_2, \\ E_3^* &= \cos 2\varphi.E_3 + \sin 2\varphi.E_4, \\ E_4^* &= -\sin 2\varphi.E_3 + \cos 2\varphi.E_4. \end{aligned}$$

As M is homogeneous and ν_1 is determined by $h(K, K)$, ν_1 must be constant.
So we see that except in case 3.1.d and case 3.2.c, the frame is uniquely determined and we can apply the Lemma 5.2.1.

8.1 Hypersurfaces of type 3.1

8.1.1 Type 3.1 : Case a, Case b and Case c

These cases can be treated simultaneously.
Such that $V_1 = 0$ and $W_2 = 0$, as given before :

$$\Gamma_{11}^1 = -\Gamma_{12}^2,\ \Gamma_{22}^2 = -\Gamma_{21}^1,\ \Gamma_{44}^4 = -\Gamma_{43}^3,\ \Gamma_{34}^4 = -\Gamma_{33}^3.$$

Note that the condition of the case 1. are given by

$$\Gamma_{12}^2 \neq 0 \quad \text{and} \quad \Gamma_{21}^1 = 0.$$

And the symmetry of $K_{\mathcal{E}_1}|_{\mathcal{E}_2}$ follow that

$$\Gamma_{34}^1 = 0.$$

From

$$G1211 : \quad -3\Gamma_{11}^2\Gamma_{12}^2$$

we first get that $\Gamma_{11}^2 = 0$. The equation $(G1341)$ now implies that

$$\Gamma_{13}^4 \Gamma_{14}^4 = 0.$$

Let us first assume that $\Gamma_{14}^4 = 0$.
Then $(G1441)$ implies that $\lambda_1 = 0$. Then $(G1212)$ and $(G1221)$ respectively imply that

$$\begin{aligned} &\Gamma_{21}^2(\Gamma_{12}^2 + \Gamma_{21}^2) = 0, \\ &-(2\Gamma_{12}^2 - \Gamma_{21}^2)(3\Gamma_{12}^2 - \Gamma_{21}^2) = 0. \end{aligned}$$

This leads to $\Gamma_{12}^2 = 0$ which is a contradiction.

Therefore we must have that

$$\Gamma_{14}^4 \neq 0 \quad \text{and} \quad \Gamma_{13}^4 = 0.$$

The equation $(G1221)$ and $(G1212)$ are given by

$$\begin{aligned} -\lambda_1 - 6(\Gamma_{12}^2)^2 + 5\Gamma_{12}^2\Gamma_{21}^2 - (\Gamma_{21}^2)^2 = 0, \\ \lambda_1 + \Gamma_{21}^2(\Gamma_{12}^2 + \Gamma_{21}^2) = 0, \end{aligned}$$

adding this equations, we get that $\Gamma_{21}^2 = \Gamma_{12}^2$. So that

$$G1212 \; : \quad \lambda_1 + 2(\Gamma_{12}^2)^2 = 0,$$

implies that $\lambda_1 = -2(\Gamma_{12}^2)^2$. From the equations

$$\begin{aligned} G1234 &: \quad \Gamma_{14}^4\Gamma_{43}^4 = 0, \\ G1243 &: \quad \Gamma_{14}^4(\Gamma_{33}^4 - 2\Gamma_{43}^3) = 0, \end{aligned}$$

we then deduce that $\Gamma_{23}^4 = \Gamma_{34}^2 = 0$. Whereas

$$\begin{aligned} G1434 &: \quad 2\Gamma_{14}^4\Gamma_{23}^4 = 0, \\ G1343 &: \quad 2\Gamma_{14}^4(\Gamma_{23}^4 - \Gamma_{34}^2) = 0, \end{aligned}$$

imply that $\Gamma_{43}^4 = 0$ and $\Gamma_{33}^4 = 2\Gamma_{43}^3$. The equations $(G1331)$ and $(G1441)$ now reduce to

$$\begin{aligned} 2(\Gamma_{12}^2 - \Gamma_{14}^4)(\Gamma_{12}^2 + 2\Gamma_{14}^4) = 0, \\ 2(\Gamma_{12}^2 - 2\Gamma_{14}^4)(\Gamma_{12}^2 + \Gamma_{14}^4) = 0, \end{aligned}$$

which again lead to the contradiction that $\Gamma_{12}^2 = 0$. Hence these cases are not possible, and there is not hypersurfaces of type.

8.1.2 Type 3.1 : Case d

Due to the form of the difference tensor we have that

$$\Gamma_{21}^1 = \Gamma_{34}^1 = \Gamma_{31}^1 = \Gamma_{41}^1 = \Gamma_{12}^3 = \Gamma_{12}^4 = \Gamma_{14}^4 = \Gamma_{24}^4 = \Gamma_{34}^2 = \Gamma_{33}^3 = \Gamma_{43}^3 = 0.$$

This implies that

$$h(\nabla_{\mathcal{E}_2}\mathcal{E}_2, \mathcal{E}_1) = 0,$$

and in view of the form of K also

$$h(\hat{\nabla}_{\mathcal{E}_2}\mathcal{E}_2, \mathcal{E}_1) = 0.$$

So by a direct computation we get that

$$(\hat{\nabla}K)(E_1, E_3, E_3) - (\hat{\nabla}K)(E_3, E_1, E_3) = h(E_1 \hat{\nabla}_{E_3} E_3) = 0.$$

On the other hand, from the Codazzi equation for K, (4.34) we deduce that

$$(\hat{\nabla}K)(E_1, E_3, E_3) - (\hat{\nabla}K)(E_3, E_1, E_3) = \tfrac{1}{2}(\lambda_1 - \lambda_2)E_1.$$

Hence this case also leads to a contradiction.

8.2 Hypersurfaces of type 3.2

8.2.1 Type 3.2 : Case a or Case b

These cases can be again treated simultaneously as we have again the possibility to define an affine invariant frame. From the choice of the frame we have that

$$\Gamma_{21}^1 = \Gamma_{12}^2 = \Gamma_{43}^3 = \Gamma_{33}^3 = \Gamma_{41}^1 = \Gamma_{12}^3 = 0.$$

Note also that in both Case a and Case b, we have the condition

$$\Gamma_{31}^1 \neq 0.$$

Case a can be recognised by

$$\Gamma_{12}^4 \neq \pm 2\Gamma_{31}^1.$$

Whereas Case b can be recognised by

$$\Gamma_{14}^4 \neq 0, \quad \Gamma_{24}^4 = 0 \quad \text{and} \quad \Gamma_{12}^4 = \pm 2\Gamma_{31}^1.$$

From the equations

$$G1313 : \quad \lambda_2 + 4(\Gamma_{31}^1)^2 = 0,$$

$$G1311 : \quad 5\Gamma_{14}^4\Gamma_{31}^1 = 0,$$

it immediately follows that $\Gamma_{14}^4 = 0$ (and therefore Case b can not occur) as well as that $\lambda_2 = -4(\Gamma_{31}^1)^2$. Similarly from

$$G2322 : \quad -5\Gamma_{24}^4\Gamma_{31}^1,$$

we deduce that $\Gamma_{24}^4 = 0$. From ($G1212$) we get that

$$\lambda_1 = -(\Gamma_{11}^2)^2 - (\Gamma_{21}^2)^2.$$

The equation ($G3443$)

$$4(\Gamma_{31}^1)^2 = (\Gamma_{33}^4)^2 + (\Gamma_{43}^4)^2,$$

implies that Γ_{33}^4 and Γ_{43}^4 can not cancel simultaneously. Therefore from

$$\begin{aligned} G1333 &: \quad \Gamma_{33}^4\Gamma_{34}^1 = 0, \\ G1444 &: \quad -\Gamma_{34}^1\Gamma_{43}^4 = 0, \end{aligned}$$

it follows that $\Gamma_{34}^1 = 0$. Similarly from

$$\begin{aligned} G2333 &: \quad \Gamma_{33}^4\Gamma_{34}^2 = 0, \\ G2444 &: \quad -\Gamma_{34}^2\Gamma_{43}^4 = 0, \end{aligned}$$

we get that $\Gamma_{34}^2 = 0$. The equations ($G1213$) and ($G1224$) become

$$\begin{aligned} -\Gamma_{12}^4\Gamma_{13}^4 + 4\Gamma_{11}^2\Gamma_{31}^1 &= 0, \\ 2\Gamma_{11}^2\Gamma_{12}^4 - 2\Gamma_{13}^4\Gamma_{31}^1 &= 0. \end{aligned}$$

As we are necessary in Case (a), we have that $(\Gamma_{12}^4)^2 - 4(\Gamma_{31}^1)^2 \neq 0$ and therefore the previous system of equations implies that $\Gamma_{13}^4 = \Gamma_{11}^2 = 0$. Similarly from

$$\begin{aligned} G1214 &: \quad 2\Gamma_{12}^4\Gamma_{21}^2 - 2\Gamma_{23}^4\Gamma_{31}^1 = 0, \\ G1223 &: \quad \Gamma_{12}^4\Gamma_{23}^4 - 4\Gamma_{21}^2\Gamma_{31}^1 = 0, \end{aligned}$$

we get that $\Gamma_{21}^2 = \Gamma_{23}^4 = 0$. Repeating the same argument respectively for ($G1314$) and ($G1323$) and ($G1423$) and ($G1414$) we have that $\Gamma_{31}^2 = \Gamma_{33}^4 = 0$ and $\Gamma_{41}^2 = 0$ and $\Gamma_{43}^4 = -2\Gamma_{31}^1$. A contradiction now follows from ($G2424$) which becomes

$$2((\Gamma_{12}^4)^2 - 4(\Gamma_{31}^1)^2) = 0.$$

8.2.2 Type 3.2 : Case c

For this type of hypersurfaces, we want to show the following theorem

Theorem 8.2.1 *Let M^4 be a locally strongly convex, locally homogeneous, affine hypersurface in $\mathbb{R}^5$ whose affine shape operator has two distinct real eigenvalues both of multiplicity 2. Assume that M is of Type 3.2.c. Then $\lambda_1 = 0$ and λ_2 is negative. Moreover M is affine congruent with an open part of the affine hypersurface given by*

$$2x_2x_3x_4 - x_4^2 - x_1\left(x_3^2 - 2x_5\right) - 2x_2^2x_5 = 1.$$

Before starting the proof of the theorem, we need the following lemma.

Lemma 8.2.1 *We can choose the frame $\{E_1, E_2, E_3, E_4\}$ such that*

$$\Gamma_{11}^3 = \sqrt{-\lambda_2}, \qquad \Gamma_{12}^4 = \sqrt{-\lambda_2}, \qquad \Gamma_{21}^4 = \sqrt{-\lambda_2},$$
$$\Gamma_{22}^3 = -\sqrt{-\lambda_2}, \qquad \Gamma_{31}^1 = \frac{\sqrt{-\lambda_2}}{2}, \qquad \Gamma_{32}^2 = -\frac{\sqrt{-\lambda_2}}{2},$$
$$\Gamma_{41}^2 = \sqrt{-\lambda_2}, \qquad \Gamma_{43}^4 = \sqrt{-\lambda_2}, \qquad \Gamma_{44}^3 = -\sqrt{-\lambda_2},$$

and all the other Gammas are vanished.

Proof : From the choice of the frame we have that

$$\Gamma_{21}^1 = \Gamma_{12}^2 = \Gamma_{43}^3 = \Gamma_{33}^3 = \Gamma_{41}^1 = \Gamma_{12}^3 = 0.$$

We also have that, if necessary after changing the sign of E_2 that $\Gamma_{12}^4 = 2\Gamma_{31}^1 \neq 0$. Moreover, we have also $\Gamma_{14}^4 = \Gamma_{24}^4 = \Gamma_{34}^1 = \Gamma_{34}^2 = 0$.

Using the Codazzi equation for K we now find by a straightforward computation that

$$\begin{aligned}
&Glc1231 && (2\Gamma_{11}^2 - \Gamma_{13}^4)\Gamma_{31}^1 = 0,\\
&Glc1241 && (2\Gamma_{21}^2 - \Gamma_{23}^4)\Gamma_{31}^1 = 0,\\
&Glc1331 && \frac{1}{2}\left(4(\Gamma_{31}^1)^2 - \lambda_1 + \lambda_2\right) = 0,\\
&Glc1341 && \Gamma_{31}^1(2\Gamma_{31}^2 - \Gamma_{33}^4) = 0,\\
&Glc2431 && \Gamma_{31}^1(2\Gamma_{31}^1 - 2\Gamma_{41}^2 + \Gamma_{43}^4) = 0.
\end{aligned}$$

Therefore we get that

$$\Gamma_{13}^4 = 2\Gamma_{11}^2, \quad \Gamma_{33}^4 = 2\Gamma_{31}^2, \quad \Gamma_{23}^4 = 2\Gamma_{21}^2, \quad \Gamma_{43}^4 = -2\Gamma_{31}^1 + 2\Gamma_{41}^2.$$

Moreover, we have that

$$\Gamma_{31}^1 = \frac{\sqrt{\lambda_1 - \lambda_2}}{2}.$$

The remaining unknown coefficients coefficients are $\Gamma_{11}^2, \Gamma_{21}^2, \Gamma_{31}^2, \Gamma_{41}^2$. As the frame is not unique these are not necessarily constants.

Now the Gauss equation $(G1331)$ is given by

$$\lambda_1 = 0.$$

The final Gauss equations now reduce to

$$\begin{array}{ll}
G1212 & (\Gamma_{11}^2)^2 + (\Gamma_{21}^2)^2 - E_2(\Gamma_{11}^2) + E_1(\Gamma_{21}^2),\\
G1312 & \Gamma_{21}^2\Gamma_{31}^2 - 2\Gamma_{11}^2\Gamma_{41}^2 + \frac{3}{2}\Gamma_{11}^2\sqrt{-\lambda_2} - E_3(\Gamma_{11}^2) + E_1(\Gamma_{31}^2) = 0,\\
G1334 & 2\Gamma_{21}^2\Gamma_{31}^2 - 4\Gamma_{11}^2\Gamma_{41}^2 + 3\Gamma_{11}^2\sqrt{-\lambda_2} - 2E_3(\Gamma_{11}^2) + 2E_1(\Gamma_{31}^2) = 0,\\
G1412 & 2\Gamma_{11}^2\Gamma_{31}^2 + \Gamma_{21}^2\Gamma_{41}^2 - E_4(\Gamma_{11}^2) + E_1(\Gamma_{41}^2) = 0,\\
G2312 & -\Gamma_{11}^2\Gamma_{31}^2 - 2\Gamma_{21}^2\Gamma_{41}^2 + +\frac{1}{2}\Gamma_{21}^2\sqrt{-\lambda_2} - E_3(\Gamma_{21}^2) + E_2(\Gamma_{31}^2) = 0,\\
G2412 & 2\Gamma_{21}^2\Gamma_{31}^2 - \Gamma_{11}^2\Gamma_{41}^2 + \Gamma_{11}^2\sqrt{-\lambda_2} - E_4(\Gamma_{21}^2) + E_2(\Gamma_{41}^2) = 0,\\
G3412 & 2\left((\Gamma_{31}^2)^2 + (\Gamma_{41}^2)^2 - \Gamma_{41}^2\sqrt{-\lambda_2}\right) - E_4(\Gamma_{31}^2) + E_3(\Gamma_{41}^2) = 0.
\end{array}$$

These equations are precisely the existence conditions, see Appendix 2 of [Lau65]

$$[E_i, E_j]\varphi = \nabla_{E_i}E_j(\varphi) - \nabla_{E_j}E_i(\varphi) = E_i(E_j(\varphi)) - E_j(E_i(\varphi)), \text{ for } 1 \leq i, j \leq 4.$$

for the following system of differential equations

$$\begin{cases}
E_1(\varphi) = -\Gamma_{11}^2,\\
E_2(\varphi) = -\Gamma_{21}^2,\\
E_3(\varphi) = -\Gamma_{31}^2 - \frac{\sqrt{-\lambda_2}}{2}\sin 2\varphi,\\
E_4(\varphi) = -\Gamma_{41}^2 + \frac{\sqrt{-\lambda_2}}{2}\cos 2\varphi + \frac{\sqrt{-\lambda_2}}{2},
\end{cases} \tag{8.1}$$

Taking therefore a solution of this system and using the freedom in the frame to define,

$$\begin{cases}
E_1^* = \cos\varphi\,.E_1 + \sin\varphi\,.E_2,\\
E_2^* = -\sin\varphi\,.E_1 + \cos\varphi\,.E_2,
\end{cases} \qquad
\begin{cases}
E_3^* = \cos 2\varphi\,.E_3 + \sin 2\varphi\,.E_4,\\
E_4^* = -\sin 2\varphi\,.E_3 + \cos 2\varphi\,.E_4,
\end{cases}$$

we see that the new frame satisfies all previous conditions and that moreover:

$$\hat{\nabla}_{E_1^*}E_1^* = \left((E_1(\varphi) + \Gamma_{11}^2)\cos\varphi + (E_2(\varphi) + \Gamma_{21}^2)\sin\varphi\right)E_2^* + \frac{\sqrt{-\lambda_2}}{2}E_3^*$$

$$
\begin{aligned}
&= \frac{\sqrt{-\lambda_2}}{2} E_3^*, \\
\hat{\nabla}_{E_2^*} E_1^* &= \left(-(E_1(\varphi) + \Gamma_{11}^2)\sin\varphi + (E_2(\varphi) + \Gamma_{21}^2)\cos\varphi\right) E_2^* + \frac{\sqrt{-\lambda_2}}{2} E_4^* \\
&= \frac{\sqrt{-\lambda_2}}{2} E_4^*, \\
\hat{\nabla}_{E_3^*} E_3^* &= 2\left((E_3(\varphi) + \Gamma_{31}^2)\cos 2\varphi + \left(E_4(\varphi) + \Gamma_{41}^2 - \frac{\sqrt{-\lambda_2}}{2}\right)\sin 2\varphi\right) E_4^* = 0, \\
\hat{\nabla}_{E_4^*} E_3^* &= 2\left(-(E_3(\varphi) + \Gamma_{31}^2)\sin 2\varphi + \left(E_4(\varphi) + \Gamma_{41}^2 - \frac{\sqrt{-\lambda_2}}{2}\right)\cos 2\varphi\right) E_4^* \\
&= \frac{\sqrt{-\lambda_2}}{2} E_4^*,
\end{aligned}
$$

Therefore we may assume that

$$\Gamma_{11}^2 = 0, \quad \Gamma_{21}^2 = 0, \quad \Gamma_{31}^2 = 0, \quad \Gamma_{41}^2 = \sqrt{-\lambda_2}.$$

This completes the proof of the lemma. □

proof Proof of Theorem 8.2.1. We use the local frame constructed in the previous lemma. By applying a homothety we may assume that $\lambda_2 = -1$. The Lie brackets of the last frame are given by

$$
\begin{aligned}
&[E_1, E_2] = 0, \quad [E_1, E_3] = -\frac{\sqrt{-\lambda_2}}{2} E_1, \quad [E_1, E_4] = -\sqrt{-\lambda_2} E_2, \\
&[E_2, E_3] = \frac{\sqrt{-\lambda_2}}{2} E_2, \quad [E_2, E_4] = 0, \quad [E_3, E_4] = -\sqrt{-\lambda_2} E_2.
\end{aligned}
$$

Therefore a straightforward computation shows that the integrabilty conditions, see Appendix 2 of [Lau65], of the following system of differential equations for the functions ρ_1, ρ_2 and ρ_3

$$
\begin{aligned}
&E_1(\rho_1) = 0, \quad E_2(\rho_1) = 0, \quad E_3(\rho_1) = \rho_1 \frac{\sqrt{-\lambda_2}}{2}, \quad E_4(\rho_1) = 0, \\
&E_1(\rho_2) = 0, \quad E_2(\rho_2) = 0, \quad E_3(\rho_2) = 0, \quad E_4(\rho_2) = \frac{\sqrt{-\lambda_2}}{\rho_1^2}, \\
&E_1(\rho_3) = \rho_1 \sqrt{-\lambda_2}, \quad E_2(\rho_3) = 0, \quad E_3(\rho_3) = 0, \quad E_4(\rho_3) = 0,
\end{aligned}
$$

are satisfied. We now define

$$E_1^* = \tfrac{1}{\rho_1} E_1 - \tfrac{1}{2}\rho_1 \rho_2 E_2,$$

$$\begin{aligned}
E_2^* &= \rho_1 E_2,\\
E_3^* &= E_3,\\
E_4^* &= \tfrac{1}{2}\rho_1\rho_3 E_2 + \rho_1^2 E_4.
\end{aligned}$$

We then get that $[E_i^*, E_j^*] = 0$. So there exist local coordinates (x, z, u, y), such that

$$(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial u}, \frac{\partial}{\partial y}) = (\tfrac{1}{\rho_1}E_1 - \tfrac{1}{2}\rho_1\rho_2 E_2, \rho_1 E_2, E_3, \tfrac{1}{2}\rho_1\rho_3 E_2 + \rho_1^2 E_4).$$

Choosing the initial conditions for the functions ρ_i appropriately we get that

$$\rho_1 = e^{\frac{\sqrt{-\lambda_2}}{2}u}, \qquad \rho_2 = y, \qquad \rho_3 = x.$$

Note that

$$\begin{cases}
E_1 = \rho_1\left(\partial_x + \dfrac{1}{2}y\partial_z\right),\\
E_2 = \dfrac{1}{\rho_1}\partial_z,\\
E_3 = \partial_u,\\
E_4 = \dfrac{1}{\rho_1^2}\left(\partial_y - \dfrac{1}{2}x\partial_z\right).
\end{cases}$$

In the same way as in the previous theorems we now determine the system of differential equations for the position vector F and the affine normal of the immersion. We get that

$$\xi_x = \xi_z = 0, \qquad \xi_u = F_u, \qquad \xi_y = F_y - \tfrac{1}{2}xF_z,$$

and

$$\begin{aligned}
F_{uu} &= \xi,\\
F_{ux} &= 0,\\
F_{uy} &= F_y - \tfrac{1}{2}xF_z,\\
F_{uz} &= 0,\\
F_{zz} &= e^u\left(-F_u + \xi\right),\\
F_{zx} &= \tfrac{1}{2}e^{-u}\left(2F_y - xF_z + ye^{2u}F_u - ye^{2u}\xi\right),\\
F_{zy} &= \tfrac{1}{2}xe^u\left(-F_u + \xi\right),\\
F_{xx} &= \frac{1}{4}e^{-u}\left(\left(4 - e^{2u}y^2\right)F_u - 4F_y + 2xyF_z + \left(-e^{2u}y^2 + 4\right)\xi\right),\\
F_{xy} &= \tfrac{1}{4}e^{-u}\left(x\left(e^{2u}yF_u + 2F_y - e^{2u}\xi y\right) + \left(2e^u - x^2\right)F_z\right),\\
F_{yy} &= \tfrac{1}{4}e^u(4e^u + x^2)\left(-F_u + \xi\right).
\end{aligned} \tag{8.2}$$

Deriving the first equation with respect to u, we get

$$F_{uuu} - F_u = 0.$$

Its solutions are given by

$$F(u, x, y, z) = A_1(x, y, z) + A_2(x, y, z)e^u + A_3(x, y, z)e^{-u}, \tag{8.3}$$

The other differential equations involving the variable u then imply that A_3 is a constant, A_2 depends only on the variable y and

$$(A_1)_y = \tfrac{1}{2}x(A_1)_z. \tag{8.4}$$

From the differential equations F_{zz} we obtain that

$$\xi = e^{-u}\left((A_1)_{zz} - A_3 + A_2e^{2u}\right).$$

Substituting this in the differential equation for F_{yy} we see that

$$(A_1)_{zz} = (A_2)_{yy}$$

The differential equations for ξ imply also that

$$(A_1)_{zz} = 2A_3.$$

Therefore it follows that we can write

$$A_2(y) = A_3y^2 + C_1y + C_2,$$

where C_1 and C_2 are constant vectors. Moreove, using also the differential equation for F_{xz} we see that we can write

$$A_1(x, y, z) = A_3z^2 + (xyA_3 + C_1x + B_1(y))z + B_2(x, y).$$

Expressing now (8.4) yields

$$xzA_3 + (B_1)_yz + (B_2)_y = xzA_3 + \tfrac{1}{2}(x^2yA_3 + C_1x^2 + B_1(y)x)).$$

Therefore $B_1(y) = C_3$ a constant vector and

$$B_2(x, y) = \tfrac{1}{4}x^2y^2A_3 + \tfrac{1}{2}x^2yC_1 + \tfrac{1}{2}xyC_3 + B_3(x).$$

The only remaining equation of our system of differential equations now becomes

$$(B_3)_{xx} = 2C_2.$$

So after applying a translation in $\mathbb{R}^5$ we see that we can write

$$B_3(x) = C_2x^2 + C_4x.$$

Mapping now $C_2, C_4, C_3, C_1, 2A_3$ to the standard basis completes the proof of the theorem □

Note that all of the examples appearing in the theorem are indeed affine homogeneous. As the first two already appear as generalised Calabi products in [DV94a] we only have to consider the final example.

We look at the set of equiaffine matrices given by

$$b(x,y,z,u) = \begin{pmatrix} e^u & 2e^{u/2}x & 0 & 0 & 0 & x^2 \\ 0 & e^{u/2} & 0 & 0 & 0 & x \\ 0 & e^{u/2}y & e^{-u/2} & 0 & 0 & \frac{1}{2}(xy+2z) \\ e^u y & e^{u/2}\left(\frac{3xy}{2}+z\right) & e^{-u/2}x & 1 & 0 & \frac{1}{2}x(xy+2z) \\ \frac{e^u y^2}{2} & \frac{1}{2}e^{u/2}y(xy+2z) & e^{-u/2}\left(\frac{xy}{2}+z\right) & y & e^{-u} & \frac{1}{8}(xy+2z)^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As

$$\begin{aligned} b(x,&y,z,u)b(x',y',z',u') \\ &= b(x+e^{\frac{u}{2}}x', y+e^{-u}y', -\tfrac{1}{2}e^{-u}xy' + \tfrac{1}{2}e^{u/2}x'y + e^{-u/2}z' + z, u+u'), \end{aligned}$$

the set of above matrices forms a group. It is clear that the image of our surface is the orbit of the point $(1,0,0,0,1,1)$.

Conclusion

In the first part of this thesis, paper 1, we have study 5-dimensional spheres with nearly Sasakian structure and nearly cosymplectic structure, who are in the almost contact manifolds case, useful for the quantum physic, especially in the proofs of the supersymmetry study.

We proved that the surfaces in the 5-sphère with nearly Sasakian structure or nearly cosymplectic structure are always minimal. This result is false in the spheres with Sasakian or Khealer structure and in many others. This gives a great asset to these structures. We also give a direct relation with Lagrangian surfaces.

De-Necola and his collaborators in there paper "On nearly sasakian and nearly cosymplectic manifolds" [DNDY16] (30 mars 2016, no published), prove that every nearly Sasakian manifold of dimension greater than five is Sasakian !, which limits the study of these manifolds to the dimension five. Moreover, they classify nearly cosymplectic manifolds of dimension greater than five.

For the second part, we have classified a 4-dimensional locally strongly convex, locally homogeneous, hypersurfaces whose affine shape operator has two distinct principal curvatures, such that the multiplicity of both eigenvalues is 2. The hypersurfaces of dimension 4 who S has two eigenvalues of multiplicity 1 and 3, has bieen classifiend by Dillem-Vrancken in [DV94b]. If we use the Dillem-Vrancken conjecture,see [LMSS96], stating that for an positive definite affine homogeneous affine hypersurface or a positive definite isoparametric affine hypersurface the affine shape operator S has at most one non zero eigenvalue, then we have a global classification of the dimension four.

For the dimensions greater to four, there are a classification of the quasi-umbilical in 5-dimensional hypersurfaces see [DV93b] and [HLZ14], bat all the others cases remains open subjects.

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